Optimal Managerial Authority

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Abstract

I develop a dynamic agency model to investigate optimal managerial authority and its interaction with managerial compensation. The model shows that when hiring a manager, the principal delegates authority that is unresponsive to either the manager's outside options or the firm's recruitment costs, in contrast to promised compensation, which increases in both. Over time, both the manager's authority and his compensation rise after good performances and decline after bad realizations. Authority-performance sensitivity decreases as the manager's authority grows, resembling entrenchment. In contrast, pay-performance sensitivity increases with the manager's authority. If managerial authority can be adjusted only infrequently, the optimal contract may allow for self-dealing. The model delineates career trajectories that lead to managerial self-dealing. Moreover, the model reveals that early-career luck plays a disproportionate role in determining the manager's authority and lifetime utility.

Key words: delegation, dynamic contracting, managerial compensation, managerial turnover, self-dealing

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1 Introduction

Properly exercising decision-making authority is crucial in the operation of firms. In a modern corporation, shareholders seldom make operational decisions. They instead delegate a majority of operational decision-making authority to professional managers who possess expertise and superior information (Dessein 2002). A manager's main duty is to properly exercise the authority delegated by corporate owners (Bolton and Dewatripont 2013). Therefore, the optimal allocation of authority is central in designing a managerial job. The following questions arise naturally: How much authority should be granted to newly hired manages? How should their level of authority evolve over time?[1] And how should their authority interact with compensation?

In this paper, I build a dynamic agency model that characterizes the optimal delegation of authority and its interaction with managerial compensation. This model explains several stylized facts that have not yet been well addressed in the literature. First, when recruiting for a managerial position, companies alter the manager's compensation level in response to varying labor market conditions, but not the level of delegated authority.[2] Second, a manager's authority is sensitive to his past performance and increases after good performance, but this sensitivity decreases as his authority grows, which resembles managerial entrenchment. In contrast, since a manager is granted more stock and options as authority grows, his pay-performance sensitivity increases with authority.[3] Furthermore, the model provides novel implications concerning the interactions between managerial authority, compensation, and career trajectories, all of which are discussed in detail in Section 4.

[1] Authority delegation is dynamic in firms. A well-performing middle-level manager will usually be assigned to lead a larger team; a CEO with good past performance can be granted a dual role as the board chairman or president. Or conversely, a manager can also be divested of part of his authority due to misconduct or poor performance. One recent example is the Volkswagen case. In June 2020, Volkswagen AG replaced the company CEO Herbert Diess's dual role as chief of namesake brand after vehicle delays and clashes with labor unions. (https://www.wsj.com/articles/volkswagen-board-considering-management-shake-up-for-vw-brand-11591632658)

[2] Empirical studies document that managerial compensation varies according to labor market conditions (see, e.g., Bizjak et al. (2008); Brookman and Thistle (2013), among others). On the other hand, an employment contract usually states "the executive shall have the duties and responsibilities **typical for such position** and may otherwise be **assigned or modified** by the CEO or the Board of Directors." This verbiage demonstrates that (1) the initial delegated authority is associated with the managerial position only and (2) the dynamic and evolving nature of authority delegation over time.

[3] Edmans et al. (2017) comprehensively review executive compensation. Figures 6, 7, and 8 give examples of where the proportions of stocks and options increase in managerial authority (by a cross-sectional comparison between CEO and non-CEO executives).

The study of optimal delegation was pioneered by Holmstrom (1977, 1984). Much of this literature focuses on delegation without monetary transfers, which limits its application to firms. In firms, performance-sensitive compensation is an important tool to align interests of the owners and managers and make delegation profitable. Understanding how optimal managerial authority and managerial compensation interact, especially in a dynamic world, is therefore of importance.[4]

To investigate optimal managerial authority and the corresponding managerial compensation in firms, I adopt a dynamic contracting approach[5] and study multi-task delegation problems in a dynamic environment, allowing for private savings and borrowing, as well as costly managerial turnover. The model is set up in discrete time to clarify the agency problems and is solved in continuous time for analytical tractability.

In the model, a risk-neutral principal ("she") has one project in each period. A project comprises a continuum of different tasks, each affecting the project's probability of success. These tasks can be understood as operational decisions, for example, about setting budgets or selecting suppliers. Each task requires a decision to be made among many different options. The principal cannot distinguish among the options. A qualified manager ("he") has expertise and can distinguish among all the options. Therefore, the principal may want to delegate some decision-making authority to the manager. Among the delegated tasks, the manager can make decisions that increase the project's probability of success; alternatively, he can pick the options that contain private benefits but decrease the project's probability of success. The principal incentivizes the manager to make good decisions by linking his current and future compensation to the project's output. She also optimally chooses the set of tasks to delegate for each period. Specifically, when hiring, the principal provides a full-commitment contract on output-contingent managerial authority and the compensation process. This contract is equivalent to a series of spot contracts provided at the beginning of each period, specifying the manager's authority and wage

[4]Ottaviani (2000) considers a static uniform-quadratic case with full delegation and action-contingent transfers. Krishna and Morgan (2008) focus on a static case in which the principal can commit to a transfer rule but retains decision-making authority.

[5]I apply the contracting approach, because, on the one hand, contracting is a common approach to managerial compensation problems. An employer contracts on compensation to incentivize a manager to make profitable decisions. On the other hand, the formal authority of a manager is delegated ultimately by firm owners through explicit or implicit contracts (Aghion and Tirole 1997). Therefore, contracting is also a natural approach to study delegation problems.

in the current period, and his output-contingent continuation value. The principal commits intertemporally to the manager's continuation value in the firm.[6] In contrast, the manager has limited commitment and can quit at any time. If the manager leaves, the principal can hire a qualified replacement with a constant cost. All managerial candidates have constant absolute risk aversion (CARA) preference and can save and borrow privately.

In the main model, managerial authority can be adjusted in each period. This setting gives rise to the following optimal mechanism: the promised continuation value increases after good performances and decreases after bad realizations; managerial authority monotonically increases in the manager's continuation value, as does his pay-performance sensitivity; and the relative magnitude of change in authority and compensation decreases with the manager's continuation value.

This mechanism demonstrates the dynamic misalignment effect on authority delegation. The intuition underlying this effect is as follows. To extract more good decisions from the manager, the principal needs to delegate more authority and make the manager's compensation more sensitive to the project's output. However, raising pay-performance sensitivity is costly, not only because of the manager's risk aversion but also due to potential managerial turnover. In this model, without a wealth effect and because the manager can smooth consumption by private savings and borrowing,[7] the dynamics of misalignment is entirely driven by the manager's limited commitment and associated turnover costs: the misalignment problem becomes more severe when the manager is closer to departure, which results in less authority delegation and lower pay-performance sensitivity. In contrast, in the benchmark case in which the manager has full commitment and never leaves the firm, dynamic misalignment disappears, and the degree of misalignment is constant over time. Therefore, the optimal managerial authority and pay-performance sensitivity are also constant and have reached a pinnacle.

[6] Spear and Srivastava (1987) and Phelan (1995) prove that efficient contracts can be written recursively with commitment on the continuation value of the manager. In Section 4, I assume that opportunities to adjust managerial authority follow a Poisson distribution and can be noncontractible. Therefore, I adopt recursive contracts.

[7] If the manager cannot save or borrow privately, the principal is able to control the manager's consumption path and will usually distort payment timing to provide additional incentives. For example, Hoffmann and Pfeil (2021) discuss how deferring compensation increases the agent's (manager's) stake in the firm and provides incentives. Grochulski and Zhang (2021) study how temporarily suspending the agent (manager) given no consumption can rebuild his "skin in the game" and restore his incentives.

The mechanism explains the stylized fact that a long-serving manager who has a high level of authority seems entrenched, in the sense that the level of his authority becomes less sensitive to performance, while his compensation becomes more equity or option based (e.g., Edmans et al. 2017). The intuition behind this result is as follows: The dynamic misalignment problem fades when the manager's continuation value is sufficiently high. Hence, the manager gains more authority, the level of which is less sensitive to the manager's performance. Meanwhile, the principal selects a contract with higher pay-performance sensitivity, which is implemented by options or additional units of stocks, to provide incentives for good decisions. The opposite evolvement of the authority-performance sensitivity and the pay-performance sensitivity is one of the main findings in this paper.

I also show that the initial authority delegated to a newly hired manager is tied to the position and is independent of the labor market conditions. The compensation level, in contrast, varies according to the labor market conditions. In other words, changes in the manager's outside options or the firm's recruitment costs do not affect the authority initially allocated to a newly hired manager. This result comes from the principal optimally offsetting the effects of labor market conditions on the severity of dynamic misalignment by adjusting the initial continuation value of the manager.

I then extend the model so as to examine the case in which authority is adjusted less frequently than is pay-performance sensitivity. In reality, managerial authority is adjusted infrequently due to various frictions.[8] In contrast, the pay-performance adjusts automatically with the firm's performance if stock options constitute part of the compensation package or if the number of stocks granted to the manager changes. The paper finds that if the opportunity to change the manager's authority arises only intermittently, while the pay-performance sensitivity can be adjusted frequently, the manager may engage in self-dealing (i.e., inefficient consumption of private benefits). Managerial self-dealing is tolerated by the principal, even though she could eliminate it by setting a sufficiently high pay-performance sensitivity. Therefore, the principal is in effect using private benefits as a cheaper alternative to compensation, at the cost of productive efficiency.

[8]For example, infrequent changes to the board composition (Adams and Ferreira 2007) or a deadlock on the board (Donaldson et al. 2020) may lead to a lag in the adjustment of a top executive's authority.

Importantly, the case of infrequent authority adjustment predicts that luck in one's early career is paramount in determining the manager's authority and lifetime utility. Moreover, with the analysis I delineate the career trajectories leading to managerial self-dealing. If a manager experiences a series of good realizations in the early stages of his career, he becomes better aligned with the firm and will be granted more authority in the future. Thereafter, the manager has more discretion in making decisions, generating more profits for the firm and gaining higher compensation for himself. Later in his career, if he suffers from negative shocks, he can take advantage of his high level of authority and engage in self-dealing, that is, acting in his own best interest, rather than in that of the firm's, thereby keeping his lifetime utility high.

In contrast, if the manager first encounters negative shocks, he will be stripped of part of his authority since the misalignment becomes more severe. Thereafter, he gets stuck in a low-authority situation even if he later experiences positive shocks and becomes better aligned with the principal. He cannot well exploit his superior knowledge. Consequently, his lifetime utility is lower. This story depicts the career trajectory that leads to managerial self-dealing: a manager who experiences good luck in the early stages and bad luck in the later years of his career is more likely to engage in self-dealing.

Related Literature This paper bridges two strands of literature, the literature of optimal delegation and the dynamic contracting literature.

The optimal delegation literature was pioneered by Holmstrom (1977, 1984). This strand of literature studies the optimal allocation of decision-making authority between the principal and the agent when the agent has private information. Alonso and Matouschek (2008) investigate conditions for interval delegation to be optimal and provide explanations for the widespread use of threshold delegation (a particular type of interval delegation). Amador and Bagwell (2013) generalize the results by considering a general class of preferences and provide necessary and sufficient conditions for the optimality of interval delegation. Some related work studies full delegation and compares delegation with communication, for example, Dessein (2002) and Ottaviani (2000). Li et al. (2017) and Lipnowski and Ramos (2020) examine dynamic delegation without monetary transfer in a repeated games setup.[9] Most of the delegation literature considers the case in which monetary transfers are unavailable. Relative to this literature, my model sheds light on the dynamic interaction between optimal multi-task delegation and optimal compensation.

Methodologically, the paper belongs to the continuous-time dynamic contracting literature. Optimal delegation is a seldom-visited topic in this strand of literature. One relevant study is done by Malenko (2019), who examines the capital allocation process in an organization when the manager has empire-building preferences. He finds that the threshold delegation of investment decisions is optimal, and the level of delegation decreases with the agent's continuation value. In my paper, the manager's operational decision-making authority increases with the manager's continuation value.

More broadly, this paper is related to the hidden-action models in the dynamic contracting literature. One strand of the literature analyzes the agent's hidden efforts, for example, Sannikov (2008), Zhu (2013), and Grochulski and Zhang (2021). My paper differs from their hidden effort models in that authority delegation is a choice variable of the principal. The principal can use authority delegation to restrict the action space of the manager.[10] Another major strand of the dynamic contract literature considers cash-flow diversion models, for example, DeMarzo and Sannikov (2006), Biais et al. (2007), Piskorski and Westerfield (2016), and Hoffmann and Pfeil (2021), among others. In these papers, cash flows are privately observed and can be diverted, thereby creating an *ex post* moral hazard problem. Besides, in these papers, the optimal aggregate incentives level is constant due to the constant diversion efficiency, and, consequently, moral hazard (i.e., diversion) is eliminated in equilibrium.[11] Noe and Rebello (2012)

[9] More generally, the delegation literature belongs to a set of theories of optimal rules: the relationship between the ultimate objective of the rule-setter and the optimal rule to commit to. Some relevant studies are Aghion and Tirole (1997), Burkart et al. (1997), Armstrong and Vickers (2010), Frankel (2014), but they consider scenarios different from the delegation literature. For example, Aghion and Tirole (1997) and Burkart et al. (1997) investigate the impact of the manager's authority on the information structure, while the delegation literature takes the information structure as given.

[10] The predictions in their paper are also different from mine. In Zhu (2013), effort levels are binary, and the effort may increase or decrease with the agent's continuation value, depending on the parameters. Sannikov (2008) and Grochulski and Zhang (2021) consider a risk-averse agent with non-monetary effort costs. It is more difficult to incentivize efforts as the agent's continuation value increases. Consequently, there exists a high retirement point.

[11] Piskorski and Westerfield (2016) study costly monitoring. Monitoring differs from delegation in that monitoring deters undesirable actions by threat of potential punishment, which is an incentive device and can substitute for pay-performance sensitivity. In contrast, delegation directly controls the agent's action space. Moreover, more intensive monitoring generally improves the firm's performance but less delegation usually leads to worse outcomes compared to the first-best case.

study managerial compensation and costly monitoring with dynamic learning of a latent firm characteristic. They find that monitoring intensity is negatively correlated with managerial compensation and the firm's fortune, and managerial private benefits may ameliorate agency conflicts.

The technical assumptions of this paper follow He (2011), who solves the double-deviation problem with private savings and borrowing by adopting the CARA preference.[12] My paper adds to that model by allowing the agent (manager) to have limited commitment.[13] Relative to the continuous-time dynamic contracting literature, I provide a discrete-time setup with an *ex ante* moral hazard problem and also disentangle the dynamic misalignment effect due to one-sided commitment from the wealth effect and the deferred compensation effect.

The job design aspect of this paper is related to Itoh (1994), Axelson and Bond (2015), Ke et al. (2018), and Ferreira and Nikolowa (2020), as well as to studies in personnel economics.[14] Axelson and Bond (2015) develop an incentive-based theory of finance jobs in an equilibrium framework. They focus on how to allocate pre-specified jobs to agents with hidden efforts, and find that the labor market conditions profoundly affect the jobs allocated to a young agent as well as his subsequent career. The timing distortion in payment and the consequent performance bond effect is a major force driving their results. Ferreira and Nikolowa (2020) develop a theory of optimal job creation technology where employees gain utility from both consumption and job prestige. Relative to this literature, my paper investigates the dynamic job design for a managerial position, emphasizing on-the-job authority dynamics and the interaction between the manager's authority-performance sensitivity and pay-performance sensitivity.

[12] When private savings and borrowing are allowed, the first-order approach may fail, that is, the first-order conditions may not guarantee full incentive compatibility. Generally speaking, the contracting problem becomes very difficult with private savings and borrowing. See Section 6 of Sannikov (2008) for more discussion.

[13] A majority of models that simultaneously consider the agent's limited commitment and private savings and borrowing in the continuous-time contracting literature assume a risk-neutral agent with limited liability and inefficient private savings technology, for example, Hoffmann and Pfeil (2010), DeMarzo et al. (2012), and Hoffmann and Pfeil (2021). In these models, the principal can manipulate payment timing and there exists deferred compensation. In my model, due to the efficient private savings and borrowing technology, the principal cannot distort the payment timing and, therefore, cannot use it as an additional tool to incentivize the agent.

[14] Lazear and Shaw (2007) and Lazear (2018) provide literature review on personnel economics.

2 The Model

The model is set up in discrete time to better clarify the agency problem. This discrete-time setting lays down a clear conceptual foundation for continuous-time modeling and allows me to derive a rich set of predictions.

2.1 Technology and authority delegation

The model considers a firm with an infinite life span. Time is partitioned into intervals with a length of $\delta > 0$, that is, $t = 0, \delta, 2\delta, \ldots$. The discount factor is $1/(1 + r\delta)$, where $r < 1$ is the common discount rate in this economy. A risk-neutral principal ("she") hires a manager ("he") to operate the firm. The manager has CARA preferences, and his instantaneous utility is represented by $u(c_t) = -e^{-\gamma c_t}$. The manager is allowed to privately save or borrow against his employment contract at the risk-free rate, r.

In each period of time, the firm undertakes a project that may succeed or fail. If the project succeeds, it generates net profits of $y_H = \sqrt{\delta}$ at the end of the period; if it fails, it generates $y_L = -\sqrt{\delta}$. In other words, $y_{t+\delta} \in \{-\sqrt{\delta}, \sqrt{\delta}\}$. A project comprises a mass-one continuum of tasks, all of which affect the project's probability of success. Take a manufacturing firm as an example. The tasks typically involve designing production lines, investing in machinery, choosing input materials, selecting suppliers, training employees, advertising, and marketing, etc.

A task is modeled as making one decision among infinitely many options. Of all the options in a task, only two are most relevant: the good one increases the project's probability of success; the bad one decreases the project's probability of success but carries private benefits; other options neither affect the probability nor contain private benefits. Both parties are aware of the task's impact on the project and the level of private benefits contained in the bad option. However, only the manager can distinguish among the options.[15] Therefore, both the good option and the bad option are picked with a probability of zero if the principal makes the decision. Moreover, if

[15] This setup is for elaboration simplicity. An equivalent and more general setup could be as follows. The manager has full information while the principal knows the distribution of the options: their effects on the project and private benefits contained. The distribution has the following characteristics: (1) the effects to the project are zero in expectation; (2) only finite many options contain private benefits; (3) the good option has the greatest positive effect on the project's probability of success

the principal retains full authority and makes decisions on all the tasks herself, the project will succeed or fail with equal probability; that is, the net present value (NPV) of the project is zero.[16]

Assumption 1 *Let $i \in [0,1]$ represent the index of a task.*
The good option in task i increases the project's probability of success by $\frac{1}{2}\sqrt{\delta}di$; the bad option in task i decreases the project's probability of success by $\frac{1}{2}\sqrt{\delta}di$ but carries private benefits of $B_i\delta di$, where $B_i = i$.

As an illustration of Assumption 1, consider that if all tasks are delegated to the manager and all decisions made by him are good (i.e., he chooses all the good options), the project's probability of success increases by $\int_0^1 \frac{1}{2}\sqrt{\delta}di = \frac{1}{2}\sqrt{\delta}$, and the project's expected profits increase by $\frac{1}{2}\sqrt{\delta}\cdot\sqrt{\delta} + (-\frac{1}{2}\sqrt{\delta})\cdot(-\sqrt{\delta}) = \delta$; if he makes all bad decisions on this continuum of tasks (i.e., he chooses all the bad options), the expected profits of the project decrease by δ, but he gains private benefits amounting to $\int_0^1 i\delta di = \frac{1}{2}\delta$.

Table 1 provides another illustration of Assumption 1. The table demonstrates how the options in Task i affect the project's expected profits. For one unit of task i, the good option increases the project's expected profits by δ, while the bad option decreases the project's expected profits by the same amount but delivers private benefits of $i\delta$. All the other options are neutral on the project and do not deliver private benefits.

Table 1: Options and Impacts of Task i (per unit)

Options in Task i:	Good Option	Bad Option	Other Options
Change in the project's expected profit	δ	$-\delta$	0
Private benefits	0	$i \cdot \delta$	0

Assumption 1 implies that each task has the same effect on the project's probability of success. They differ in the level of potential private benefits and are sorted according

but contains no private benefits; and (4) the bad option contains the highest private benefits but has an inverse effect on the project compared to the good one.

[16]The assumption is realistic because, in practice, firm owners are usually aware of the levels of potential private benefits associated wth different tasks, but they lack available attention/expertise to identify all the options to reach the optimal decision for each task. For instance, they know that the supplier choice may deliver more private benefits than employee training arrangements to the manager. However, they usually lack the information, expertise, and attention to select the most suitable supplier or to design the best employee training program.

to the level of potential private benefits. The assumption $B_i = i \leqslant 1$ makes sure that the bad option is indeed the worst of all the options.

Let $\alpha_t \in [0, 1]$ denote the measure of authority (i.e., the number of tasks) delegated to the manager at the beginning of period t. It is intuitive that the tasks delegated are those with the smallest private benefits, because it is easier to incentivize good decisions in these tasks. The principal retains the decision-making authority on tasks with high private benefits to prevent the manager from making bad decisions. Let $\theta_t \in [0, \alpha_t]$ denote the measure of good decisions made by the manager in period t. Figure 1 visualizes the allocation of authority for the period t.

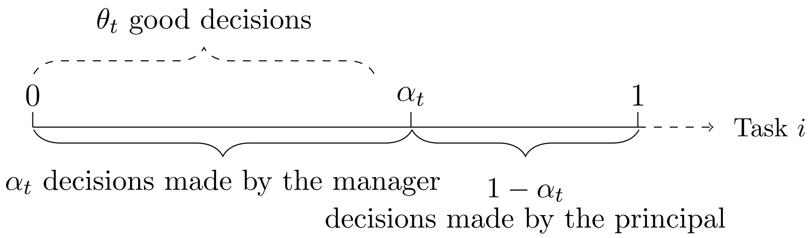


Figure 1: Authority delegation at time t

By simple calculations, the distribution of cash flows at the end of period t can be expressed in the manager's authority α_t, and the number of good decisions θ_t:

$$y_{t+\delta} = \begin{cases} y_H = \sqrt{\delta}, & Pr(y_H) = \dfrac{1}{2}[1 + (2\theta_t - \alpha_t)\sqrt{\delta}], \\ y_L = -\sqrt{\delta}, & Pr(y_L) = \dfrac{1}{2}[1 - (2\theta_t - \alpha_t)\sqrt{\delta}]. \end{cases} \tag{1}$$

REMARK 1 *The authority delegation considered in this paper can be viewed as an example of the general formulation of the delegation problem defined in Holmstrom (1977, 1984). I term the decision on a task i a "sub-decision", and the sequence of decisions on all the tasks $i \in [0, 1]$ a "joint decision". Let $\mathbf{d}$ denote the joint decision. The decision space D is the set for all feasible joint decisions. The delegation process in this paper, where the principal makes decisions on tasks $i \in (\alpha_t, 1]$ and the manager makes decisions on tasks $i \in [0, \alpha_t]$, starts from the principal choosing the control set C, where $C \subseteq D$. The control set C contains all the joint decisions $\mathbf{d}$ where the sub-decisions on tasks $i \in (\alpha_t, 1]$ are fixed and predetermined by the principal. Then, the manager chooses a joint decision $\mathbf{d} \in C$, completing this delegation process.*

2.2 The contract

The contract provided by the principal specifies an authority delegation process $\alpha = \{\alpha_t\}_{t \geqslant 0}$ and an output-contingent wage process $w = \{w_t\}_{t \geqslant 0}$. Given the contract, the manager maximizes his expected discounted utility by choosing a decision process $\theta = \{\theta_t\}_{t \geqslant 0}$ and a consumption process $c = \{c_t\}_{t \geqslant 0}$.[17] Here, w_t and c_t are written as rates, so as to be consistent with the notations used for the continuous-time model that I will investigate later. That is, the wage for the period $[t, t + \delta)$ is $w_t \delta$, and the manager's consumption is $c_t \delta$. Without loss of generality, I assume both are end-of-period values.

The quadruple $(\alpha, w; \theta, c)$ is referred to as an **incentive-compatible contract**[18], where (θ, c) is the process of the manager's recommended decisions and consumption. The maximized expected discounted utility at time t is referred to as the manager's continuation value at t, denoted by V_t, which is a start-of-period value.

While the principal can commit to the above long-term contract, the manager will only stay in the relationship when his continuation value, V_t, derived from his future consumption in the firm, is greater than his outside option, $\underline{V}$. In other words, the model assumes a one-sided commitment by the principal. This assumption is consistent with the realities present in labor markets.[19] If the incumbent manager leaves, the principal can hire a replacement. Each time a manager is hired, the firm incurs a recruitment cost, $q \geqslant 0$, which can be understood as a search cost, like fees paid to headhunters, or an orientation cost.

Following Spear and Srivastava (1987) and Phelan (1995), the efficient contract can be written recursively. Since the incumbent manager's continuation value, V_t, is the only state variable in this model, the contract can be equivalently written in the following way. At the beginning of period t, the manager's continuation value V_t is given, and the principal writes an incentive-compatible spot contract $(\alpha_t, w_t, V_{t+\delta}(y_H), V_{t+\delta}(y_L); \theta_t, c_t)$, which keeps the principal's promise on V_t conditional

[17] Given the wage process and the consumption process, the savings and borrowing process is pinned down. To simply the notation, I haven't written down that process explicitly.

[18] An incentive compatible contract is a contract including the agent's recommended strategies. For example, see DeMarzo and Sannikov (2006).

[19] Phelan (1995) points out that many long-term economic relationships are characterized by parties' differing abilities to commit to long-term contracts. In labor markets, while an employer could conceivably sign a contract that offers a worker a job for life, workers cannot promise to never quit or work for another firm.

on the manager behaving as suggested in the contract. This commitment on V_t can be expressed in the form of **the promise-keeping constraint** (Phelan 1995; Fernandes and Phelan 2000):

$$V_t = \frac{1}{1 + r\delta} \cdot \left\{ u(c_t)\delta + [Pr(y_H) \cdot V_{t+\delta}(y_H) + Pr(y_L) \cdot V_{t+\delta}(y_L)] \right\}, \tag{2}$$

where $Pr(y_H)$ and $Pr(y_L)$ are defined in Equation (1).

He (2011) proves that if the manager can save or borrow privately, it is without loss of generality to focus on the incentive-compatible no-savings contracts.[20] In this paper, I follow He (2011) and focus on the contracts that lead to zero savings or borrowing in equilibrium.

REMARK 2 *Rewriting the contract in this fashion demonstrates a realistic way to implement the full commitment by the principal. Rather than providing an extremely complex state-contingent contract that covers the length of the employment relation, the principal only needs to write a spot contract* $(w_t, \alpha_t, V_{t+\delta}(y_H), V_{t+\delta}(y_L))$ *with recommendations on* (θ_t, c_t)*, which together satisfy* ***the promise-keeping constraint*** *(2), at each time* t*.*

3 Model Solutions and Analysis

In this section, I derive the continuous-time limit of the model and solve for the optimal contract. The technical advantages of the continuous-time methods lead to a simpler computational procedure and make the optimal contracting tractable.

3.1 Manager's optimization problem

The manager's problem is to find the optimal choices of (θ, c) given the contract.

First, define

$$\beta_t = \frac{(V_{t+\delta}(y_H) - V_{t+\delta}(y_L))/u'(c_t)}{y_H - y_L}. \tag{3}$$

[20] See He (2011) Lemma 2.

β_t measures the sensitivity of the manager's continuation value with respect to output normalized by his marginal utility. The incentive-compatible spot contract $(\alpha_t, w_t, V_{t+\delta}(y_H), V_{t+\delta}(y_L); \theta_t, c_t)$ with the promise-keeping constraint (2) can now be equivalently summarized as $(\alpha_t, w_t, \beta_t; \theta_t, c_t)$ with the same constraint. By the pay, the equilibrium β_t represents the sensitivity of the manager's certainty-equivalent pay with respect to the output when δ goes to zero in the model. Therefore, I term β_t as the "pay-performance sensitivity" hereafter. See Subsection 3.6 for the derivation and detailed discussion.

Following He (2011), the first-order approach applies, and the manager's consumption and operational decisions can be examined separately. Appendix A.1 adapts the proof of He (2011) to this model. Lemma 1 summarizes the manager's optimal decisions at each time t.

Lemma 1 *Given the contract* (α_t, w_t, β_t) *and the promised continuation value* V_t*:*
if $V_t > \underline{V}$*, the manager chooses* $\theta_t = \min\{2\beta_t,\ \alpha_t\}$*,* $c_t = -\frac{1}{\gamma}\ln(-rV_t)$*;*
if $V_t \leqslant \underline{V}$*, the manager quits.*

Lemma 1 shows that conditional on the manager staying in the firm, the number of good decisions he makes is determined by the pay-performance sensitivity. Given a certain level of authority, the manager makes more good decisions when his pay-performance sensitivity is higher. Intuitively, a higher pay-performance sensitivity makes bad decisions more costly for the manager and, therefore, can provide incentives for higher moral-hazard tasks. Allowing private savings and borrowing, the manager smooths consumption over time. The policies (θ_t, c_t) summarized in Lemma 1 satisfy the incentive compatibility constraint of the manager, and, therefore, are indeed the recommended decisions and consumption in the incentive-compatible spot contract at time t.

3.2 Principal's problem in recursive form

The principal's objective is to maximize her expected discounted profits by optimally designing the contract. Following the literature, I solve the principal's problem in a recursive way. Let $F(V_t)$ represent the principal's continuation value at the beginning of time t. The principal's problem at time t is summarized as follows:

$$F(V_t) = \max_{(\alpha_t, w_t, \beta_t)} \frac{1}{1+r\delta} \cdot \big\{ -w_t\delta + E_t[y_{t+\delta}] + E_t[F(V_{t+\delta})] \big\} \tag{4}$$

$$s.t. \quad V_t = \frac{1}{1+r\delta} \cdot \big\{ u(c_t)\delta + [Pr(y_H) \cdot V_{t+\delta}(y_H) + Pr(y_L) \cdot V_{t+\delta}(y_L)] \big\},$$

where

$$E_t[y_{t+\delta}] = (2\theta_t - \alpha_t)\delta,$$

$$E_t[F(V_{t+\delta})] = Pr(y_H) \cdot F(V_{t+\delta}(y_H)) + Pr(y_L) \cdot F(V_{t+\delta}(y_L)),$$

$$Pr(y_H) = \frac{1}{2}[1 + (2\theta_t - \alpha_t)\sqrt{\delta}], \;\; Pr(y_L) = \frac{1}{2}[1 - (2\theta_t - \alpha_t)\sqrt{\delta}],$$

$$\beta_t = \frac{(V_{t+\delta}(y_H) - V_{t+\delta}(y_L))/u'(c_t)}{y_H - y_L},$$

$$c_t = w_t + \int_{\theta_t}^{\alpha_t} i di.$$

That is, the principal maximizes the discounted profits by providing a spot contract (α_t, w_t, β_t) with recommendations on (c_t, θ_t), subject to the promise-keeping constraint. The last equation above comes from the fact that the principal provides a wage level at which there is no saving or borrowing in equilibrium, and the manager fully consumes the wage and private benefits at each time.

3.3 Continuous-time version of the problem

Taking the limit $\delta \to 0$, I derive the continuous-time version of the model (see Appendix A.2 for the detailed derivation), which is summarized in the following three points.

(i) The cumulative output Y_t evolves according to

$$dY_t = y_{t+\delta} = (2\theta_t - \alpha_t)dt + dZ_t, \tag{5}$$

where $\{Z_t\}_{t\geqslant 0}$ is a standard Brownian motion and dZ_t represents the limit of unexpected component of output $(y_{t+\delta} - E_t[y_{t+\delta}])$ when $\delta \to 0$.

(ii) The manager's continuation value evolves according to

$$dV_t = (rV_t - u(c_t))dt + \beta_t u'(c_t)dZ_t. \tag{6}$$

(iii) The principal's continuation value satisfies the Hamilton-Jacobian-Bellman (HJB) equation:

$$rF(V_t) = \max_{(\alpha_t, w_t, \beta_t)} \{-w_t + (2\theta_t - \alpha_t) + F'(V_t) \cdot [rV_t - u(c_t)] + \frac{1}{2}F''(V_t)\beta_t^2[u'(c_t)]^2\}. \tag{7}$$

3.4 Optimal contract

I first solve for the relationship between the optimal authority level, α_t, and the pay-performance sensitivity, β_t.

Proposition 1 *If α_t and β_t can be freely chosen from the set $[0, 1]$, $\alpha_t = 2\beta_t$.*

Proposition 1 demonstrates that more managerial authority should be accompanied by a higher pay-performance sensitivity. Combining Proposition 1 and Lemma 1, it's easy to find that $\alpha_t = \theta_t$. That is, there is no managerial self-dealing if the principal can freely choose the level of managerial authority and the pay-performance sensitivity in each period. Intuitively, bad decisions are inefficient and are dominated by the uninformed decisions made by the principal herself. Therefore, it is optimal for the principal to set the level of managerial authority and the corresponding pay-performance sensitivity so to eliminate self-dealing.

Corollary 1 *If α_t and β_t can be freely chosen by the principal at the beginning of time t, $\forall t \geqslant 0$, there is no managerial self-dealing. That is, all the decisions made by the manager are good and increase the project's probability of success.*

Applying the results in Lemma 1 and Proposition 1, the principal's Hamilton-Jacobi-Bellman (HJB) equation (7) simplifies to:

$$rF(V_t) = \max_{\alpha_t} \left\{ \frac{1}{\gamma} \ln(-rV_t) + \alpha_t + \frac{1}{8}F''(V_t)\alpha_t^2 \cdot (r\gamma V_t)^2 \right\}. \tag{8}$$

Taking the first-order condition with respect to α_t yields

$$\alpha_t = -\frac{4}{(r\gamma V_t)^2 F''(V_t)}. \tag{9}$$

Plugging the expression of α_t back into the HJB equation gives the ordinary differential equation (ODE):

$$rF(V_t) + \frac{2}{(r\gamma V_t)^2 F''(V_t)} - \frac{1}{\gamma}\ln(-rV_t) = 0. \tag{10}$$

It remains to find the boundary conditions to fully characterize the optimal contract. The first boundary condition is the "value-matching condition" at the manager's turnover point,

$$F(\underline{V}) = F(V_0) - q, \text{ with } V_0 \in \arg\max_V F(V). \tag{11}$$

This boundary condition reflects that, at the managerial turnover point in time, the principal hires a replacement with a recruitment cost q, and he optimally chooses the initial level of continuation value promised to the new manager, V_0.

The second boundary condition comes from the fact that if the manager's expected compensation level goes to infinity, or equivalently, his continuation value tends to the zero upper bound (since CARA utility is a negative exponential utility), he has no incentives to leave the firm, and the principal's continuation value converges to the level in the benchmark case without managerial turnover, denoted by $\bar{F}(V_t)$:

$$\lim_{V_t \to 0} \left[\bar{F}(V_t) - F(V_t)\right] = 0. \tag{12}$$

The optimal contract is fully characterized by the above equations and conditions.

3.5 Dynamics of managerial authority

Before investigating how the managerial authority evolves under the optimal contract, I first solve for the benchmark case, where the manager always stays with the firm, or in other words, the manager has full commitment. The proposition below summarizes the equilibrium results in this benchmark case.

Proposition 2 (Benchmark Case)
If the incumbent manager has full commitment (i.e., always stays with the firm), then for any level of his continuation value V_t*,*
1) the principal's policies are: $\alpha_t = \bar{\alpha} \equiv \min\{\frac{4}{r\gamma}, 1\}$*,* $\beta_t = \frac{\bar{\alpha}}{2}$*,* $w_t = -\frac{1}{\gamma}\ln(-rV_t)$*;*
2) the manager's policies are: $\theta_t = \bar{\alpha}$*,* $c_t = w_t$*;*
3) the principal's continuation value is: $\bar{F}(V_t) \equiv \frac{2}{r^2\gamma} + \frac{1}{r\gamma}\ln(-rV_t)$*.*

Proposition 2 shows that if the manager has full commitment, the principal will optimally delegate a constant authority level $\bar{\alpha}$, independent of the manager's continuation value. Pay-performance sensitivity would also be at a corresponding constant level.

Compared with the benchmark, the limited commitment by the manager creates an additional dynamic layer of misalignment. Furthermore, the severity of the dynamic misalignment grows as the manager's continuation value decreases, reducing optimal authority delegation. The following proposition confirms this intuition.

Proposition 3 (Dynamics of Managerial Authority)
1) Managerial authority α_t *monotonically increases in the manager's continuation value* V_t *if there exists managerial turnover:* $\frac{d\alpha_t}{dV_t} > 0$*.*
2) The upper limit of α_t *is* $\bar{\alpha}$*, where* $\bar{\alpha}$ *is the authority level when the manager has full commitment (as shown in Proposition 2):* $\lim_{V_t \to 0} \alpha_t = \bar{\alpha}$*.*

Part (1) of Proposition 3 states that, under the optimal contract, the principal delegates more authority to the manager when his continuation value is higher, i.e., the manager is dynamically better aligned with her. Part (2) shows that when the manager's continuation value tends toward the zero upper bound, or equivalently, his consumption level goes to infinity, his authority converges to the same level as in the case with full commitment (Proposition 2).

Figure 2 visualizes the optimal dynamic relationship between the manager's consumption level, the manager's authority, and the principal's continuation value. The manager's consumption level is equivalent to his wage level as there is no managerial self-dealing and no saving or borrowing in equilibrium. Moreover, according to Lemma 1, there is a one-to-one increasing relationship between the manager's consumption c_t and his continuation value V_t.

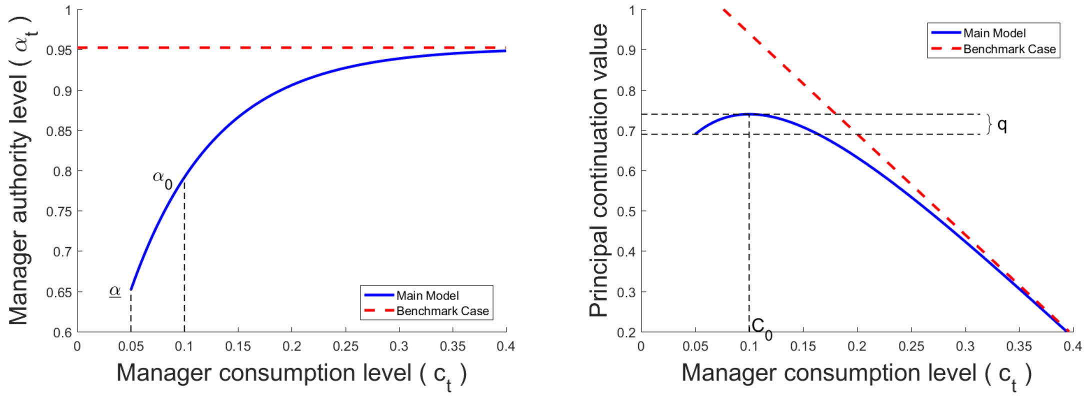


Figure 2: Dynamics of managerial authority

The left panel plots the manager's authority, α_t, as a function of the manager's consumption level c_t. According to Lemma 1, the manager's consumption is a monotonically increasing transformation of his continuation value: $c_t = -\frac{1}{\gamma}\ln(-rV_t)$. Additionally, the manager's consumption is equal to his compensation in the main model: $c_t = w_t$. Therefore, the horizontal axis also represents the manager's compensation. The right panel plots the principal's continuation value, $F(V_t)$, as a function of the manager's consumption level, c_t. The parameters are $r = 0.4$, $\gamma = 10.5$, $q = 0.05$, and $\underline{c} = 0.05$ ($\underline{V} = -1.4789$).

The left panel of Figure 2 provides an example of authority delegation as outlined in Proposition 2 and Proposition 3, respectively. The red dashed line represents the authority level without managerial turnover. Consistent with Proposition 2, the manager's authority is at a constant high level if he always stays with the firm. The blue curve illustrates how the manager's authority evolves when there is managerial turnover. It is obvious from the figure that $\bar{\alpha}$ is the highest level of authority the principal would ever delegate. This is precisely because the possibility of turnover creates an additional dynamic layer of misalignment and drives down optimal delegation. This result is consistent with the *Ally Principle* in the delegation literature, which states that the principal gives more discretion to a more aligned agent.[21]

The right panel of Figure 2 presents how the principal's continuation value evolves with the manager's consumption level. The principal's continuation value in the benchmark case, $\bar{F}(V_t)$, is higher than its counterpart, $F(V_t)$, in the main model, for any level of V_t. The intuition undergirding this result is simple: the principal is better off if the misalignment problem is less severe.

ANALYSIS: Degree of Misalignment

[21]See, for example, Holmstrom (1977) and Huber and Shipan (2006). Holmstrom (1977) shows that if the delegation set is a single interval, the Ally Principal holds under general conditions.

To confirm that it is indeed dynamics of misalignment that drives optimal delegation of authority, I now explicitly identify the magnitude of misalignment and analyze how the static and the dynamic components affect optimal managerial authority.

Let M_t denote the degree of misalignment at time t. It is defined as the loss in the principal's continuation value from one bad decision by the manager (i.e., one-step deviation of the manager). To obtain M_t, I take the partial derivative of the right-hand side of Equation (4) with respect to θ_t and normalize it by the time interval δ.

$$M_t \equiv \lim_{\delta \to 0} \frac{1}{\delta} \cdot \frac{\partial \left[\frac{1}{1+r\delta} \cdot \left\{ -w_t \delta + E_t[y_{t+\delta}] + E_t[F(V_{t+\delta})] \right\} \right]}{\partial \theta_t}$$

Simplifying this expression,[22] we obtain

$$M_t = 2\left[1 - \beta_t \cdot r\gamma V_t \cdot F'(V_t)\right],$$

which demonstrates that, for any level of the pay-performance sensitivity, β_t, the degree of misalignment, M_t, monotonically decreases in the manager's continuation value, V_t.[23]

Use M_t^{static} to denote the degree of misalignment in the benchmark case. The following equation shows that, indeed, only the static component of misalignment remains, and it is independent of V_t:

$$M_t^{static} = 2\left[1 - \beta_t \cdot r\gamma V_t \cdot \bar{F}'(V_t)\right] = 2[1 - \beta_t].$$

Hence, the degree of misalignment due to the manager's limited commitment is

$$M_t^{dynamic} = M_t - M_t^{static} = 2\beta_t\left[1 - r\gamma V_t \cdot F'(V_t)\right],$$

which monotonically decreases in the manager's continuation value, V_t.

[22]Proof: $M_t = \lim_{\delta\to 0} \frac{1}{\delta} \cdot \frac{1}{1+r\delta}\left[2\delta + \sqrt{\delta}[F(V^H_{t+\delta}) - F(V^L_{t+\delta})]\right] = \lim_{\delta\to 0} \frac{1}{\delta} \cdot \frac{1}{1+r\delta}\left[2\delta + \sqrt{\delta} \cdot 2\beta_t u'(c_t)\sqrt{\delta}F'(V_t)\right] = 2\left[1 + \beta_t u'(c_t)F'(V_t)\right] = 2\left[1 - \beta_t \cdot r\gamma V_t \cdot F'(V_t)\right]$, where the second equation is derived from Equations (A-1) in Appendix A.2

[23]Proof: $\frac{\partial M_t}{\partial V_t} = -2\beta_t r\gamma[F'(V_t) + V_t F''(V_t)] < -2\beta_t r\gamma[\bar{F}'(V_t) + V_t \bar{F}''(V_t)] = 0$, where the inequality comes from the fact that $F'(V_t) > \bar{F}'(V_t)$, $F''(V_t) < \bar{F}''(V_t)$, and $V_t < 0$ (See Proof of Proposition 3 in Appendix B for details).

The above analysis shows that a lower continuation value of the manager spells more severe dynamic misalignment and thus leads the principal to delegate less authority to the manager.

3.6 Authority-performance sensitivity

Extensive studies have been done on a manager's pay-performance sensitivity. However, the authority-performance sensitivity is much less investigated. One of the few related topics is managerial entrenchment. A long-serving manager who has moved up the corporate ladder maintains his authority or his authority may be slightly affected by the firm's bad performance, and this phenomenon is often explained by "managerial entrenchment". However, the entrenchment explanation makes it difficult to reconcile the fact that the compensation of a top manager is more equity based or option based (e.g., Edmans et al. 2017), which implies that a top manager's compensation is more sensitive to the firm's performance.

In this subsection, I apply the model to explain the puzzling phenomenon of simultaneous low authority-performance sensitivity and high pay-performance sensitivity for a manager with high authority. Moreover, I characterize the dynamics of the authority-performance sensitivity and his pay-performance sensitivity over the manager's tenure, and predict that near managerial turnover, the incumbent manager will be largely stripped of his authority after a bad performance, while his compensation, although being relatively low, is less affected.

First, let's revisit the definition of the pay-performance sensitivity, β_t, in this model. Applying Ito's lemma to the equilibrium wage expression, $w_t = -\frac{1}{\gamma}\ln(-rV_t)$, I obtain that

$$dw_t = \frac{1}{2}r^2\gamma\beta_t^2 dt + r\beta_t dZ_t. \tag{13}$$

Therefore,

$$\beta_t = \frac{1}{r}\cdot\frac{dw_t}{dY_t} = \frac{d(w_t/r)}{dY_t}. \tag{14}$$

β_t is the sensitivity of the discount-rate scaled compensation, w_t/r, to the output. A consumption stream of $\{w_t\}$ for all future periods delivers V_t to the manager. Therefore, w_t/r is the present value of the certainty-equivalent wage that generates V_t. Hence, β_t is the sensitivity of the present value of the certainty-equivalent pay stream with respect to the firm's performance. I refer to β_t as the "pay-performance sensitivity" in this paper.

Similar to β_t, I define the authority-performance sensitivity, ψ_t, as

$$\psi_t = \frac{d\alpha_t}{dY_t}.$$

From the previous analysis, it's known that the pay-performance sensitivity decreases with misalignment. What about the authority-performance sensitivity? To better investigate this problem, I first decompose ψ_t into the product of two parts, the authority level α_t and the relative sensitivity of authority to compensation ψ_t/β_t:

$$\psi_t = \frac{1}{2} \cdot \alpha_t \cdot \frac{\psi_t}{\beta_t}, \tag{15}$$

and proves that ψ_t/β_t monotonically decreases in the manager's continuation value.

Proposition 4 *The ratio between authority-performance sensitivity and pay-performance sensitivity monotonically decreases in the manager's continuation value:* $\frac{d}{dV_t}(\frac{\psi_t}{\beta_t}) < 0$.

Proposition 4 predicts that the lower the manager's continuation value, the swifter are changes in the manager's authority compared to his compensation level when the firm's performance changes. That is, the principal primarily adjusts authority delegation to maximize her profits when the dynamic misalignment problem is severe, while he relies more on performance-sensitive compensation when the manager is dynamically better aligned with her. Figure 3 below provides two examples of this proposition. For both cases, the ratio of the manager's authority-performance sensitivity and pay-performance sensitivity quickly declines towards zero as the manager's consumption increases.

The result in Proposition 4 implies that when a manager is near departure, he should be stripped off a large fraction of this authority after bad performance, while his compensation level is less affected. In contrast, when the manager has a high continuation value and is less likely to leave, his authority is at a high level and should be less sensitive to the firm's performance, while at the same time, his compensation is also high but should become more sensitive to the firm's performance.

Proposition 3 shows that α_t increases in V_t. Proposition 4 states that ψ_t/β_t decreases in V_t. Therefore, the shape of ψ_t is determined by the relative strength

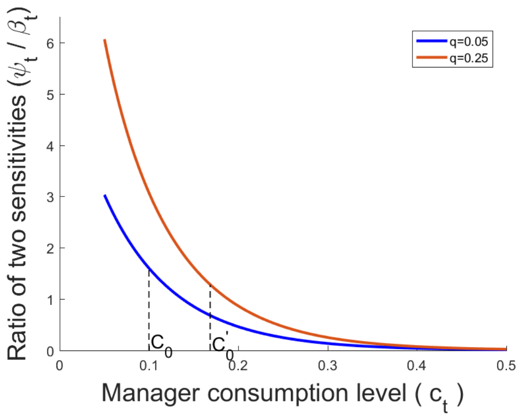


Figure 3: Ratio of the two sensitivities

This figure depicts the ratio of authority-performance sensitivity to pay-performance sensitivity, ψ_t/β_t. This ratio is monotonically decreasing in the manager's continuation value. According to Lemma 1, the manager's consumption is a monotonically increasing transformation of his continuation value: $c_t = -\frac{1}{\gamma}\ln(-rV_t)$. Additionally, the manager's consumption is equal to his compensation in the main model: $c_t = w_t$. Therefore, the horizontal axes also represent the manager's compensation. The parameters are $r = 0.4$, $\gamma = 10.5$, $\underline{c} = 0.05$ ($\underline{V} = -1.4789$), $q = 0.05$, and $q = 0.25$.

of these two components. On the one hand, the manager's authority should be less sensitive to the firm's performance when his authority is lower, because the firm's performance is largely out of his control and has little to do with his decisions. Thus, authority-performance sensitivity decreases with misalignment. On the other hand, the principal tends to increase the relative change in authority delegation and managerial compensation when the misalignment problem deteriorates. This force drives authority-performance sensitivity to increase with misalignment. The resultant authority-performance sensitivity monotonically decreases in the manager's continuation value for a wide range of parameters. Proposition 5 provides a sufficient condition for authority-performance sensitivity ψ_t to be monotonically decreasing.

Proposition 5 *The range of recruitment costs q for the authority-performance sensitivity ψ_t to be monotonically decreasing in V_t takes a threshold form: $q \leqslant q^*$. A lower bound for the threshold q^* as a function of the parameters r and γ is given in the appendix.*

Proposition 5 shows that there exists a positive recruitment cost q^*, such that when $q \leqslant q^*$, the authority-performance sensitivity ψ_t monotonically decreases in the manager's continuation value, V_t. That means that if the authority-performance sensitivity monotonically decreases for a given level of recruitment cost q, it does so for any lower levels of recruitment costs $q' < q$.

The left panel of Figure 4 provides two examples of how the authority-performance sensitivity evolves with the manager's consumption or compensation levels. The blue curve depicts the case in which the recruitment cost is relatively small, $q = 0.05$. Then, the manager's authority-performance sensitivity is indeed monotonically decreasing with the manager's consumption levels, or equivalently the manager's continuation value. The orange curve depicts the case in which the recruitment cost is high, $q = 0.25$. In this case, the manager's authority-performance sensitivity mainly decreases in his continuation value, but increases for a narrow range of values near his departure. For both cases, the manager's authority-performance sensitivity quickly declines towards zero as his consumption increases, resembling the phenomenon of "managerial entrenchment". The right panel of Figure 4 depicts the principal's continuation values in these two cases.

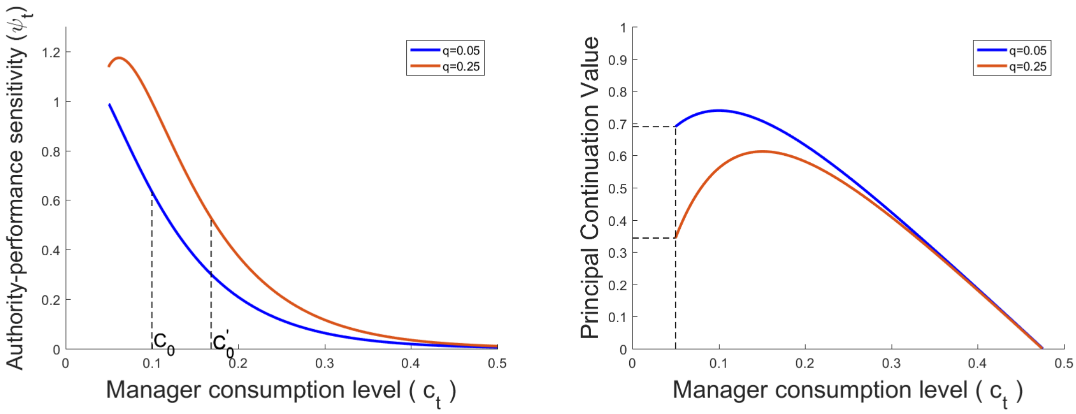


Figure 4: Authority-performance Sensitivity

The left panel depicts the authority-performance sensitivity as a function of the manager's consumption level, with the recruitment costs $q = 0.05$ and $q = 0.25$ respectively. The right panel depicts the principal's continuation value in these two cases. According to Lemma 1, the manager's consumption is a monotonically increasing transformation of his continuation value: $c_t = -\frac{1}{\gamma}\ln(-rV_t)$. Additionally, the manager's consumption is equal to his compensation in the main model: $c_t = w_t$. Therefore, the horizontal axes also represent the manager's compensation. The parameters are $r = 0.4$, $\gamma = 10.5$, $\underline{c} = 0.05$ ($\underline{V} = -1.4789$), $q = 0.05$, and $q = 0.25$.

3.7 Manager's initial authority

An old proverb goes "a new broom sweeps clean". Then, how much authority should be granted to a new manager? Moreover, is the manager's initial authority affected by managerial labor market conditions? And if so, how? The second question arises because empirical studies have documented that the managerial compensation outlined in executive contracts varies in accordance with labor market conditions (e.g., Bizjak et al. 2008; Brookman and Thistle 2013), while how his initial authority reacts to labor market conditions is not clear. Although people tend to focus on compensation more when signing an employment contract, the initial authority level is also an important aspect of the job and deserves attention. In this subsection, I apply the model to an analysis of the initial authority of a manager and an interaction of initial authority with managerial compensation to investigate how the managerial labor market influences both.

To begin with, the initially promised continuation value, V_0, should be within $[\underline{V}, 0)$. Hence, according to Proposition 3, the manager's initial authority, α_0, must satisfy $\underline{\alpha} \leqslant \alpha_0 < \bar{\alpha}$, where $\underline{\alpha}$ is an endogenously determined authority level at the point of managerial turnover, and $\bar{\alpha} = \min\{\frac{4}{r\gamma}, 1\}$, is the upper limit of the admissible managerial authority, as is defined in Proposition 2.

The managerial labor market influences authority and compensation through variations in the principal's recruitment costs and the manager's outside options. Before looking into its effect on the initial managerial authority, I first demonstrate how it affects the manager's expected compensation, or equivalently, the manager's initial continuation value, V_0. Lemma 2 states the results.

Lemma 2 *The manager's initial continuation value, V_0, increases in the recruitment cost, q, and the manager's outside option ,$\underline{V}$.*

The intuition behind this lemma is simple. *Ceteris paribus*, a higher recruitment cost, q, makes managerial turnover more costly to the principal and worsens the dynamic misalignment problem. Therefore, the principal is willing to give the manager more rents to align their interests and induce him to stay with the firm longer. Similarly, a better outside option also worsens the dynamic misalignment problem as the manager is more likely to leave. Hence, also in this case, the principal counteracts by promising a higher V_0.

Next, Proposition 6 summarizes the effects of the recruitment cost and the manager's outside option on the initial authority of a new manager.

Proposition 6 (Manager's Initial Authority)
The initial authority of a new manager, α_0, is independent of both the recruitment cost q, and the manager's outside option, $\underline{V}$.

Proposition 6 implies that α_0 is independent of managerial labor market conditions. This result is due to the canceling effect of the two opposite forces. On the one hand, a higher recruitment cost, q, or a better outside option, $\underline{V}$, aggravates the dynamic misalignment and makes the principal less willing to delegate authority. On the other hand, the principal counteracts these effects by promising a higher continuation value, V_0, to the incoming manager, and thus can delegate more authority compared to when the continuation value is at a nonreactive lower level of V_0. To put it another way, the principal optimally counteracts the impacts of the managerial labor market conditions to the extent that the initial delegated authority stays the same.

Results in this subsection help to explain the interesting phenomenon that compensation packages for managerial positions vary markedly with labor market conditions, while the initial authority granted to newly hired managers is usually unresponsive to labor market conditions.[24]

4 Managerial Self-dealing

Managerial self-dealing is socially inefficient and inimical to the firm's overall productivity. The previous model demonstrates that there is no self-dealing if both the manager's authority and his pay-performance sensitivity can be freely adjusted. However, in real-world situations, self-dealing takes place from time to time. A manager may use the company's aircrafts for private purposes, select a costly supplier to gain perks, hire employees on the grounds of friendship or kinship grounds, and so on. In this section, I analyze the reasons and occasions for managerial self-dealing to take place and delineate the managerial career trajectories more likely to lead to massive self-dealing.

4.1 Infrequent adjustment of authority

The previous model assumed authority delegation can be adjusted in every period. In reality, however, managerial authority generally changes less frequently than manager's pay-performance sensitivity. On the one hand, managerial authority is adjusted infrequently because of various frictions. For example, a deadlock on the

[24] Refer to Footnote 2 in the introduction for evidence.

board (Donaldson et al. 2020) may lead to a lag in the adjustment of a top manager's authority. Additionally, the board's composition also affects how authority is delegated to a top manager, and the board's composition changes once every several years (e.g., Adams and Ferreira 2007). Technological constraints could also limit the frequency of authority adjustments. For example, a manager often works on similar projects over some period of time, and authority adjustment is sensible only when the group of projects is completed. On the other hand, a manager's pay-performance sensitivity adjusts automatically with the firm's performance if stock options constitute part of the manager's compensation package. If the number of stocks granted to the manager changes, his pay-performance sensitivity also varies accordingly.

In this section, I assume the opportunity to change an incumbent manager's authority is exogenous and arrives at a rate of λ. Authority delegation is adjusted only when the opportunity to change the manager's authority arrives or when a new manager is hired. At all other times, managerial authority stays constant. The principal still needs to keep her promise on the manager's continuation value each time when writing the contract. The other assumptions are the same as those made in the previous model.

I investigate the constrained-optimal behaviors of the principal and the manager in this setup. I let α_t represent the current level of managerial authority. Therefore, α_t stays constant when opportunities to adjust managerial authority have not arrived.

The principal's HJB equation becomes

$$rF(\alpha_t, V_t) = \max_{(w_t,\beta_t)} \left\{ -w_t + (2\theta_t - \alpha_t) + \frac{1}{2}\frac{\partial^2 F(\alpha_t, V_t)}{\partial V_t^2}\beta_t^2 (u'(c_t))^2 + \lambda[\max_{\alpha} F(\alpha, V_t) - F(\alpha_t, V_t)] \right\}, \tag{16}$$

and the boundary conditions are

$$F(\alpha_t, \underline{V}) = \max_{(\alpha, V_0)} F(\alpha, V_0) - q \ ,$$

$$\lim_{V_t \to 0} [\bar{F}(\alpha_t, V_t) - F(\alpha_t, V_t)] = 0 \ .$$

The principal's HJB equation now has two state variables, α_t and V_t, and three choice variables, w_t, β_t, and α, where α represents the managerial authority level the principal would choose to delegate when the opportunity to change authority arrives. The first boundary condition suggests that the principal would reset the authority to the optimal level when hiring a new manager. The second boundary condition suggests that as V_t tends to the upper bound zero, the principal's value function approaches the level in the benchmark case without managerial turnover.

4.2 Results and analysis

In this section, I analyze constrained-optimal managerial authority, the corresponding pay-performance sensitivity, and the manager's behaviors in the above setting. With the results, I discover those managerial career trajectories that are more likely to lead to self-dealing. Moreover, I demonstrate that early-career luck is paramount in determining the manager's authority and lifetime utility.

Proposition 7 *For any $\lambda > 0$, the adjusted managerial authority level, α, increases in the manager's continuation value: $\frac{d\alpha}{dV_t} > 0$.*

Proposition 7 generalizes the results in Proposition 3 to any frequency of authority adjustment. It shows that when the manager's authority can be adjusted, the reset level is monotonically increasing with his continuation value. Figure 5 provides an example. The horizontal axis depicts the manager's consumption, which is a monotonically increasing function of the manager's continuation value, $c_t = -\frac{1}{\gamma}\ln(-rV_t)$. However, the manager's consumption level may be greater than his wage within that period if his authority is adjusted infrequently, because the manager may also engage in self-dealing and consume private benefits. Proposition 8 below provides the conditions by which managerial self-dealing ($\theta_t < \alpha_t$) occurs.

Proposition 8 *There exists an authority level $\alpha^* \in [0, \bar{\alpha}]$, such that*
(1) if $\alpha_t \leqslant \alpha^*$, $\forall\ V_t \in [\underline{V}, 0)$, $\beta_t = \frac{\alpha_t}{2}$, and therefore, $\theta_t = \alpha_t$;
(2) if $\alpha_t > \alpha^*$, there exists a continuation value level $V^*(\alpha_t)$, such that
(2.1) if $V_t \geqslant V^*(\alpha_t)$, $\beta_t = \frac{\alpha_t}{2}$, and therefore, $\theta_t = \alpha_t$;
(2.2) if $V_t < V^*(\alpha_t)$, $\beta_t < \frac{\alpha_t}{2}$, and therefore, $\theta_t < \alpha_t$.

Proposition 8 demonstrates that a necessary condition for managerial self-dealing is that the manager possesses a relatively high level of authority ($\alpha_t > \alpha^*$). With a high level of authority, the manager engages in self-dealing when his continuation value is relatively low. One thing worth noting is that the principal can always eliminate managerial self-dealing by making the manager's compensation sufficiently

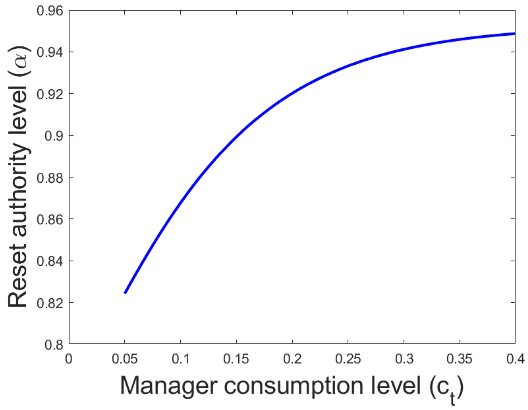


Figure 5: Reset levels of managerial authority, α

The figure plots the reset level of delegated authority, α, as a function of the manager's consumption level, c_t, when the opportunity to adjust managerial authority arises. According to Lemma 1, the manager's consumption is a monotonically increasing transformation of his continuation value: $c_t = -\frac{1}{\gamma}\ln(-rV_t)$. The parameters are $r = 0.4$, $\gamma = 10.5$, $q = 0.05$, $\underline{c} = 0.05$ ($\underline{V} = -1.4789$), and $\lambda = 2$.

sensitive to the firm's output, or more concretely, by setting $\beta_t = \alpha_t/2$. However, in certain situations, acquiescing to managerial self-dealing by providing a less-sensitive compensation is a superior strategy for the principal. This is because providing highly sensitive compensation may lead to frequent and costly managerial turnover. Therefore, by allowing managerial self-dealing, in the essence, the principal is making use of private benefits from the manager's self-dealing as a cheaper alternative to managerial compensation, even at the cost of overall firm efficiency.

Figure 6 provides a numerical example of Proposition 8. The flat surface represents the region of no managerial self-dealing in equilibrium, where $\alpha_t = 2\beta_t = \theta_t$. The curved surface is the region with self-dealing, where $\alpha_t > 2\beta_t = \theta_t$. The manager may engage in self-dealing when his authority level is higher than a certain level, α^*. At a particular authority level, $\alpha_t > \alpha^*$, self-dealing takes place if the manager's consumption level is relatively low, or equivalently if his continuation value is relatively low.

Together, Propositions 7 and 8 predict a time-series feature of a manager's authority and self-dealing. A manager who has performed well in the past will be conferred a high level of authority when there is an opportunity to do so. The manager will not abuse his authority if he continues to do well and is well-aligned with the principal. If, instead, the manager becomes unlucky and his performance trends downward, so that the misalignment problem becomes more severe and he is more likely to leave the firm,

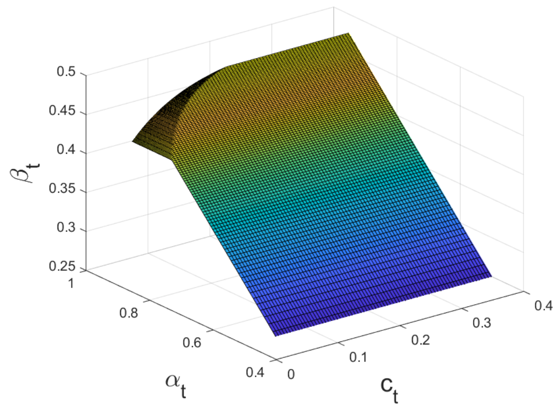


Figure 6: Pay-performance sensitivity and managerial self-dealing

The figure depicts the optimal choice of pay-performance sensitivity, β_t, as a function of the current authority, a_t, and consumption, c_t. The current authority, a_t, is a state variable and cannot be changed when the adjustment opportunity does not arise. Consumption, c_t, is a monotonically increasing transformation of the state variable, V_t: $c_t = -\frac{1}{\gamma}\ln(-rV_t)$, according to Lemma 1. The curved surface of this figure represents the region of managerial self-dealing, where $\beta_t = \theta_t/2 < \alpha_t/2$. The flat surface represents the region with no managerial self-dealing, where $\beta_t = \theta_t/2 = \alpha_t/2$. The parameters are $r = 0.4$, $\gamma = 10.5$, $q = 0.05$, $\underline{c} = 0.05$ ($\underline{V} = -1.4789$), and $\lambda = 2$.

the manager may engage in self-dealing before his authority is adjusted downward. In summary, a manager who has a high level of authority but grim career prospects tends to engage in self-dealing.

4.3 Early luck and managerial self-dealing

With the above analysis, I can demonstrate that if managerial authority can be adjusted only infrequently, early-career luck plays a disproportionate role in determining the manager's lifetime authority and utility. Besides, I can delineate the career trajectories that lead to massive managerial self-dealing.

To illustrate the role of early-career luck, I consider the following two opposite career trajectories, which are summarized as "*two fates of a manager*". If a manager experiences a series of good realizations in the early stages of his career, he becomes better aligned with the firm and will be granted more authority when the opportunity to delegate more authority arrives. Thereafter, the manager will have more discretion in making decisions, generating more profits for the firm and gaining higher compensation for himself. Later in his career, if he suffers from negative shocks, he can take advantage of the high level of authority and engage in self-dealing, keeping a high lifetime utility. In contrast, if the manager encounters negative shocks in the early stage of his career, he will be stripped of part of his authority when there is an opportunity, since his continuation value becomes lower, and the misalignment problem becomes more severe. Thereafter, he gets stuck in a low-authority situation even if he later experiences positive shocks and becomes better aligned with the principal. He cannot well exploit his superior knowledge because his authority is restricted. Consequently, his lifetime utility is lower.

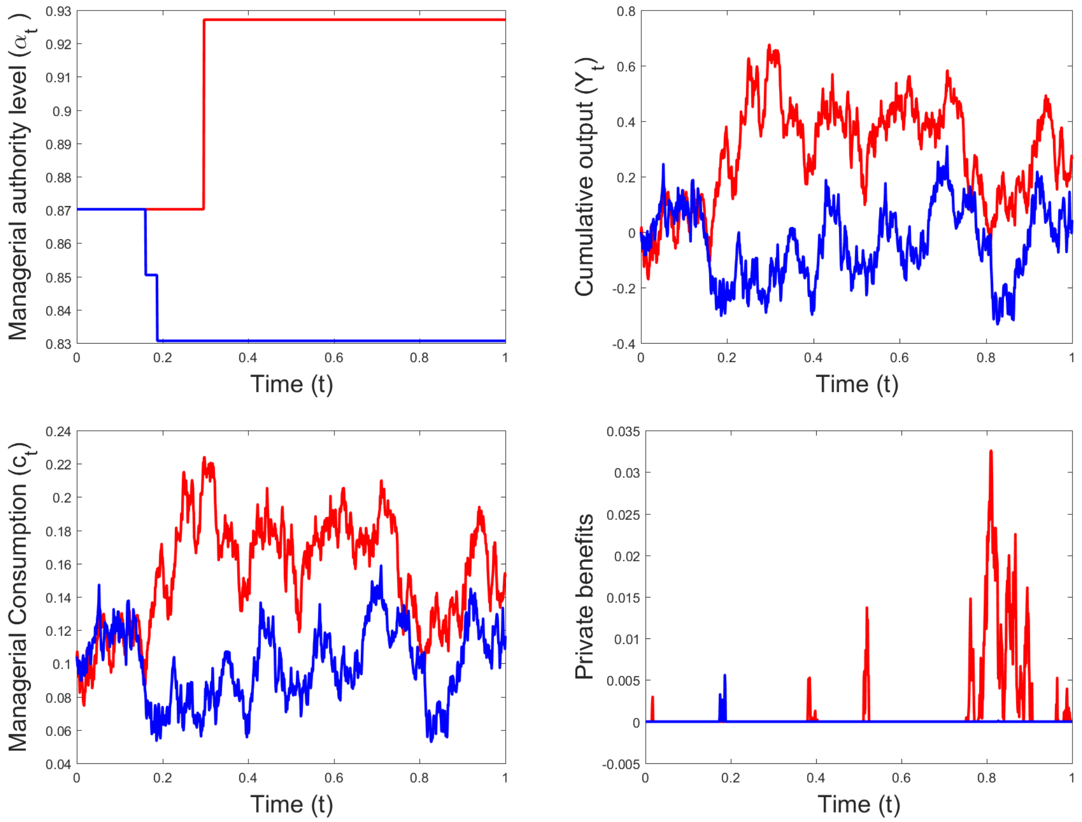


Figure 7: Two fates of a manager

The figure provides a simulated example of the “two fates of a manager” story. The red curves represent a manager with good luck in his early career; this manager encounters bad luck later. The blue curves represent a manager with bad luck in his early career; this manager experiences good luck later. To make a good comparison, I normalize the aggregate level of shocks for either of them to zero. The parameters for this simulation are $r = 0.4$, $\gamma = 10.5$, $q = 0.05$, $\underline{c} = 0.05$ ($\underline{V} = -1.4789$), and $\lambda = 2$.

Figure 7 provides simulated career trajectories of two managers. The manager with early-career good luck is represented in red, and the manager with early-career bad luck is represented in blue. The top-left panel shows their authority levels over time. They start with the same level of authority. Then, the authority levels are adjusted in opposite directions after experiencing good luck and bad luck, respectively. The firm’s cumulative output and the managerial consumption are higher for the manager with early luck, and the comparative advantage persists even after the aggregate shocks they experience converge ($t = 1$ in the figure). These results exhibit the disproportionate role of early-career luck on the firm's overall performance and the manager's lifetime utility.

The last panel of Figure 7 depicts the two manager's engagement with self-dealing and the levels of private benefits they obtain. Consistent with the analysis in the previous subsection, a manager who has a high level of authority but experiences bad luck and thus a grim prospect in the firm tends to engage in large-scale self-dealing. Combining this with the early-career experiences, I delineate the career trajectories that lead to massive self-dealing: a manager who experiences good luck in the early stages of his career and bad luck in the later years of his career is more likely to engage in massive self-dealing when he still holds a high level of authority but has a grim future with the firm.

5 Conclusion

In this paper, I develop a dynamic multi-task delegation model to analyze optimal managerial authority and its dynamic interaction with managerial compensation. With parsimonious assumptions, the model generates rich results and characterizes initial values and the dynamics of optimal managerial authority and consumption, which are consistent with real-world observations.

The model shows that when hiring a manager, the principle's delegation of authority is unresponsive to either the manager's outside options or the firm's recruitment costs, in contrast to promised compensation, which increases in both. Over time, both the manager's authority and his compensation rise after good performance and decline after bad realizations. Authority-performance sensitivity decreases as the manager's authority grows, resembling entrenchment. In contrast, pay-performance sensitivity increases with the manager's authority, consistent with the fact that firms grant more stocks or stock options to top managers. By exploiting the infrequent adjustment of authority, the model sheds light on managerial self-dealing and the impact of early-career luck: early luck plays a disproportionate role in the manager's career and lifetime utility, and a manager who experiences good luck in the early stages and bad luck in the later years of his career is more likely to engage in massive self-dealing.

The driving force of the model is the dynamic misalignment effect resulting from costly managerial turnover. Because of his limited commitment, the manager leaves the firm if past performances are bad and his continuation value to the firm drops below his outside options, as is consistent with empirical findings (e.g., Jenter and Lewellen 2020). Therefore, firm owners trade off among the projects' probabilities of success, the performance-sensitive compensation paid to the incumbent manager, and the cost from potential managerial turnover. Good past performance lowers the probability of managerial turnover, making the manager dynamically better aligned with the firm, and thus invites higher pay-performance sensitivity and more delegated authority. In the model, I have abstracted from unobservable skill differences and focused on the moral hazard problem. I also exclude the wealth effect by assuming CARA preference and the timing distortion of managerial compensation by allowing private savings and borrowing to isolate the dynamic misalignment effect. I explicitly identify the degree of misalignment and decompose it into dynamic and static components to aid the analysis.

The relatively simple structure of this model leaves several directions for future research. For instance, incorporating the search and matching model to provide a general equilibrium analysis on the effects of the managerial labor market would be desirable. Additionally, it might also be worth studying the dynamic trade-off between authority and costly efforts by considering a "power-hungry" manager with costly decision-making processes. Furthermore, the model can also be generalized to study the multiple hierarchical structure and the equilibrium authority distribution of a firm. I leave the full development of a richer model of this sort for future research.

References

Adams, R. B., Almeida, H., and Ferreira, D. (2005). Powerful CEOs and their impact on corporate performance. *The Review of Financial Studies*, 18(4):1403–1432.

Adams, R. B. and Ferreira, D. (2007). A theory of friendly boards. *The journal of finance*, 62(1):217–250.

Aghion, P. and Tirole, J. (1997). Formal and real authority in organizations. *Journal of political economy*, 105(1):1–29.

Alonso, R. and Matouschek, N. (2007). Relational delegation. *The RAND Journal of Economics*, 38(4):1070–1089.

Alonso, R. and Matouschek, N. (2008). Optimal delegation. *The Review of Economic Studies*, 75(1):259–293.

Amador, M. and Bagwell, K. (2013). The theory of optimal delegation with an application to tariff caps. *Econometrica*, 81(4):1541–1599.

Armstrong, M. and Vickers, J. (2010). A model of delegated project choice. *Econometrica*, 78(1):213–244.

Axelson, U. and Bond, P. (2015). Wall street occupations. *The Journal of Finance*, 70(5):1949–1996.

Biais, B. and Landier, A. (2020). Endogenous agency problems and the dynamics of rents. *The Review of Economic Studies*, 87(6):2542–2567.

Biais, B., Mariotti, T., Plantin, G., and Rochet, J.-C. (2007). Dynamic security design: Convergence to continuous time and asset pricing implications. *The Review of Economic Studies*, 74(2):345–390.

Bizjak, J. M., Lemmon, M. L., and Naveen, L. (2008). Does the use of peer groups contribute to higher pay and less efficient compensation? *Journal of Financial Economics*, 90(2):152–168.

Bolton, P. and Dewatripont, M. (1994). The firm as a communication network. *The Quarterly Journal of Economics*, 109(4):809–839.

Bolton, P. and Dewatripont, M. (2013). Authority in organizations. *Handbook of organizational Economics*, pages 342–372.

Brookman, J. T. and Thistle, P. D. (2013). Managerial compensation: Luck, skill or labor markets? *Journal of corporate Finance*, 21:252–268.

Burkart, M., Gromb, D., and Panunzi, F. (1997). Large shareholders, monitoring, and the value of the firm. *The quarterly journal of economics*, 112(3):693–728.

Crawford, V. P. and Sobel, J. (1982). Strategic information transmission. *Econometrica: Journal of the Econometric Society*, pages 1431–1451.

Cziraki, P. and Jenter, D. (2020). The market for CEOs. working paper.

DeMarzo, P. M. and Fishman, M. J. (2007). Optimal long-term financial contracting. *The Review of Financial Studies*, 20(6):2079–2128.

DeMarzo, P. M., Fishman, M. J., He, Z., and Wang, N. (2012). Dynamic agency and the q theory of investment. *The Journal of Finance*, 67(6):2295–2340.

DeMarzo, P. M. and Sannikov, Y. (2006). Optimal security design and dynamic capital structure in a continuous-time agency model. *The Journal of Finance*, 61(6):2681–2724.

Dessein, W. (2002). Authority and communication in organizations. *The Review of Economic Studies*, 69(4):811–838.

Donaldson, J. R., Malenko, N., and Piacentino, G. (2020). Deadlock on the Board. *The Review of Financial Studies*, 33(10):4445–4488.

Edmans, A. and Gabaix, X. (2016). Executive compensation: A modern primer. *Journal of Economic literature*, 54(4):1232–87.

Edmans, A., Gabaix, X., and Jenter, D. (2017). Executive compensation: A survey of theory and evidence. *The handbook of the economics of corporate governance*, 1:383–539.

Edmans, A., Gabaix, X., Sadzik, T., and Sannikov, Y. (2012). Dynamic CEO compensation. *The Journal of Finance*, 67(5):1603–1647.

Fernandes, A. and Phelan, C. (2000). A recursive formulation for repeated agency with history dependence. *Journal of Economic Theory*, 91(2):223–247.

Ferreira, D. and Nikolowa, R. (2020). Prestige, Promotion, and Pay. *Promotion, and Pay (June 29, 2020).*

Frankel, A. (2014). Aligned delegation. *American Economic Review*, 104(1):66–83.

Garrett, D. F. and Pavan, A. (2012). Managerial turnover in a changing world. *Journal of Political Economy*, 120(5):879–925.

Graham, J. R., Harvey, C. R., and Puri, M. (2015). Capital allocation and delegation of decision-making authority within firms. *Journal of financial economics*, 115(3):449–470.

Graham, J. R., Kim, D., and Kim, H. (2020a). Executive Mobility in the United States, 1920 to 2011. Working Paper.

Graham, J. R., Kim, H., and Leary, M. (2020b). CEO-board dynamics. *Journal of Financial Economics*.

Grochulski, B. and Zhang, Y. (2021). Termination as an Incentive Device. Technical report, Working paper.

Hall, P. and Heyde, C. C. (2014). *Martingale limit theory and its application*. Academic press.

He, Z. (2011). A model of dynamic compensation and capital structure. *Journal of Financial Economics*, 100(2):351–366.

He, Z. (2012). Dynamic compensation contracts with private savings. *The Review of Financial Studies*, 25(5):1494–1549.

Hoffmann, F. and Pfeil, S. (2010). Reward for luck in a dynamic agency model. *The Review of Financial Studies*, 23(9):3329–3345.

Hoffmann, F. and Pfeil, S. (2021). Dynamic multitasking and managerial investment incentives. *Journal of Financial Economics*.

Holmstrom, B. (1984). On the theory of delegation. Technical report. Bayesian Models in Economic Theory.

Holmstrom, B. R. (1977). On Incentives and Control in Organizations. Ph.D. Thesis, Stanford University.

Huber, J. D. and Shipan, C. R. (2006). Politics, delegation, and bureaucracy. In *The Oxford handbook of political science*.

Itoh, H. (1994). Job design, delegation and cooperation: A principal-agent analysis. *European Economic Review*, 38(3-4):691–700.

Jensen, M. C. and Murphy, K. J. (1990). Performance pay and top-management incentives. *Journal of political economy*, 98(2):225–264.

Jenter, D. and Lewellen, K. (2020). Performance-induced CEO turnover. The Review of Financial Studies, forthcoming.

Katz, L. F. (1986). Efficiency wage theories: A partial evaluation. *NBER macroeconomics annual*, 1:235–276.

Ke, R., Li, J., and Powell, M. (2018). Managing careers in organizations. *Journal of Labor Economics*, 36(1):197–252.

Kini, O., Williams, R., and Yin, D. (2020). CEO non-compete agreements, job risk, and compensation. *The Review of Financial Studies*.

Krishna, V. and Morgan, J. (2008). Contracting for information under imperfect commitment. *The RAND Journal of Economics*, 39(4):905–925.

Lazear, E. P. (2018). Compensation and Incentives in the Workplace. *Journal of Economic Perspectives*, 32(3):195–214.

Lazear, E. P. and Shaw, K. L. (2007). Personnel economics: The economist's view of human resources. *Journal of economic perspectives*, 21(4):91–114.

Li, J., Matouschek, N., and Powell, M. (2017). Power dynamics in organizations. *American Economic Journal: Microeconomics*, 9(1):217–41.

Lipnowski, E. and Ramos, J. (2020). Repeated delegation. *Journal of Economic Theory*, 188:105040.

Malenko, A. (2019). Optimal dynamic capital budgeting. *The Review of Economic Studies*, 86(4):1747–1778.

Martimort, D. and Semenov, A. (2006). Continuity in mechanism design without transfers. *Economics Letters*, 93(2):182–189.

Melumad, N. D. and Shibano, T. (1991). Communication in settings with no transfers. *The RAND Journal of Economics*, pages 173–198.

Noe, T. H. and Rebello, M. J. (2012). Optimal corporate governance and compensation in a dynamic world. *The Review of Financial Studies*, 25(2):480–521.

Ottaviani, M. (2000). The economics of advice. *University College London, mimeo.*

Phelan, C. (1995). Repeated moral hazard and one-sided commitment. *Journal of Economic Theory*, 66(2):488–506.

Piskorski, T. and Westerfield, M. M. (2016). Optimal dynamic contracts with moral hazard and costly monitoring. *Journal of Economic Theory*, 166:242–281.

Radner, R. (1993). The organization of decentralized information processing. *Econometrica: Journal of the Econometric Society*, pages 1109–1146.

Sannikov, Y. (2008). A continuous-time version of the principal-agent problem. *The Review of Economic Studies*, 75(3):957–984.

Shapiro, C. and Stiglitz, J. E. (1984). Equilibrium unemployment as a worker discipline device. *The American Economic Review*, 74(3):433–444.

Simon, H. A. (1951). A formal theory of the employment relationship. *Econometrica: Journal of the Econometric Society*, pages 293–305.

Spear, S. E. and Srivastava, S. (1987). On repeated moral hazard with discounting. *The Review of Economic Studies*, 54(4):599–617.

Tirole, J. (1999). Incomplete contracts: Where do we stand? *Econometrica*, 67(4):741–781.

Tirole, J. (2006). *The theory of corporate finance*. Princeton University Press.

Van Zandt, T. (1999). Real-time decentralized information processing as a model of organizations with boundedly rational agents. *The Review of Economic Studies*, 66(3):633–658.

Zhu, J. Y. (2013). Optimal contracts with shirking. *Review of Economic Studies*, 80(2):812–839.

Appendix A.1

CARA Preference and the First-order Approach

The analysis closely follows that of Lemma 3 and Section 2.3.3 in He (2011).

Consider a deviating manager with savings S who faces the contract $(\alpha_t, w_t, \beta_t; \theta_t, c_t)$. The principal is unaware of this deviation. Therefore, there exists a gap between the principal's promise V_t and the deviating manager's actual continuation value. Let $\widehat{V}_t(S, V_t)$ denote this actual continuation value. Then

$$\widehat{V}_t(S, V_t) = V_t \cdot e^{-r\gamma S}.$$

For a CARA agent without wealth effect,given the private savings S, his new optimal policy is to take the optimal decision-consumption policy without savings but to consume an extra rS more for all future dates $s \geqslant t$. $u(\theta_s, c_s + rS) = e^{-r\gamma S} u(\theta_s, c_s)$ explains the factor $e^{-r\gamma S}$.

Actual continuation value $\widehat{V}_t$ is hidden from the principal's view, and the authority and pay-performance sensitivity decisions only depend on the principal's promise V_t. Departure happens when the principal's promise V_t hits $\underline{V}$. I use τ to represent this departure time.

Let $\{\widehat{\theta_s}\}, s \in [t, \tau]$ represent the manager's optimal decision choice and $\{\widehat{c_s}\}, s \in [t, \infty)$ represent the optimal consumption choice with this private saving level S. $\{\theta_s\}$ and $\{c_s + rS\}$ also represent a feasible decision-consumption choice for this optimization problem and therefore,

$$\begin{aligned}
\widehat{V}_t &= \mathrm{E}_t \left[\sum_{n=0}^{\infty} \frac{1}{(1 + r\delta)^{(n+1)}} u(\widehat{c_{t+n\delta}}) \delta \right] \\
&\geqslant \mathrm{E}_t \left[\sum_{n=0}^{\infty} \frac{1}{(1 + r\delta)^{(n+1)}} u(c_{t+n\delta} + rS) \delta \right] = V_t \cdot e^{-r\gamma S}.
\end{aligned}$$

Similarly, $\{\widehat{\theta_s}\}$ and $\{\widehat{c_s} - rS\}$ represent a feasible decision-consumption choice for the optimization problem without private saving. Thus,

$$
\begin{aligned}
V_t &= \mathrm{E}_t \left[\sum_{n=0}^{\infty} \frac{1}{(1+r\delta)^{(n+1)}} u(c_{t+n\delta})\delta \right] \\
&\geqslant \mathrm{E}_t \left[\sum_{n=0}^{\infty} \frac{1}{(1+r\delta)^{(n+1)}} u(\widehat{c_{t+n\delta}} - rS)\delta \right] = \widehat{V_t} \cdot e^{r\gamma S}
\end{aligned}
$$

The above two inequalities complete the proof $\widehat{V_t} = V_t \cdot e^{-r\gamma S}$ and show that $\widehat{\theta_s} = \theta_s$, $\widehat{c_s} = c_s + rS$. Optimal decision is not affected by the presence of private savings, while optimal consumption is higher by rS at all dates.

Now we return to the manager's consumption choice problem. The marginal utility from consumption must be equal to the marginal value of hidden wealth

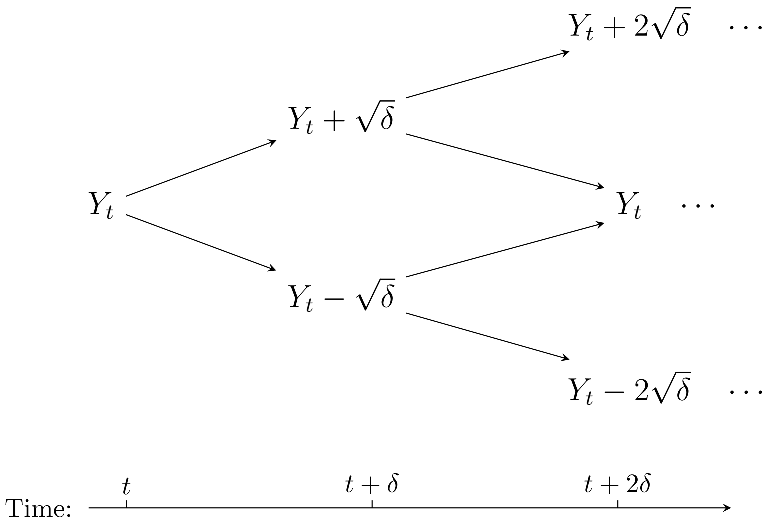
$$
\begin{aligned}
u'(c_t) &= \frac{\partial}{\partial S} \widehat{V_t}(S, V_t) = -r\gamma V_t \\
c_t &= -\frac{1}{\gamma} \ln(-rV_t)
\end{aligned}
$$

Hence, the first-order approach applies in this setup.

Appendix A.2

Continuous-time Limit of the Discrete-time Model

To begin with, we derive the continuous-time dynamics of the firm's cash flows. Let Y_t denote the cumulative cash flows till time t. From (1) we find that the dynamics of Y_t forms a binomial tree:



Use dY_t to denote the increment of the cumulative cash flows at time t: $dY_t = Y_{t+\delta} - Y_t$. dY_t follows a two-point distribution with mean $(2\theta_t - \alpha_t)\delta$ and variance $\delta + O(\delta^2)$, $\forall t \geqslant 0$.
Define $dZ_t = dY_t - E_t(dY_t) = dY_t - (2\theta_t - \alpha_t)\delta$. It has mean 0 and variance $\delta + O(\delta^2)$. Following Martingale central limit theorem (reference: Hall and Heyde (2014)), $\{Z_t\}_{t\geqslant 0}$ converges in distribution to the Wiener process (standard Brownian motion).

Therefore, the stochastic process of cash flows in continuous time becomes:

$$dY_t = (2\theta_t - \alpha_t)dt + dZ_t \ ,$$

where $\{Z_t\}_{t\geqslant 0}$ is a standard Brownian motion.

To get the continuous-time dynamics of the manager's continuation value, first combine Equation (2) and (3):

$$\begin{cases} V_t = \dfrac{1}{1+r\delta} \cdot \left\{ u(c_t)\delta + [Pr(y_H) \cdot V_{t+\delta}(y_H) + Pr(y_L) \cdot V_{t+\delta}(y_L)] \right\} \\ \beta_t = \dfrac{(V_{t+\delta}(y_H) - V_{t+\delta}(y_L))/u'(c_t)}{y_H - y_L} \end{cases}$$

where

$$Pr(y_H) = \frac{1}{2}[1 + (2\theta_t - \alpha_t)\sqrt{\delta}] \ , \ Pr(y_L) = \frac{1}{2}[1 - (2\theta_t - \alpha_t)\sqrt{\delta}] \ , \ y_H = \sqrt{\delta} \ , \ y_L = -\sqrt{\delta}.$$

We get:

$$\begin{cases} V^H_{t+\delta} \equiv V_{t+\delta}(y_H) = (1+r\delta)V_t - u(c_t)\delta - (2\theta_t - \alpha_t)\beta_t u'(c_t)\delta + \beta_t u'(c_t)\sqrt{\delta} \\ V^L_{t+\delta} \equiv V_{t+\delta}(y_L) = (1+r\delta)V_t - u(c_t)\delta - (2\theta_t - \alpha_t)\beta_t u'(c_t)\delta - \beta_t u'(c_t)\sqrt{\delta} \end{cases}$$

$$\Rightarrow \quad \begin{cases} V^H_{t+\delta} - V_t = [rV_t - u(c_t)]\delta + \beta_t u'(c_t)[y_H - (2\theta_t - \alpha_t)\delta] \\ V^L_{t+\delta} - V_t = [rV_t - u(c_t)]\delta + \beta_t u'(c_t)[y_L - (2\theta_t - \alpha_t)\delta] \end{cases}$$

When $\delta \to 0$, the process for the manager's continuation value becomes:

$$dV_t = (rV_t - u(c_t))dt + \beta_t u'(c_t) dZ_t \ .$$

To derive the continuous-time version of the principal's problem, we first apply Taylor-Young approximation to $F(V^H_{t+\delta})$ and $F(V^L_{t+\delta})$, and get

$$\begin{cases} F(V^H_{t+\delta}) = F(V_t) + F'(V_t)\cdot[rV_t - u(c_t) - (2\theta_t - \alpha_t)\beta_t u'(c_t)]\delta + \frac{1}{2}F''(V_t)\beta_t^2[u'(c_t)]^2\delta \\ \qquad\qquad + F'(V_t)\cdot \beta_t u'(c_t)\sqrt{\delta} + o(\delta) \ ; \\ F(V^L_{t+\delta}) = F(V_t) + F'(V_t)\cdot[rV_t - u(c_t) - (2\theta_t - \alpha_t)\beta_t u'(c_t)]\delta + \frac{1}{2}F''(V_t)\beta_t^2[u'(c_t)]^2\delta \\ \qquad\qquad - F'(V_t)\cdot \beta_t u'(c_t)\sqrt{\delta} + o(\delta) \ . \end{cases} \tag{A-1}$$

Plugging into the principal's problem expressed in Equation (4):

$$F(V_t) = \max_{(\alpha_t, w_t, \beta_t)} \frac{1}{1+r\delta}\cdot\{-w_t\delta + (2\theta_t - \alpha_t)\delta + F(V_t) + F'(V_t)[rV_t - u(c_t)]\delta + \frac{1}{2}F''(V_t)\beta_t^2[u'(c_t)]^2\delta + o(\delta)\}$$

Let $\delta \to 0$, we get the principal's Hamilton-Jacobian-Bellman (HJB) equation:

$$rF(V_t) = \max_{(\alpha_t, w_t, \beta_t)} \{-w_t + (2\theta_t - \alpha_t) + F'(V_t)\cdot[rV_t - u(c_t)] + \frac{1}{2}F''(V_t)\beta_t^2[u'(c_t)]^2\} \ .$$

Appendix B

Proof of Lemma 1

The manager maximizes his continuation value by choosing θ_t, taken the contract as given. As is proven in Appendix A.1, CARA preference prevents double deviation. Therefore, I do not need to consider the joint deviation of θ_t and c_t.

$$
\begin{aligned}
V_t =& \frac{1}{1+r\delta} \cdot \{u(c_t)\delta + [Pr(y_H) \cdot V_{t+\delta}(y_H) + Pr(y_L) \cdot V_{t+\delta}(y_L)]\} \\
=& \frac{1}{1+r\delta} \cdot \{u(w_t + \frac{\alpha_t^2 - \theta_t^2}{2})\delta + \frac{1+(2\theta_t - \alpha_t)\sqrt{\delta}}{2} \cdot V_{t+\delta}(y_H) + \frac{1-(2\theta_t - \alpha_t)\sqrt{\delta}}{2} \cdot V_{t+\delta}(y_L)\},
\end{aligned}
$$

where the first term of the second equality comes from the fact that the manager consumes his wage and private benefits in the no-saving equilibrium:

$$
c_t = w_t + \int_{\theta_t}^{\alpha_t} i di = w_t + \frac{\alpha_t^2 - \theta_t^2}{2}.
$$

Now, take F.O.C. of V_t with respect to θ_t and we get:

$$
\begin{aligned}
& u'(c_t) \cdot (-\theta_t)\delta + \sqrt{\delta} \cdot [V_{t+\delta}(y_H) - V_{t+\delta}(y_L)] = 0 \\
\Rightarrow \quad & u'(c_t) \cdot (-\theta_t)\delta + \sqrt{\delta} \cdot \beta_t u'(c_t) \cdot 2\sqrt{\delta} = 0 \\
\Rightarrow \quad & \theta_t = 2\beta_t.
\end{aligned}
$$

The above equation gives the interior solution of θ_t. If $2\beta_t > \alpha_t$, θ_t takes the upper bound value α_t. Therefore, $\theta_t = \min\{2\beta_t,\ \alpha_t\}$.

To find the manager's optimal consumption decision, we take a short cut and apply the Euler equation. (More through proof could be found in He (2011).)

With efficient private savings and borrowing technology, the manager can smooth his consumption intertemporally. Therefore, the Euler equation holds:

$$
u'(c_t) = \mathrm{E}_t[u'(c_\tau)],\ \forall \tau > t
$$

Apply the Euler equation to the manager's CARA utility, we find that

$$u(c_t) = \mathrm{E}_t[u(c_\tau)], \ \forall \tau > t.$$

Then,

$$\begin{aligned} V_t &= \mathrm{E}_t\left[\sum_{n=0}^{\infty} \frac{1}{(1+r\delta)^{(n+1)}} u(c_{t+n\delta})\delta\right] \\ &= \sum_{n=0}^{\infty} \frac{1}{(1+r\delta)^{(n+1)}} u(c_t)\delta \\ &= \frac{u(c_t)}{r}. \end{aligned}$$

Therefore, $c_t = -\frac{1}{\gamma}\ln(-rV_t), \ \forall t \geqslant 0.$ ■

Proof of Proposition 1

At time t, the principal's dynamic optimization problem is summarized in the HJB equation (7):

$$rF(V_t) = \max_{(\alpha_t, w_t, \beta_t)} \{-w_t + (2\theta_t - \alpha_t) + F'(V_t)\cdot[rV_t - u(c_t)] + \frac{1}{2}F''(V_t)\beta_t^2[u'(c_t)]^2\},$$

where the wage level w_t satisfies that there is no saving or borrowing in equilibrium by the manager:

$$w_t = c_t - \int_{\theta_t}^{\alpha_t} i di = c_t - \frac{\alpha_t^2 - \theta_t^2}{2},$$

and the manager's decision θ_t satisfies (according to Lemma 1)

$$\theta_t = \min\{2\beta_t, \ \alpha_t\}.$$

When $\alpha_t \in [0, 2\beta_t)$, the RHS of Equation (7) increases with α_t; when $\alpha_t \in [2\beta_t, 1]$, the RHS of Equation (7) decreases with α_t. Therefore, $\alpha_t = 2\beta_t$ maximizes Equation (7).

Together with Lemma 1, we get $\alpha_t = 2\beta_t = \theta_t$. ■

Proof of Proposition 2

Let $\bar{F}(V_t)$ represents the principal's continuation value at t if there is no potential manager departure. By definition,

$$\bar{F}(V_t) = \max_{\{\alpha_\tau, w_\tau, \beta_\tau\}_{\tau \geqslant t}} \mathrm{E}_t \left[\int_t^\infty e^{-r(\tau - t)} (2\theta_\tau - \alpha_\tau - w_\tau) d\tau \right].$$

According to Proposition 1, $\theta_t = \alpha_t$, and there is no private benefits in equilibrium. Thus, $w_t = c_t$, $\forall t \geqslant 0$, since the contract implies no saving.

Also, applying Ito's lemma to the equation $c_t = -\frac{1}{\gamma}\ln(-rV_t)$ in Lemma 1, we get $dc_t = \frac{1}{8}r^2\gamma\alpha_t^2 dt + \frac{1}{2}r\alpha_t dZ_t$. Thus, $E_t[c_\tau] = c_t + E_t[\int_t^\tau \frac{1}{8}r^2\gamma\alpha_s^2 ds]$.

Applying the above results into the expression of $\bar{F}(V_t)$, we find:

$$\begin{aligned}
\bar{F}(V_t) &= \max_{\{\alpha_\tau, c_\tau\}_{\tau \geqslant t}} \mathrm{E}_t \big[\int_t^\infty e^{-r(\tau - t)} (\alpha_\tau - c_\tau) d\tau \big] \\
&= \max_{\{\alpha_\tau, c_\tau\}_{\tau \geqslant t}} \mathrm{E}_t \big[\int_t^\infty e^{-r(\tau - t)} (\alpha_\tau - c_t - \int_t^\tau \frac{1}{8} r^2 \gamma \alpha_s^2 ds) d\tau \big] \\
&= \max_{\{\alpha_\tau, c_\tau\}_{\tau \geqslant t}} -\frac{c_t}{r} + \mathrm{E}_t \big[\int_t^\infty e^{-r(\tau - t)} (\alpha_\tau - \frac{1}{8} r \gamma \alpha_\tau^2) d\tau \big],
\end{aligned}$$

The problem now becomes an intra period maximization problem. It's obvious that we should set

$$\alpha_t \equiv \bar{\alpha} = \min\{\frac{4}{r\gamma}, 1\}.$$

Then,

$$\bar{F}(V_t) = \frac{2}{r^2\gamma} - \frac{c_t}{r} = \frac{2}{r^2\gamma} + \frac{1}{r\gamma}\ln(-rV_t), \ \forall t.$$

Also, the transversality condition $\lim_{\tau \to \infty} e^{-r(\tau - t)} \mathrm{E}_t \big[\bar{F}(V_\tau)\big] = 0$ is satisfied.■

Proof of Proposition 3

We've already got $\alpha_t = -\frac{4}{(r\gamma V_t)^2 F''(V_t)}$. So, the ODE could be rewritten as:

$$\alpha_t = 2rF(V_t) - \frac{2}{\gamma}\ln(-rV_t).$$

Differentiating with respect to V_t,

$$\frac{d\alpha_t}{dV_t} = 2rF'(V_t) - \frac{2}{\gamma V_t}$$

Remember that we have defined $\bar{F}(V_t)$ in Section 2.2.4.. If we can prove that $F(V_t) < \bar{F}(V_t)$, $F'(V_t) > \bar{F}'(V_t)$, and $F''(V_t) < \bar{F}''(V_t)$, then

$$\frac{1}{2}\frac{d\alpha_t}{dV_t} > r\bar{F}'(V_t) - \frac{1}{\gamma V_t} = \frac{1}{\gamma V_t} - \frac{1}{\gamma V_t} = 0 \ ,$$

and

$$\alpha_t = -\frac{4}{(r\gamma V_t)^2 F''(V_t)} < -\frac{4}{(r\gamma V_t)^2 \bar{F}''(V_t)} = \bar{\alpha}_t \ .$$

We now prove that $F(V_t) < \bar{F}(V_t)$, $F'(V_t) > \bar{F}'(V_t)$, and $F''(V_t) < \bar{F}''(V_t)$.

Economic arguments dictate that $F(V_t) < \bar{F}(V_t)$. Intuitively, the principal's continuation value tend to be lower when the agent has an option to quit compared to the case where the agent does not have this option.

Mathematically, for any path of consumptions, the principal's obligation c_τ jumps upwards from $\underline{c} = -\ln(-r\underline{V})/\gamma$ to $c_0 = -\ln(-rV_0)/\gamma$ when the quit happens. As a result,

$$\mathrm{E}_t[c_\tau] > c_t + \int_t^\tau \frac{1}{8} r^2 \gamma \alpha_s^2 ds \ .$$

Thus,

$$\begin{aligned}
F(V_t) &= \mathrm{E}_t\left[\int_t^\infty e^{-r(\tau-t)}(\alpha_\tau - c_\tau)d\tau\right] \\
&< \mathrm{E}_t\left[\int_t^\infty e^{-r(\tau-t)}(\alpha_\tau - c_t - \int_t^\tau \frac{1}{8} r^2 \gamma \alpha_s^2 ds)d\tau\right] \\
&= -\frac{c_t}{r} + \mathrm{E}_t\left[\int_t^\infty e^{-r(\tau-t)}\left(\alpha_\tau - \frac{1}{8} r \gamma \alpha_\tau^2\right) d\tau\right] \\
&\leqslant -\frac{c_t}{r} + \frac{2}{r^2\gamma} \\
&= \bar{F}(V_t)
\end{aligned}$$

On the other hand, $F(V_t) > \bar{F}(V_t) - 2/r^2\gamma$. The principal achieves the latter continuation value by setting all future consumption levels constant $c \equiv -\ln(-rV_t)/\gamma$

and zero power $\alpha \equiv 0$, which is obviously sub-optimal.

$$\begin{aligned}
F(V_t) &= \mathrm{E}_t \left[\int_t^{\infty} e^{-r(\tau-t)} (\alpha_\tau - c_\tau) d\tau \right] \\
&> \mathrm{E}_t \left[\int_t^{\infty} e^{-r(\tau-t)} \ln(-rV_t)/\gamma d\tau \right] \\
&= \frac{1}{r\gamma} \ln(-rV_t) \\
&= \bar{F}(V_t) - 2/r^2\gamma
\end{aligned}$$

The ODE could be rewritten as

$$F''(V_t) = -\frac{2}{(r\gamma V_t)^2} \cdot \frac{1}{rF(V_t) - \ln(-rV_t)/\gamma} \; .$$

Define $G(V_t) = \bar{F}(V_t) - F(V_t)$. From the above analysis, $0 < G(V_t) < 2/r^2\gamma$, $\forall V_t$.

$$\begin{aligned}
G''(V_t) &= \bar{F}''(V_t) - F''(V_t) \\
&= -\frac{1}{r\gamma V_t^2} + \frac{2}{(r\gamma V_t)^2} \cdot \frac{1}{2/r\gamma - rG(V_t)} \\
&= \frac{1}{V_t^2} \cdot \frac{rG(V_t)}{2 - r^2\gamma G(V_t)} \\
&> 0 \; .
\end{aligned}$$

The limit of $G(V_t)$ as $V_t \to 0$ must be equal to 0, because otherwise $G''(V_t)$ explodes, and if we integrate, $G(V_t)$ also explodes. Since $G(V_t) > 0$ for any $V_t < 0$,

$$\lim_{V_t \to 0} G'(V_t) \leqslant 0 \; .$$

And because $G(V_t)$ is convex, $G'(V_t)$ is non-decreasing. Therefore, $G'(V_t) < 0$, $\forall V_t$. This completes the proof.■

Proof of Proposition 4

According to (10), $\alpha_t = 2rF(V_t) - \frac{2}{\gamma}\ln(-rV_t) = 2rF(V_t) + 2c_t$. Then, $\frac{d\alpha_t}{dw_t} = \frac{d\alpha_t}{dc_t} = 2rF'(V_t)\frac{dV_t}{dc_t} + 2 = 2F'(V_t)u'(c_t) + 2 = 2F'(V_t)(-r\gamma V_t) + 2$, since we already know that $w_t = c_t$, $V_t = \frac{u(c_t)}{r}$, and $u'(c_t) = -r\gamma V_t$. Then,

$$\frac{d}{dV_t}(\frac{d\alpha_t}{dw_t}) = 2F''(V_t)(-r\gamma V_t) - 2r\gamma F'(V_t).$$

Besides, we know that

$$\frac{d}{dV_t}(\frac{d\bar{\alpha}}{dw_t}) = 2\bar{F}''(V_t)(-r\gamma V_t) - 2r\gamma \bar{F}'(V_t) = 0.$$

Therefore,

$$\frac{d}{dV_t}(\frac{d\alpha_t}{dw_t}) = 2(-r\gamma V_t)[F''(V_t) - \bar{F}''(V_t)] - 2r\gamma[F'(V_t) - \bar{F}'(V_t)] < 0,$$

because we have shown that $F''(V_t) < \bar{F}''(V_t)$ and $F'(V_t) > \bar{F}'(V_t)$ in the proof of Proposition 3. ■

Proof of Proposition 5

The authority-performance sensitivity is given by

$$\begin{aligned}\psi_t &= \frac{1}{2}\alpha_t \cdot \frac{\psi_t}{\beta_t} \\ &= \alpha_t \cdot (1 - r\gamma V_t F'(V_t))\end{aligned}$$

Differentiating with respect to V_t

$$\begin{aligned}\frac{d\psi_t}{dV_t} &= \frac{d\alpha_t}{dV_t}(1 - r\gamma V_t F'(V_t)) + \alpha_t \cdot (-r\gamma F'(V_t) - r\gamma V_t F''(V_t)) \\ &= (2rF'(V_t) - \frac{2}{\gamma V_t})(1 - r\gamma V_t F'(V_t)) - r\gamma\alpha_t F'(V_t) - r\gamma V_t \alpha_t F''(V_t) \\ &= \frac{1}{-\gamma V_t}\Big[2(1 - r\gamma V_t F'(V_t))^2 - \gamma\alpha_t(-r\gamma V_t F'(V_t)) - r\gamma^2 V_t^2 \alpha_t F''(V_t)\Big]\end{aligned}$$

Substituting in $\alpha_t = -\dfrac{4}{r^2\gamma^2 V_t^2 F''(V_t)}$,

$$\frac{d\psi_t}{dV_t} = \frac{1}{-\gamma V_t}\Big[2(1 - r\gamma V_t F'(V_t))^2 - \gamma\alpha_t(-r\gamma V_t F'(V_t)) - \frac{4}{r}\Big]$$

ψ_t is decreasing in V_t if and only if

$$2(1 - r\gamma V_t F'(V_t))^2 - \gamma\alpha_t(-r\gamma V_t F'(V_t)) - \frac{4}{r} < 0 \tag{A-2}$$

The expression is a quadratic function of $-r\gamma V_t F'(V_t)$ and is easy to see that a sufficient condition for $d\psi_t/d\alpha_t < 0$ is

$$-1 < -r\gamma V_t F'(V_t) < \sqrt{\frac{2}{r}} - 1$$

If the authority-performance sensitivity monotonically decreased for a given level of recruitment cost q_A, so does it for any lower levels of recruitment costs $q_B < q_A$. The proof of this statement uses equation (A-6) in the proof of Proposition 6. Differentiating (A-6) with respect to V_t,

$$\begin{aligned} F_B'(V_t) &= \frac{V_0^A}{V_0^B} F_A'(V_t) \\ -r\gamma V_t F_B'(V_t) &= -r\gamma \frac{V_0^A}{V_0^B} V_t \cdot F_A'\left(\frac{V_0^A}{V_0^B} V_t\right) \end{aligned}$$

$-r\gamma V_t F_A'(V_t)$ satisfies equation (A-2) for $V_t \in [\underline{V}, 0)$ and in particular for the sub-interval $V_t \in [V_0^A/V_0^B \cdot \underline{V}, 0)$. Therefore $-r\gamma V_t F_B'(V_t)$ satisfies equation (A-2) for the entire range of V_t, $[\underline{V}, 0)$, and ψ_t is monotonically decreasing in V_t for the same range.

The previous section proves that the range of q for ψ_t to be monotonically decreasing in V_t takes a threshold form: $q \leqslant q^*$. Next we provide a lower bound for the threshold q^* as a function of parameters r and γ.

First observe that $F''(V_t) < \overline{F}''(V_t)$ and for $V_t < V_0$,

$$\begin{aligned} F'(V_t) &= F'(V_0) - \int_{V_t}^{V_0} F''(V) dV \\ &> F'(V_0) - \int_{V_t}^{V_0} \overline{F}''(V) dV \\ &= 0 - (\overline{F}'(V_0) - \overline{F}'(V_t)) \\ &= \frac{1}{r\gamma V_t} - \frac{1}{r\gamma V_0} \end{aligned}$$

Therefore,

$$
\begin{aligned}
q = F(V_0) - F(\underline{V}) &= \int_{\underline{V}}^{V_0} F'(V)dV \\
&> \int_{\underline{V}}^{V_0} \left(\frac{1}{r\gamma V} - \frac{1}{r\gamma V_0} \right) dV \\
&= \frac{1}{r\gamma} \left(-\ln\left(\frac{\underline{V}}{V_0} \right) + \frac{\underline{V} - V_0}{V_0} \right)
\end{aligned}
$$

This inequality gives us an upper bound for $\underline{V}/V_0$:

$$
\frac{\underline{V}}{V_0} < -W_{-1}(-e^{-1-r\gamma q}) \tag{A-3}
$$

where W_{-1} is the Lambert W function.

Next we establish lower bounds for α_0 and $\underline{\alpha}$. $G(V_t) = \overline{F}(V_t) - F(V_t)$ is a convex function. Because $F'(V_0) = 0$, $G'(V_0) = \overline{F}'(V_0) - F'(V_0) = -1/r\gamma V_0$.

$$
\begin{aligned}
G(V_0) &= -\int_{V_0}^{0} G'(V)dV < -G'(V_0) \cdot (0 - V_0) = \frac{1}{r\gamma} \\
\alpha_0 &= 2rF(V_0) + 2c_0 = 2rF(V_0) - 2r\overline{F}(V_0) + \frac{4}{r\gamma} = \frac{4}{r\gamma} - 2rG(V_0) > \frac{2}{r\gamma}
\end{aligned}
$$

where the last inequality used the assumption $r < 1$. Therefore $\underline{\alpha}$ is bounded below by,

$$
\begin{aligned}
\underline{\alpha} &= \alpha_0 - 2r(F(V_0) - F(\underline{V})) - 2(c_0 - \underline{c}) \\
&\geqslant \frac{2}{r\gamma} - 2rq - \frac{2}{\gamma} \ln(\frac{\underline{V}}{V_0})
\end{aligned}
$$

The upper bound for $-r\gamma V_t F'(V_t)$ follows from the above inequalities,

$$
\begin{aligned}
F'(V_t) &= F'(V_0) - \int_{V_t}^{V_0} F''(V)dV \\
&= 0 + \int_{V_t}^{V_0} \frac{4}{r^2\gamma^2V^2\alpha(V)} dV \\
&< \int_{V_t}^{V_0} \frac{4}{r^2\gamma^2V^2\underline{\alpha}} dV \\
&= \frac{4}{r^2\gamma^2\underline{\alpha}} \left(\frac{1}{V_t} - \frac{1}{V_0} \right)
\end{aligned}
$$

Therefore,

$$-r\gamma V_t F'(V_t) < \frac{4}{r\gamma\underline{\alpha}}\left(\frac{V_t}{V_0}-1\right) \leqslant \frac{2\left(-W_{-1}(-e^{-1-r\gamma q})-1\right)}{1-r^2\gamma q+rW_{-1}(-e^{-1-r\gamma q})}$$

The right hand side of the above inequality is an increasing function of q and is equal to 0 when $q=0$. Therefore q^* is higher than the solution to the following equation

$$\frac{2\left(-W_{-1}(-e^{-1-r\gamma q})-1\right)}{1-r^2\gamma q+rW_{-1}(-e^{-1-r\gamma q})} = \sqrt{\frac{2}{r}}-1 \tag{A-4}$$

and this completes the proof. ∎

Proof of Lemma 2

First, we prove that V_0 increases with q.

Consider two firms, Firm A and Firm B. They have different recruitment costs, $0 \leqslant q_A < q_B$, but are otherwise identical. Then, the corresponding continuation values of the principal must satisfy $F_A(V_t) > F_B(V_t)$. This is because Firm A can at least always use the same strategy as Firm B, and saves the recruitment costs. Therefore, $F''_A(V_t) > F''_B(V_t)$, according to Equation (10).

Similar to the logic in Proof of Proposition 3, we define $g(V_t) = F_A(V_t) - F_B(V_t)$. So, $g(V_t) > 0$ and $g''(V_t) > 0$, $\forall\ V_t < 0$. Since $\lim_{V_t \to 0} g(V_t) = 0$ and $g(V_t) > 0$, $\lim_{V_t \to 0} g'(V_t) \leqslant 0$. $g(V_t)$ is a convex function, $g'(V_t)$ is non-decreasing. Therefore, $g'(V_t) < 0$, $\forall\ V_t < 0$.

From the proof of Proposition 3, we know that $0 > \bar{F}''(V_t) > F''(V_t)$, since $G''(V_t) > 0$. Therefore, $F_A(V_t)$ and $F_B(V_t)$ are concave. Besides, we know that $F'_A(V_0^A) = 0$ and $F'_B(V_0^B) = 0$. Therefore, $F'_A(V_0^B) < F'_B(V_0^B) < 0$, since we've got that $g'(V_t) < 0$. This leads to the result that $F'_A(V_0^B) < F'_A(V_0^A)$, and thus, $V_0^A < V_0^B$, i.e., V_0 increases with q.

To prove that V_0 increases with $\underline{V}$, consider two firms, Firm C and Firm D. They are in two different industries where the managers' outside options are different, $\underline{V_C} < \underline{V_D}$. Following the same steps as above, we reach the conclusion that V_0 increases with $\underline{V}$. ∎

Proof of Proposition 6

To prove that α_0 is independent of q, adopt the same setting as in the proof of Lemma 2, when proving prove that V_0 increases with q.

According to Equation (9), we only need to prove that

$$(V_0^A)^2 F_A''(V_0^A) = (V_0^B)^2 F_B''(V_0^B) \tag{A-5}$$

If we could prove that

$$F_B(V_t) = F_A\left(\frac{V_0^A}{V_0^B} \cdot V_t\right) + \frac{1}{r\gamma}\big[\ln(-rV_0^B) - \ln(-rV_0^A)\big], \tag{A-6}$$

the above equality (A-5) is satisfied. We then prove that the relationship between $F_A(V_t)$ and $F_B(V_t)$ satisfies Equation (A-6). Therefore, it boils down to prove that if $F_A(V_t)$ is the solution to Firm A's problem, $F_B(V_t)$ as expressed in Equation (A-6) is the solution to Firm B's problem.

From the previous proof for the relationship between V_0 and q, we know that there is one-to-one mapping between V_0 and q. Therefore, the boundary conditions $F'(V_0) = 0$ and $\lim_{V_t \to 0}[\bar{F}(V_t) - F(V_t)] = 0$ together with the ODE (10) pin down the solution $F(V_t)$.

Suppose $F_A(V_t)$ satisfies the above boundary conditions and the ODE. We prove that $F_B(V_t)$ as expressed in Equation (A-6) also satisfies the boundary conditions and the ODE.

First,

$$F_B'(V)\Big|_{V=V_0^B} = \frac{V_0^A}{V_0^B} \cdot F_A'(V)\Big|_{V=V_0^A} = 0$$

Second,

$$\begin{aligned}
&\lim_{V\to 0}[F_B(V) - \bar{F}(V)] \\
=&\lim_{V\to 0}\left[F_A\left(\frac{V_0^A}{V_0^B}\cdot V\right) - \bar{F}(\frac{V_0^A}{V_0^B}\cdot V)\right] + \lim_{V\to 0}\left[\bar{F}\left(\frac{V_0^A}{V_0^B}\cdot V\right) - \bar{F}(V)\right] + \frac{1}{r\gamma}\left(\ln(-rV_0^B) - \ln(-rV_0^A)\right) \\
=&0 + \frac{1}{r\gamma}\ln\left(\frac{V_0^A}{V_0^B}\right) + \frac{1}{r\gamma}\ln\left(\frac{V_0^B}{V_0^A}\right) \\
=&0
\end{aligned}$$

Third,

$$\begin{aligned}\frac{2}{r\gamma V^2 F_B''(V)} &= \frac{2}{r\gamma\left(\frac{V_0^A}{V_0^B}\cdot V\right)^2 F_A''\left(\frac{V_0^A}{V_0^B}\cdot V\right)} \\ &= \frac{1}{\gamma}\ln\left(-r\cdot\frac{V_0^A}{V_0^B}\cdot V\right) - rF_A\left(\frac{V_0^A}{V_0^B}\cdot V\right) \\ &= \frac{1}{\gamma}\ln(-rV) - rF_B(V)\end{aligned}$$

Therefore, if $F_A(V_t)$ is the solution to Firm A's problem, $F_B(V_t)$ as expressed in Equation (A-6) is the solution to Firm B's problem. We have proven that that α_0 is independent of q.

To prove α_0 is independent of $\underline{V}$, again, as in the proof of Lemma 2, consider two firms, Firm C and Firm D. They are in two different industries where the managers' outside options are different, $\underline{V_C} < \underline{V_D}$. Following the same steps as in the above proof, we derive that α_0 is independent of $\underline{V}$. ■

Proof of Proposition 7

Suppose that the opportunity to change authority arrives at a rate λ. In this notes I use α_t to represent the current level of authority.

The principal's HJB equation is

$$rF(\alpha_t, V_t) = \max_{\beta_t}\left\{-w_t + (2\theta_t - \alpha_t) + \frac{1}{2}\frac{\partial^2 F(\alpha_t, V_t)}{\partial V_t^2}\left[u'(c_t)\right]^2\beta_t^2 + \lambda\left(\max_{\alpha} F(\alpha, V_t) - F(\alpha_t, V_t)\right)\right\}$$

where

$$\begin{aligned}\theta_t &= \min\{2\beta_t, \alpha_t\} \\ w_t &= c_t - \left(\frac{\alpha_t^2}{2} - \frac{\theta_t^2}{2}\right)\end{aligned}$$

Therefore

$$\beta_t = \min\left\{\frac{4}{4 - F_{VV}\left[u'(c_t)\right]^2}, \frac{\alpha_t}{2}\right\} \tag{A-7}$$

$$rF(\alpha_t, V_t) = - c_t + \lambda\left(\max_{\alpha} F(\alpha, V_t) - F(\alpha_t, V_t)\right) \tag{A-8}$$

$$+ \begin{cases} \frac{\alpha_t^2}{2} - \alpha_t + \frac{8}{4 - F_{VV}\left[u'(c_t)\right]^2}, & \text{for } \beta_t < \frac{\alpha_t}{2} \\ \alpha_t + \frac{\alpha_t^2}{8} F_{VV}\left[u'(c_t)\right]^2, & \text{for } \beta_t = \frac{\alpha_t}{2} \end{cases} \tag{A-9}$$

The principal gets a chance to reset the authority after turnover. This gives rise to one boundary condition

$$F(\alpha_t, \underline{V}) = \max_{(\alpha, V_0)} F(\alpha, V_0) - q$$

Now let $\bar{F}(\alpha_t, V_t)$ represent the discounted profit of the principal if the current authority is α and there are no exits opportunities ($\underline{V} \to -\infty$). The principal would set $\beta_t = \alpha_t/2$ before the opportunity to change authority arrives. When the opportunity arrives, the principal would set the authority to $\bar{\alpha} = 4/r\gamma$, and pay-performance sensitivity to $\bar{\alpha}/2$.

$$\bar{F}(\bar{\alpha}, V_t) = -\frac{c_t}{r} + \frac{2}{r^2\gamma}$$

$$\bar{F}(\alpha_t, V_t) = -\frac{c_t}{r} + \frac{1}{r+\lambda}\left(\alpha_t - \frac{1}{8} r\gamma\alpha_t^2\right) + \frac{\lambda}{r+\lambda}\frac{2}{r^2\gamma}$$

$\bar{F}(\bar{\alpha}, V_t)$ is equal to the flexible authority no-exit value function. $\bar{F}(\alpha_t, V_t)$ consists of 2 components: expected profits before and after the change of authority.

As V_t tends to 0, the with-exit value function approaches the no-exit value function. Therefore the other boundary condition is given by

$$\lim_{V_t \to 0} (\bar{F}(\alpha_t, V_t) - F(\alpha_t, V_t)) = 0$$

Define $G(\alpha_t, V_t) = \bar{F}(\bar{\alpha}, V_t) - F(\alpha_t, V_t)$,

$$G_{VV} = -\frac{1}{r\gamma V_t^2} - F_{VV} \tag{A-10}$$

Substituting in $F(\alpha_t, V_t) = \bar{F}(\bar{\alpha}, V_t) - G(\alpha_t, V_t)$ and using $u'(c_t) = -r\gamma V_t$

$$
\begin{aligned}
(r+\lambda)G(\alpha_t, V_t) =& \frac{2}{r\gamma} + \lambda \min_{\alpha} G(\alpha, V_t) \\
& - \begin{cases} \dfrac{\alpha_t^2}{2} - \alpha_t + \dfrac{8}{4 + r\gamma + (r\gamma V_t)^2 G_{VV}}, & \text{for } \beta_t < \dfrac{\alpha_t}{2} \\ \alpha_t - \dfrac{\alpha_t^2}{8}\left(r\gamma + (r\gamma V_t)^2 G_{VV}\right), & \text{for } \beta_t = \dfrac{\alpha_t}{2} \end{cases}
\end{aligned}
\tag{A-11}
$$

with boundary conditions

$$
\begin{aligned}
G(\alpha_t, \underline{V}) &= \min_{(\alpha, V_0)} \left[G(\alpha, V_0) + \frac{1}{r}(c_0 - \underline{c})\right] + q \\
\lim_{V_t \to 0} G(\alpha_t, V_t) &= \frac{1}{r+\lambda} \frac{(\alpha_t - \bar{\alpha})^2}{2\bar{\alpha}}
\end{aligned}
$$

Solving for G_{VV},

$$
G_{VV} = \begin{cases} \left((r+\lambda)G - \lambda \min_\alpha G(\alpha, V_t) - \dfrac{2}{r\gamma} + \alpha_t - \dfrac{r\gamma}{8}\alpha_t^2\right) \Big/ \dfrac{\alpha_t^2}{8} r^2\gamma^2 V_t^2 \,, \\ \text{if } (r+\lambda)G - \lambda \min_\alpha G(\alpha, V_t) < \dfrac{2}{r\gamma} - \dfrac{\alpha_t^2}{2} \\ \dfrac{1}{r^2\gamma^2 V_t^2}\left[8 \Big/ \left(\lambda \min_\alpha G(\alpha, V_t) - (r+\lambda)G + \dfrac{2}{r\gamma} + \alpha_t - \dfrac{\alpha_t^2}{2}\right) - 4 - r\gamma\right], \\ \text{if } (r+\lambda)G - \lambda \min_\alpha G(\alpha, V_t) > \dfrac{2}{r\gamma} - \dfrac{\alpha_t^2}{2} \end{cases}
\tag{A-12}
$$

G_{VV} is an increasing function of $(r+\lambda)G - \lambda \min_\alpha G(\alpha, V_t)$. G_{VV} is increasing in α_t for $(r+\lambda)G - \lambda \min_\alpha G(\alpha, V_t) < 2/r\gamma - \alpha_t/2$ and decreasing in α_t for $(r+\lambda)G - \lambda \min_\alpha G(\alpha, V_t) > 2/r\gamma - \alpha_t/2$.

Lemma A1. *Consider 2 levels of authority $\alpha_1 < \alpha_2$. Suppose $G(\alpha_1, V_t)$ and $G(\alpha_2, V_t)$ intersects at $V_t = \tilde{V}$. Then $G(\alpha_2, V_t)$ could only cross $G(\alpha_1, V_t)$ from above, not below*

$$
G_V(\alpha_1, \tilde{V}) \geqslant G_V(\alpha_2, \tilde{V})
$$

From Lemma A1. to Proposition 7

Because $G(\alpha_t, V_t)$ is inversely related to the principal's continuation value $F(\alpha_t, V_t)$, the value functions $F(\alpha_1, V_t)$ and $F(\alpha_2, V_t)$ intersects only once, and

$$\begin{aligned} F(\alpha_1, V_t) > F(\alpha_2, V_t), &\quad \text{for } V_t \in (\underline{V}, \tilde{V}) \\ F(\alpha_1, V_t) < F(\alpha_2, V_t), &\quad \text{for } V_t \in (\tilde{V}, 0) \end{aligned}$$

If α_2 is reset authority level when $V_t = \hat{V} \in (\tilde{V}, 0)$, then $F(\alpha_2, \hat{V}) > F(\alpha, \hat{V})$ for any other authority level α. Therefore for all $V_t \in (\hat{V}, 0)$ and any authority level α_1 that is below α_2,

$$F(\alpha_2, V_t) > F(\alpha_1, V_t) \tag{A-13}$$

As a consequence the reset authority level for all $V_t \in (\hat{V}, 0)$ is above α_2. ■

Proof of Lemma A1: Suppose the opposite is true and

$$G_V(\alpha_1, \tilde{V}) < G_V(\alpha_2, \tilde{V}) \tag{A-14}$$

We consider two cases $G_{VV}(\alpha_1, \tilde{V}) < G_{VV}(\alpha_2, \tilde{V})$ and $G_{VV}(\alpha_1, \tilde{V}) \geqslant G_{VV}(\alpha_2, \tilde{V})$. If $G_{VV}(\alpha_1, \tilde{V}) < G_{VV}(\alpha_2, \tilde{V})$ then from equation (A-12),

$$(r+\lambda)G(\alpha_1, \tilde{V}) - \lambda \min_{\alpha} G(\alpha, \tilde{V}) = (r+\lambda)G(\alpha_2, \tilde{V}) - \lambda \min_{\alpha} G(\alpha, \tilde{V}) < \frac{2}{r\gamma} - \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}$$

Similar to the proof of $G'(V_t) < 0$ in Proposition 3, in this setup with infrequent adjustment of authority, $(r+\lambda)G(\alpha_t, V_t) - \lambda \min_\alpha G(\alpha, V_t)$ is decreasing in V_t,

$$(r+\lambda)\frac{\partial G(\alpha_t, V_t)}{\partial V_t} - \lambda \frac{d}{dV_t} \min_{\alpha} G(\alpha, V_t) < 0 \tag{A-15}$$

for any α_t, V_t.

Assumption (A-14) guarantees that $G(\alpha_1, V_t) < G(\alpha_2, V_t)$ for V_t between $\tilde{V}$ and the next intersection of 2 functions to the right of $\tilde{V}$. In this range,

$$(r+\lambda)G(\alpha_1, V_t) - \lambda \min_{\alpha} G(\alpha, V_t) < (r+\lambda)G(\alpha_2, V_t) - \lambda \min_{\alpha} G(\alpha, V_t) < \frac{2}{r\gamma} - \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}$$

From equation (A-12),

$$G_{VV}(\alpha_1, V_t) < G_{VV}(\alpha_2, V_t) \tag{A-16}$$

Combining (A-14) and (A-16), we find that

$$G_V(\alpha_1, V_t) < G_V(\alpha_2, V_t) \tag{A-17}$$

for any V_t between $\tilde{V}$ and the next intersection. However, this inequality indicates that $G_V(\alpha_1, \cdot)$ and $G_V(\alpha_2, \cdot)$ will diverge and $G(\alpha_1, V_t)$ will always be below $G_V(\alpha_2, V_t)$, contradicting

$$\begin{aligned}\lim_{V_t \to 0} G(\alpha_1, V_t) &= \frac{1}{r+\lambda} \frac{(\alpha_1 - \hat{\alpha})^2}{2\hat{\alpha}} \\ &> \frac{1}{r+\lambda} \frac{(\alpha_1 - \hat{\alpha})^2}{2\hat{\alpha}} = \lim_{V_t \to 0} G(\alpha_2, V_t)\end{aligned}$$

If $G_{VV}(\alpha_1, \tilde{V}) \geqslant G_{VV}(\alpha_2, \tilde{V})$, then from equation (A-12),

$$(r+\lambda)G(\alpha_1, \tilde{V}) - \lambda \min_{\alpha} G(\alpha, \tilde{V}) = (r+\lambda)G(\alpha_2, \tilde{V}) - \lambda \min_{\alpha} G(\alpha, \tilde{V}) > \frac{2}{r\gamma} - \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}$$

Between $\tilde{V}$ and the next intersection of 2 functions to the left of $\tilde{V}$,

$$(r+\lambda)G(\alpha_1, V_t) - \lambda \min_{\alpha} G(\alpha, V_t) > (r+\lambda)G(\alpha_2, V_t) - \lambda \min_{\alpha} G(\alpha, V_t) > \frac{2}{r\gamma} - \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}$$
$$G_{VV}(\alpha_1, V_t) > G_{VV}(\alpha_2, V_t)$$

Therefore,

$$G_V(\alpha_1, V_t) < G_V(\alpha_2, V_t)$$

$G_V(\alpha_1, \cdot)$ and $G_V(\alpha_2, \cdot)$ will diverge and $G(\alpha_1, V_t)$ will always be above $G_V(\alpha_2, V_t)$, contradicting

$$G(\alpha_1, \underline{V}) = \min_{(\alpha, V_0)} \left[G(\alpha, V_0) + \frac{1}{r}(c_0 - \underline{c})\right] + q = G(\alpha_2, \underline{V})$$

Neither $G_{VV}(\alpha_1, \tilde{V}) < G_{VV}(\alpha_2, \tilde{V})$ nor $G_{VV}(\alpha_1, \tilde{V}) \geqslant G_{VV}(\alpha_2, \tilde{V})$ are consistent with assumption (A-14). ■

Proof of Proposition 8

Substituting (A-10) into (A-7), we obtain

$$\beta_t(\alpha_t, V_t) = \min\left\{\frac{4}{4 + r\gamma + (r\gamma V_t)^2 G_{VV}}, \frac{\alpha_t}{2}\right\}$$

$\beta_t = \alpha_t/2$ if and only if

$$G_{VV} \leqslant \frac{1}{r\gamma V_t^2}\left[\frac{8}{\alpha_t} - 4 - r\gamma\right]$$

or equivalently from (A-12),

$$(r+\lambda)G(\alpha_t, V_t) - \lambda \min_{\alpha} G(\alpha, V_t) \leqslant \frac{2}{r\gamma} - \frac{\alpha_t^2}{2} \tag{A-18}$$

At $V_t = \underline{V}$, $G(\alpha_t, V_t)$ is the same across α_t and equal to

$$\min_{(\alpha, V_0)} \left[G(\alpha, V_0) + \frac{c_0 - \underline{c}}{r}\right] + q.$$

Let

$$\alpha^* = \sqrt{\frac{4}{r\gamma} - 2r \min_{(\alpha, V_0)} \left[G(\alpha, V_0) - 2(c_0 - \underline{c})\right] - 2rq}$$

From the monotonicity of $(r+\lambda)G(\alpha_t, V_t) - \lambda \min_\alpha G(\alpha, V_t)$ in equation (A-15), for any $\alpha_t \in [0, \alpha^*]$ and $V_t \in [\underline{V}, 0)$,

$$\begin{aligned}
(r+\lambda)G(\alpha_t, V_t) - \lambda \min_{\alpha} G(\alpha, V_t) &\leqslant (r+\lambda)G(\alpha_t, \underline{V}) - \lambda \min_{\alpha} G(\alpha, \underline{V}) \\
&= \frac{2}{r\gamma} - \frac{(\alpha^*)^2}{2} \\
&\leqslant \frac{2}{r\gamma} - \frac{\alpha_t^2}{2}
\end{aligned}$$

Similarly, for $\alpha_t \in (\alpha^*, \hat{\alpha}]$, there exists a continuation level $V^*(\alpha_t)$ such that

$$(r+\lambda)G(\alpha_t, V_t) - \lambda \min_{\alpha} G(\alpha, V_t) \leqslant \frac{2}{r\gamma} - \frac{\alpha_t^2}{2}$$

if and only if $V_t \in [V^*(\alpha_t), 0)$. ■