Insider Trading with Penalties

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Abstract

We establish existence and uniqueness of equilibrium in a generalised one-period Kyle (1985) model where insider trades can be subject to a *penalty*, i.e. a cost that is non-decreasing in the trade size. The result is obtained by considering uniform noise and holds for virtually any penalty function. Uniqueness is among *all* non-decreasing strategies. The insider demand and price functions are in general non-linear, yet tractable.

We apply this result to a problem of optimal insider trading regulation. We show analytically that the penalty functions maximising price informativeness for given noise traders' losses eliminate small rather than large trades. We generalise this result to cases where a budget constraint distorts the set of implementable regulations.

Keywords: Kyle model, non-linear equilibria, existence and uniqueness, market microstructure, insider trading, market regulation, efficient penalties.
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1 Introduction

In his seminal 1985 contribution, Albert Kyle developed a tractable and elegant framework to study how the insider information of strategic traders incorporates into market prices. Because of these virtues, his model has become and remains one of the most popular tools in the microstructure literature. Kyle (1985) operated under a Gaussian noise assumption. Bagnoli, Viswanathan, and Holden (2001) showed that, in the one-period setting, this distributional choice is not critical: as soon as the law of the noise traders' demand equals an affine transformation of the law of the fundamental, the linear Kyle equilibrium can be constructed.

In this paper, we show that one particular noise—uniform— has a special appeal, because it allows to extend Kyle's one-period framework to the case where the insider is subject to *any* cost which is non-decreasing in the size of her order. We call such costs *penalties*. This extension is desirable in a variety of contexts: one may want to model transaction costs, or, when insider trading is illegal, to capture the fact that insiders might be punished for trading. When introducing penalties, how do insider demand and price functions modify?

Our main theorem states the existence and uniqueness of an equilibrium in the presence of penalties. Importantly, it is analytically tractable: the equilibrium objects—the insider demand schedule and the price function—can easily be constructed and manipulated.

The theorem is proved using an intuitive lemma stating that under uniform noise, the market maker price function is in general non-linear, but the *expected* price function must be linear. Insider demand schedules can be computed explicitly even though they are also in general non-linear. Finally, uniqueness is among virtually *all* non-decreasing strategies and obtains as a straightforward byproduct of the analysis.

A noticeable feature of our model outcomes is that they are largely robust if one reverts to the standard Gaussian noise assumption. Of course, in that case, our analytical apparatus does not apply, but the Kyle model with penalties can be solved numerically and similar results obtain as detailed in Appendix B.

As an application of our main theorem, we solve a problem of optimal insider trading regulation. Proponents of legal insider trading wonder why one should ban an activity that contributes to the efficiency of prices. To capture this potential benefit, we define a measure of price efficiency by the expected post-trade standard deviation of the asset fundamental value. Opponents recall that allowing insider trading imposes costs upon the less informed agents. If these agents are sufficiently concerned by losses due to insiders, they could refuse to enter trades altogether, leading to a market breakdown. To capture this potential downside,

we measure the cost of insider trading by the expected losses of the uninformed traders.[1]

We answer the following questions. In the function space of all penalties, which one should a regulator select to maximise price efficiency for any given level of noise traders' losses? How do insiders trade when facing an optimal penalty?

We show analytically that optimal penalties eliminate small rather than large trades and characterise exactly these penalties. In cases where the fundamental realises at sufficiently high or sufficiently low values, the insider finds it optimal to trade; in fact, in that case she trades as much as she would without regulation, despite the large penalty prevailing at this large trading volume. Although such trades—if they occur—are costly for uninformed traders, they signal extreme events and therefore incorporate a lot of information into prices.

Implementing insider trading regulation requires to conduct investigations. Since these investigations are costly, a budget constraint on the regulator can distort the set of implementable regulations. Moreover, prosecution of insider trading can be carried out at the civil or penal level. The regulator collects a fine upon conviction only in the former case. Hence, the distinction between these prosecution modes is especially relevant when the regulator faces a budget constraint.

To take into account these important features, we recast our analysis when the regulator is subject to a budget constraint. We consider both the cases of pecuniary and non-pecuniary penalties.

In the non-pecuniary case, either a level of noise traders losses is too low to be implemented, or it can be implemented and in that case, the regulator uses the same penalty function as in the unconstrained case. In the pecuniary case, the regulator selects new penalties. When such a penalty is observed, one can theoretically infer the extent of the regulators constraint. New patterns emerge in the demand schedules of the insider trader and the associated price functions.

Our results contrast sharply with those of the related paper by DeMarzo, Fishman, and Hagerty (1998). They characterise optimal insider trading regulation when the regulator is subject to a budget constraint, levies pecuniary fines, and is solely concerned about the protection of uninformed traders. They use a model à la Easley and O'Hara (1992). Narrowing the bid-ask spread increases the utility of risk-averse uninformed traders, but the regulator cannot induce market makers to do so at will because the budget constraint limitates its ability to deter insider trades. One advantage of their approach over ours is that they determine

[1] An approach such as Leland (1992)'s can motivate explicitly the use of such measures of costs and benefits of insider trading. In his paper, information incorporation has positive real effects through its impact on investment decisions, but insider trading harms the welfare of liquidity traders. He compares two cases, with or without insiders, and abstracts from intermediary regimes where insider trading is regulated but does not disappear altogether.

both optimal investigation strategies and penalty schedules, while we focus on the latter. To do so, they assume that the regulator can ex post perfectly verify the fundamental and can use a menu of order size-dependent penalty functions, one for each possible realisation of the fundamental. A peculiar feature of their analysis is that no penalties are ever collected in equilibrium, and that the regulator initiates costly investigations although it knows with certainty that it will eventually levy no fine. The optimal penalties involve no fine on small enough trades, and so insiders trade quantities just as small as necessary to avoid a sanction even if investigated. As opposed to DeMarzo, Fishman, and Hagerty (1998), our model predicts the existence of large insider trades, and a positive rate of conviction in equilibrium. Another difference is that we find the insider's demand schedules to be monotonic in the fundamental, in line with the empirical evidence in Frino, Satchell, Wong, and Zheng (2013).

Besides the articles already mentioned, our paper is connected to several contributions.

One-period Kyle models. Rochet and Vila (1994) modified the one-period Kyle setting by assuming that the informed trader can observe the noise traders' demand. In that context, they showed existence and uniqueness of equilibrium regardless of the distributional assumptions on the noise. However, their approach does not allow in general to construct explicitly the equilibrium. Boulatov, Kyle, and Livdan (2013) and McLennan, Monteiro, and Tourky (2017) proved the general uniqueness result in the original Kyle model, using involved mathematical techniques tailored to Gaussian noise. As for the paper of Rochet and Vila, extending the existence and uniqueness result to the case of penalties seems very challenging.

Insider trading regulation. Kacperczyk and Pagnotta (2019) study a sample of insider trades found illegal by the Securities and Exchange Commission (SEC). They evidence that illegal insider traders do internalise legal risk: hence, they confirm that the regulator can act on market outcomes through the selection and implementation of penalties on insider trading. Augustin and Subrahmanyam (2019) provide a useful review on insider trading. For instance, they report that from 2011 to 2019, the SEC has spent \$ 300 million in whistleblower rewards as part of its efforts to "pursue high profile insider traders". This evidences the costly nature of insider trading monitoring, which we account for in Section 4.3, devoted to budget constraints. In a speech of the SEC, Newkirk and Robertson (1998) gather important insights from the U.S. regulator regarding insider trading. They emphasise various constraints that regulators face in implementing their laws. Again, we discuss and incorporate some of those in Section 4.3.

2 A One-Period Kyle Model with Penalties

As in the one-period version of Kyle (1985), the model features a risk-neutral insider trader (IT), noise traders (NT) and a market maker (MM). Agents are trading an asset with fundamental value v. The IT perfectly observes v and places an order $X(v)$. NT have a stochastic demand u independent of v. MM observes the total demand $X(v) + u$ and executes orders at a price P such that he breaks even on average.

The first difference of our model with Kyle (1985) is that we consider uniform—instead of Gaussian—noise:

$$u, v \sim U(-1, 1), \qquad u \perp v.$$

The choice of $[-1, 1]$ as the support is for clarity and without loss of generality: all our constructions and results carry to the case $u \sim U(-a, a)$ and $v \sim U(b, c)$.

The second difference is that a trade of size x involves a cost $C(x)$. C can be interpreted as a transaction cost or as an expected penalty imposed on illegal insider trades: $C = \alpha\tilde{C}$. In this intepretation, which we investigate in Section 4, α is the exogenous probability that the regulator starts and successfully completes an investigation, while $\tilde{C}(x)$ is the cost imposed to the IT conditional on the investigation being successful and the order of the IT being x.

2.1 Benchmark Equilibrium without Penalties

In the absence of penalties, the IT solves

$$\max_{x \in I} x\mathbb{E}_u[v - P(x + u)] \tag{1}$$

taking the price function P of the MM as given. The MM breaks even on average:

$$P(d) = \mathbb{E}[v | X(v) + u = d]. \tag{2}$$

An equilibrium is a pair (X, P) that satisfies (1) and (2). For example, (X, P) defined by

$$\begin{aligned} X(v) &= v & (3) \\ P(x + u) &= \frac{x + u}{2}, & (4) \end{aligned}$$

is an equilibrium of the one-period Kyle model without penalty. We call it the *(linear) mimicking equilibrium* since $X(v)$ and u are equal in distribution. We will prove later that this equilibrium is unique among all equilibria featuring a non-decreasing demand whose image lies in $[-1, 1]$. (With penalties, the optimal demand is no longer mimicking the

random demand u.)

2.2 Equilibrium with Penalties

The IT solves

$$\max_{x\in I} x\mathbb{E}_u[v - P(x+u)] - C(x), \tag{5}$$

taking the price function P of the MM as given. The MM breaks even on average:

$$P(d) = \mathbb{E}[v|X(v) + u = d]. \tag{6}$$

This game involving the IT and the MM is denoted $\mathcal{K}(C)$. An equilibrium of $\mathcal{K}(C)$ is a pair (X, P) such that X solves (5) and P satisfies (6).

The interval $I \subset \mathbb{R}$ in the maximisation program (5) is the set of admissible insider's demands. We will use the following assumption:

Assumption 1 $I = [-1, 1]$.

The bounds of I are those that obtain in the linear mimicking equilibrium when there is no penalty function. They are therefore natural: a demand function X whose image is not contained in $[-1, 1]$ would imply that for some values of the fundamental v, the magnitude of the IT order is *higher* when there is a penalty, compared to the linear equilibrium without penalty.[2]

To conclude this section, we state two remarks and introduce some notation.

(i) The data of a strategy X implies a pricing function P via Equation (6). We denote the pricing function associated with a demand schedule X by $P(X)$.

(ii) In the IT's maximisation program (5), the pricing function P only intervenes through the *expected price function*, denoted $\hat{P}$ and defined by

$$\hat{P}(x) = \mathbb{E}_u[P(x+u)]. \tag{7}$$

$\hat{P}$ represents the price that the IT will face on average if she places an order x. The program (5) can be rewritten in terms of the expected price function only:

$$\max_{x\in I} x(v - \hat{P}(x)) - C(x). \tag{8}$$

[2] An interesting question is whether there exists an equilibrium for which Assumption 1 does not hold.

2.3 Out-of-Equilibrium Pricing

The noise u we consider has bounded support. Moreover, from Assumption 1, the equilibrium demand functions X we consider satisfy $|X|\leq 1$. This means that the aggregate order, $d = X(v) + u$ belongs to a bounded set D. The conditional expectation in (6) is not defined for values of $d \notin D$, meaning that we must make an assumption on the out-of-equilibrium pricing of the MM:

Assumption 2 *For any equilibrium (X, P) of $\mathcal{K}(C)$ we consider, with X non-decreasing and $X([-1, 1]) \subset [-1, 1]$, we always impose the following out-of-equilibrium pricing (letting $x_M = X(1)$ and $x_m = X(-1)$):*

$$\begin{aligned} P(d) &= 1 \quad \textit{for} \quad d > 1 + x_M, \\ P(d) &= -1 \quad \textit{for} \quad d < -1 + x_m. \end{aligned}$$

This assumption states that when the MM observes an aggregate order larger than the maximal possible equilibrium order, he prices the asset as if it had realised at its maximal value, $v = 1$. Similarly, when the aggregate order is smaller than the minimal possible equilibrium order, the MM prices as if $v = -1$. When verifying that (X, P) is an equilibrium, one must not only check that $X(v)$ maximises the IT's program (5) among all x in the candidate support $[x_m, x_M]$, but also among values of x in $I \setminus [x_m; x_M]$. For these values of x, the aggregate order $d = x + u$ realises in the out-of-equilibrium region with positive probability, in which case Assumption 2 defines the price $P(d)$.

Finally, notice that Assumption 2 fully characterises out-of-equilibrium pricing: indeed, any $d \in [-1 + x_m, 1 + x_M]$ belongs to the support of $u + X(v)$, since u is $U(-1, 1)$ and $-1 \leq x_m \leq x_M \leq 1$.

2.4 Indistinguishable Equilibria

In the presence of penalties, we should expect the existence of realisations of v such that the IT is indifferent between two strategies: placing a small order and undergoing a small cost, or placing a larger order associated with a larger cost. However, as long as the set of v such that the maximisation program of the IT (5) admits several solutions has measure zero, these indifference points will almost surely not be reached. The equilibrium will therefore be independent of the choice of the maximiser $X(v)$, in the sense that any *ex post* model observable, such as the IT demand $X(v)$ or the observed price $P(d)$, is almost surely the same, and any *ex ante* model quantity, such as the IT expected profit, is the same. In that case, we wish to consider that any choice of maximiser X induces the same equilibrium.

We formalise this by introducing an equivalence relation between equilibria that we call indistinguishability.

Definition 1 *Let (X, P) and (X', P') be two equilibria of $\mathcal{K}(C)$. We say that (X, P) and (X', P') are* indistinguishable *if X and X' agree outside of a countable set.*

From now on, we identify an equilibrium of $\mathcal{K}(C)$ to its equivalence class. Definition 1 is useful because we will see that maximisers of (5) agree outside of a countable set, so the equilibria they induce belong to the same equivalence class.

2.5 Admissible penalty functions

We allow the regulator to select any penalty function that only depends in a non-decreasing manner on the magnitude of the order of the insider trader.

Definition 2 *$C : [-1; 1] \to \mathbb{R}_+$ is a penalty function if it is symmetric and non-decreasing, left-continuous over $[0; 1]$ and satisfies $C(0) = 0$. The set of penalty functions is denoted $\mathcal{C}$.*

The monotonicity assumption reflects the fact that it might be difficult to implement a higher sanction on a smaller trade for political reasons. The left-continuity assumption makes sure that the supremum of the possible IT profits is attainable.

3 Existence and uniqueness of equilibrium for $\mathcal{K}(C)$

In this section, we set out to prove our main theorem:

Theorem 1 *For any $C \in \mathcal{C}$, the Kyle game $\mathcal{K}(C)$ with penalty function C admits a unique equilibrium $(X(C), P(C))$ such that X is non-decreasing. In general, $X(C)$ and $P(C)$ are non-linear.*

3.1 Analysis of the expected price function

3.1.1 The expected price function is linear regardless of the IT demand

Lemma 1 contains the key observation at the root of our analysis.

Lemma 1 *Let $X : [-1, 1] \to [-1, 1]$ be a non-decreasing function, $x_M = X(1)$ and $x_m = X(-1)$. The expected price function $\hat{P}$ is linear on $[x_m, x_M]$:*

$$\hat{P}(x) = \frac{x}{2}.$$

Lemma 1 is crucial because it makes the surprising statement that the expected price function that must prevail in equilibrium is $\hat{P}(x) = x/2$ for $x_m \le x \le x_M$; neither the form of C nor guesses about X or P are needed. In fact, using Assumption 2, we will prove that the equality $\hat{P}(x) = x/2$ holds for any admissible demand x.
Hence, the equilibrium demand of the IT, $X(v)$, must be a maximiser of

$$\psi_C(.,v) : x \mapsto x\left(v - \frac{x}{2}\right) - C(x). \tag{9}$$

If X is such a maximiser, we claim that $(X, P(X))$ is the unique equilibrium of $\mathcal{K}(C)$ such that X is non-decreasing. To reach this conclusion, several issues remain to be addressed. First, we need to show that $\hat{P}(x) = x/2$ for any x as claimed above. Second, to make sure that Lemma 1 applies, we must check that any (selection of) maximiser is non-decreasing. Third, in order to obtain uniqueness, we need to show that $\psi_C(.,v)$ admits a unique maximiser except for a countable number of values of v.

We now provide the proof of this lemma. Section 3.1.2 clarifies the main intuitions.

Proof of Lemma 1. We use the notation $p(.)$ for a density and $p(.|.)$ for a conditional density. Write

$$\begin{aligned} p(v|d) &\propto p(d|v)p(v) \\ &\propto \mathbb{I}_{X(v)\in[d-1;d+1]}\mathbb{I}_{v\in[-1;1]}. \end{aligned}$$

That is, for $-1 + x_m \le d \le 1 + x_M$, $v|d$ is uniform over

$$\begin{aligned} \{v \in [-1;1] | X(v) \in [d-1;d+1]\} &= \{v \in [-1;1] | X(v) \in [d-1;d+1] \cap [x_m;x_M]\} \\ &= \left[\left(X_\ell^{-1}\left((d-1)\vee x_m\right); X_r^{-1}((d+1)\wedge x_M)\right)\right] \end{aligned} \tag{10}$$

where $X_\ell^{-1}(x) = \inf\{v | X(v) \ge x\}$ and $X_r^{-1}(x) = \sup\{v | X(v) \le x\}$. X_ℓ^{-1} and X_r^{-1} only disagree when there is v such that $X(v) = x$ and X is locally constant at v, i.e. they agree outside of a countable set. Then, letting $P = P(X)$,

$$P(d) = \frac{1}{2}\left(X_\ell^{-1}\left((d-1)\vee x_m\right) + X_r^{-1}\left((d+1)\wedge x_M\right)\right).$$

Now since

$$\hat{P}(x) = \frac{1}{2}\int_{x-1}^{x+1} P(z)\,\mathrm{d}z,$$

it is enough to show that $P(x+1) - P(x-1) = 1$ a.e.. Using the expression of P found above, we

obtain that for $x_m \leq x \leq x_M$:

$$\begin{aligned}2(P(x+1)-P(x-1)) &= X_\ell^{-1}(x \vee x_m) + X_r^{-1}((x+2) \wedge x_M) \\ &- X_\ell^{-1}((x-2) \vee x_m) - X_r^{-1}(x \wedge x_M) \\ &= X_r^{-1}(x_M) - X_\ell^{-1}(x_m) \\ &= 2\end{aligned}$$

a.e.. This is because $X_\ell^{-1} = X_r^{-1}$ a.e., $X_r^{-1}(x_M) = 1$, and $X_\ell^{-1}(x_m) = -1$. ■

Having identified $\hat{P}$, we know that the insider trader's problem is to maximise $\psi_C(.,v)$ as defined in (9). Because we will use this function throughout the paper, we repeat its definition here:

Definition 3 *The insider's profit function (under the expected price function $\hat{P}(x) = x/2$) for a demand x when the fundamental value is v is*

$$\psi_C(x,v) = x\left(v - \frac{x}{2}\right) - C(x). \tag{11}$$

3.1.2 Intuition

In order to isolate the intuition behind Lemma 1, let us consider the case where X is continuous and strictly increasing.

Assume that the market maker observes an aggregate order $d > 0$. Since the demand of the noise traders u takes values in $[-1,1]$, the possible demands of the IT $X(v)$ consistent with the observation of d are exactly the admissible demands such that $d-1 \leq X(v) \leq d+1$. Because admissible demands satisfy $X(v) \leq 1$ and $d+1 > 1$, the information obtained by the market maker when he observes d is that $X(v) \geq d-1$. Thus, he knows that $v \geq X^{-1}(d-1)$. Intuitively, the fact that the aggregate order is positive rules out extreme negative values of v and the MM deduces a lower bound on v, $X^{-1}(d-1)$.

Moreover, due to the uniform noise assumption, all values of v above this lower bound are equally likely. Therefore, the price $P(d)$ is given by the midpoint of the interval $[X^{-1}(d-1), 1]$.

In a similar manner, when $d < 0$, the price $P(d)$ is given by the midpoint of the interval $[-1, X^{-1}(d+1)]$.

Now, assume that the IT wants to place an order x. The IT is only concerned by the expected price impact, $\hat{P}(x)$, which is a uniform average of the $P(d)$ over $d \in [x-1, x+1]$, the set of possible aggregate demands given an IT demand x. If, instead, the IT decides to

place an order $x+\Delta x$, the set of possible aggregate demands d is $d \in [x-1+\Delta x, x+1+\Delta x]$: see Figure 1.

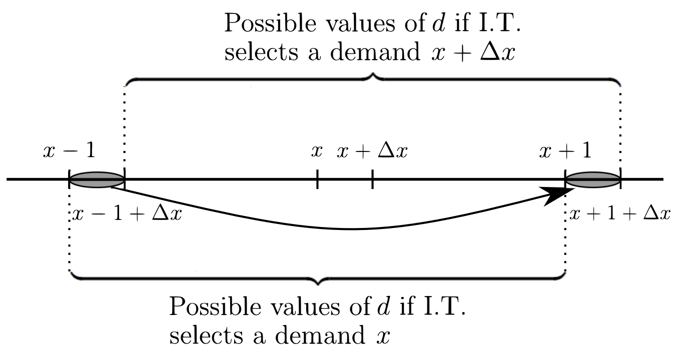


Figure 1: Marginal expected price impact of an increase in x

Thus, the only contribution to the marginal increase in expected price $\hat{P}(x+\Delta x) - \hat{P}(x)$ is due to the fact that the weight that was attributed to the interval $[x-1, x-1+\Delta x]$ is now attributed to the interval $[x+1, x+1+\Delta x]$. Crucially, this weight is the same due to the uniform noise assumption. Considering a vanishing Δx, one concludes that the marginal impact of increasing demand on expected price is proportional to $P(x+1) - P(x-1)$.

We have seen above that $P(x+1)$ is the midpoint of $[X^{-1}((x+1)-1), 1] = [X^{-1}(x), 1]$, and that $P(x-1)$ is the midpoint of $[-1, X^{-1}((x-1)+1)] = [-1, X^{-1}(x)]$. Therefore, the marginal impact on the expected price is proportional to the distance between these two midpoints:

$$\frac{d}{dx}\hat{P}(x) \propto P(x+1) - P(x-1) = \frac{1+X^{-1}(x)}{2} - \frac{X^{-1}(x)-1}{2} = 1. \tag{12}$$

Figure 2 provides an illustration of this result. From (12), we see that the expected price function is linear.

3.2 Candidate optimal demands are unique up to changes on a countable set

In this section, we set out to obtain an unambiguous definition of the strategy X that will be the maximiser of ψ_C defined in (11).

Definition 4 *Let V, I be two intervals of $\mathbb{R}$. A correspondence $\mathcal{X} : V \to \mathcal{P}(I) \setminus \emptyset$ is* non-decreasing *if for any $v_1 < v_2$ in V,* $\sup \mathcal{X}(v_1) \leq \inf \mathcal{X}(v_2)$.

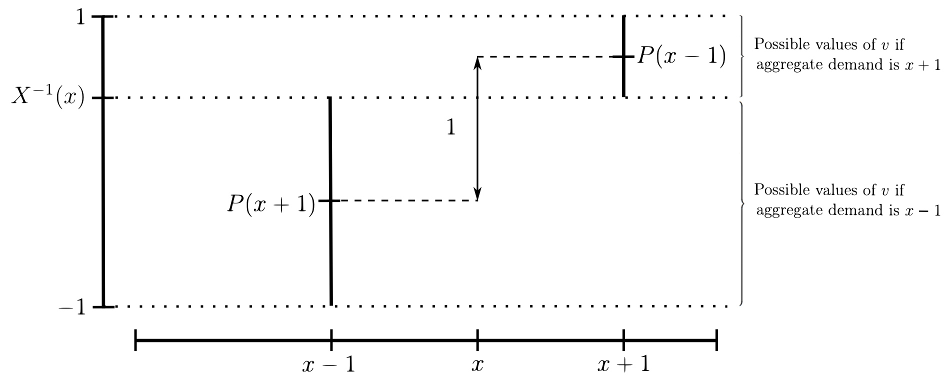


Figure 2: The marginal expected price impact is constant

If $\mathcal{X}$ is a one-to-one mapping, we recover the usual notion of a non-decreasing function.

Lemma 2 *Let $\mathcal{X} : V \to \mathcal{P}(I) \setminus \emptyset$ be a non-decreasing correspondence. Then for all v in V except on a countable set, $\mathcal{X}(v)$ is a singleton.*

Proof. The argument is the same as for the proof that a non-decreasing function has at most a countable number of discontinuities. ■

For a given penalty $C \in \mathcal{C}$, let $\mathcal{X}_C$ be the correspondence mapping $v \in [-1; 1]$ to the set of maximisers of the insider trader's profit function when she observes a realisation v of the fundamental:

$$\mathcal{X}_C(v) = \arg\max_x \ \psi_C(x, v).$$

Lemma 3 *For any $v \in [-1, 1]$, $\mathcal{X}_C(v) \neq \emptyset$, and $\mathcal{X}_C$ is a non-decreasing correspondence.*

Proof. See Appendix A.1. ■

Lemma 3 reflects the fact the IT trades more aggressively for large values of the fundamental. The combination of Lemmas 2 and 3 ensures that the maximiser of the IT's expected profit is unique except for a countable number of values of v:

Lemma 4 *There exists a non-decreasing function X_C such that for all $v \in [-1, 1]$ except on a countable set, $\mathcal{X}_C(v) = \{X_C(v)\}$. All such X_C agree outside of a countable set.*

As we identify equilibria in a same equivalence class, as introduced in Definition 1, we do not need to specify which particular X_C we consider: we can unambiguously talk about *the* maximiser of the expected profit. We are now ready to prove our main result.

3.3 Existence and uniqueness of the equilibrium of $\mathcal{K}(C)$

We recast the statement of Theorem 1 using the expression of $\hat{P}$ obtained in Lemma 1.

Let $C \in \mathcal{C}$ and $X_C(v)$ be the maximiser of $x \mapsto x\left(v - \frac{x}{2}\right) - C(x)$. Then $(X_C, P(X_C))$ is an equilibrium of $\mathcal{K}(C)$. This is the unique equilibrium among the pairs (X, P) such that $X : [-1,1] \to [-1,1]$ is non-decreasing.

Proof of Theorem 1. From Lemma 1, $\hat{P}(x) = \frac{x}{2}$ for $x_m \leq x \leq x_M$. Since $X_C(v)$ is a maximiser of $x\left(v - \frac{x}{2}\right) - C(x)$, $x = X(v)$ is an optimal response to the expected price function $\hat{P}$ among all $x \in [x_m, x_M]$. To confirm that $(X_C, P(X_C))$ is an equilibrium, we need to check what happens if the IT makes a choice outside of the candidate support $[x_m, x_M]$, knowing that the out-of-equilibrium pricing is defined by Assumption 2. Consider for instance the case $x \in (x_M, 1]$, as the case $x \in [-1, x_m]$ works identically. Then

$$
\begin{aligned}
\hat{P}(x) &= \frac{1}{2}\int_{x-1}^{x+1} P(z)\,\mathrm{d}z \\
&= \frac{1}{2}(x - x_M) + \frac{1}{2}\int_{x_M-1}^{x_M+1} P(z)\,\mathrm{d}z - \frac{1}{2}\int_{x_M-1}^{x-1} P(z)\,\mathrm{d}z \\
&= \frac{1}{2}(x - x_M) + \hat{P}(x_M) - \frac{1}{2}\int_{x_M-1}^{x-1} P(z)\,\mathrm{d}z \\
&= \frac{1}{2}(x - x_M) + \frac{x_M}{2} - \frac{1}{2}\int_{x_M-1}^{x-1} P(z)\,\mathrm{d}z \\
&= \frac{x}{2}.
\end{aligned} \tag{13}
$$

This is because when $z \in [x_M - 1, x - 1]$, $z - 1 < x - 2 \leq -1 \leq x_m$ and $z + 1 \geq x_M$ so from (10), $v|z$ is uniform over $[-1, 1]$ and $P(z) = 0$.

As $X(v)$ maximises $x \mapsto x\left(v - \frac{x}{2}\right) - C(x)$, and $\hat{P}(x) = \frac{x}{2}$ for $x \in [-1, x_m) \cup (x_M, 1]$, $X(v)$ maximises $x \mapsto x(v - \hat{P}(x)) - C(x)$ over $[-1, 1]$: $(X_C, P(X_C))$ is an equilibrium.

We now prove uniqueness. Let $X' : [-1,1] \to [x'_m, x'_M]$ be a non-decreasing strategy of the IT. By Lemma 1, the expected price $\hat{P}'$ associated with X' is $\frac{x}{2}$ for $x \in [x'_m, x'_M]$. But the computation of $\hat{P}'$ outside of $[x'_m, x'_M]$ is the same as the computation of $\hat{P}$ in (13). Hence, for all $x \in [-1, 1]$, $\hat{P}'(x) = \frac{x}{2}$. So, if $(X', P(X'))$ is an equilibrium of $\mathcal{K}(C)$ such that X' is non-decreasing, X_C and X' maximise the same objective ψ_C over $[-1; 1]$. By Lemma 4 the maximisers agree outside of a countable set, hence so do X_C and X'. In turn, we have $P(X') = P_C$. Therefore (X_C, P_C) and $(X', P(X'))$ are the same equilibrium, which establishes uniqueness. ■

Since $\psi_C(x, v) = \psi_C(-x, -v)$, the Theorem implies that the equilibrium demand function of the IT must be an odd function. In particular we know that the minimal demand x_m equals $-x_M$.

3.4 Examples of equilibria

In this section, we use Theorem 1 in order to understand how the presence of penalties affects the trading strategy of the IT and the pricing function.

Consistent with intuition, penalties reduce the demand of the IT. By how much $X(v)$ is reduced depends on the functional form of the cost C and the realisation of v. This leads in general to a non-linear demand schedule. In the following examples, we will illustrate some important determinants of the IT demand.

The price function can be very flat in some regions and increase sharply in others. In particular, the price impact of a marginal uninformed trade $\frac{d}{du}P(X(v)+u)$ strongly depends on both the realisations of u and v. By constrast, in the mimicking equilibrium of the model without penalties, this price impact is constant, regardless of the distributional assumptions on the noise.

In this section, we consider three examples of penalties: quadratic, linear, and constant over large trades. Linear and quadratic specifications are widespread in the studies of transaction costs. Hence, this section can also be intepreted as an answer to the question: how do insiders trade and how do price functions modify in the presence of these commonly used transaction costs?

3.4.1 Quadratic penalty

In this particular instance, X remains linear after the introduction of the penalty (but not P). [3] Imposing quadratic costs is akin to increasing the perceived expected price impact. Since this cost is in x^2 while the gross gains of trading are in x, the IT always trade as soon as $v \neq 0$, and the magnitude of the trade increases with the absolute value of v.

[3] In the special case of a quadratic penalty, the uniform noise assumption is not necessary to obtain a tractable solution of the Kyle problem, as the model with Gaussian noise admits a linear equilibrium.

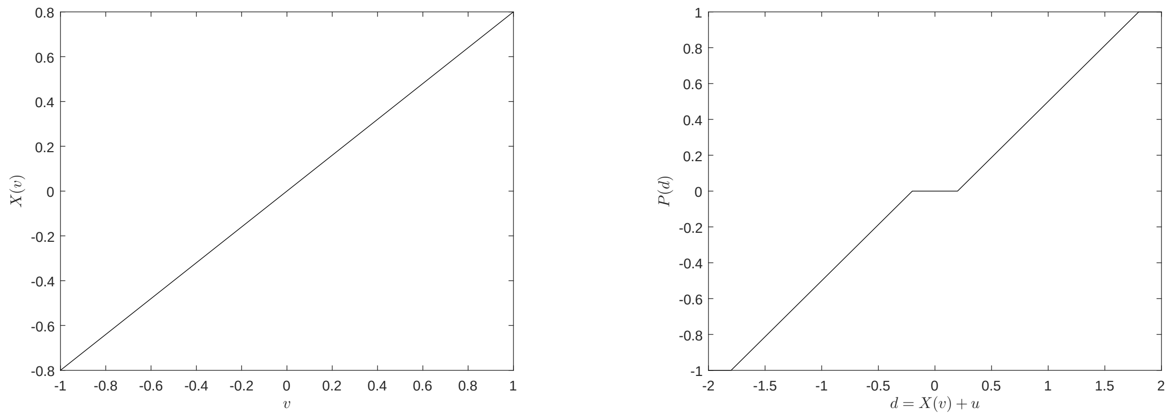


Figure 3: Insider's demand and pricing under quadratic penalty

$C(x) = \alpha x^2$, $\alpha = 0.125$. Left panel: IT demand X. Right panel: price function P.

Due to the presence of the penalty, the insider trades less than in the linear mimicking equilibrium, so that $X(1) = x_M < 1$ ($= 0.8$ in this example).

When $|d| \leq 1 - x_M (= 0.2)$, any demand of the IT is compatible with the observed aggregate order, so all values v remain equally likely, as explained in Section 3.1.2. No information is incorporated and the price remains at the initial expected value of the asset: 0. When $d > 1 - x_M$, one knows that v has not realised at a very low value. This provides a lower bound on v and the price becomes positive. As d increases, so do the lower bound and the price, until $d = 1 + x_M (= 1.8)$. In that case, one knows for sure that the IT has placed an order x_M, which means that $v = 1$, and P reaches 1. The situation is symmetrical for values of d below $x_M - 1 (= -0.2)$.

3.4.2 Linear penalty

When the penalty is linear, $C(x) = \alpha|x|$, for positive values of v, the maximisation program of the IT can be rewritten as

$$\max_{x \in [0,1]} x \left((v - \alpha) - \frac{x}{2} \right).$$

If $v \geq \alpha$, one sees that a linear cost has the same effect as reducing the value of the fundamental v by an amount α, and having no cost. Therefore, the strategy of the IT for values $v \in [\alpha, 1]$ is a translation of the linear mimicking strategy over $v \in [0, 1-\alpha]$. Similarly, the strategy of the IT for values $v \in [-1, -\alpha]$ is a translation of the linear mimicking strategy over $v \in [\alpha - 1, 0]$. This creates the two increasing linear segments in the left panel of Figure 4. In the flat middle section, v is not sufficient to cover the penalty: the IT does not trade.

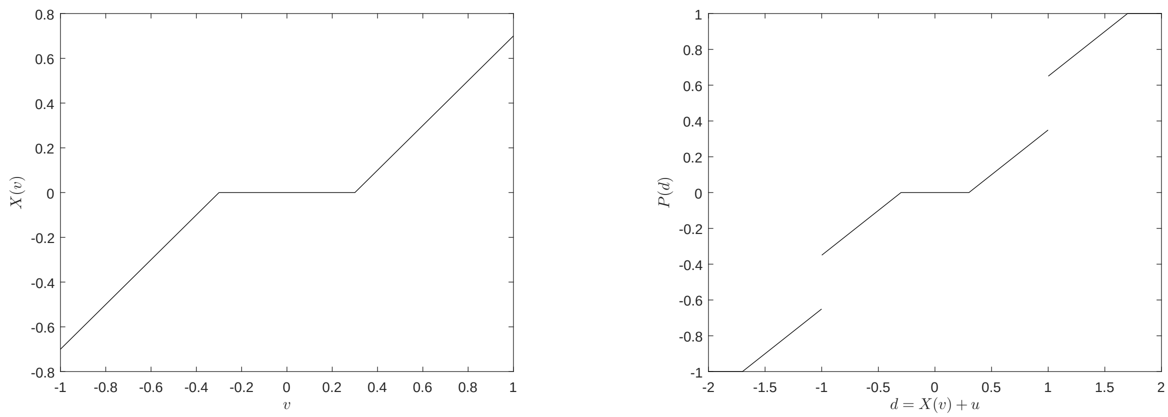


Figure 4: Insider's demand and pricing under linear penalty

$C(x) = \alpha|x|$, $\alpha = 0.3$. Left panel: IT demand X. Right panel: price function P.

The price function depicted in the right panel of Figure 4 exhibits a flat section in the center surrounded by increasing linear segments. The intuition is exactly the same as in the quadratic penalty case: when the magnitude of d is small ($|d| \leq \alpha(= 0.3)$), all values of v remain equally possible and no information is incorporated. As d grows, a lower bound on v can be deduced and the price increases. The key difference with the quadratic penalty case is that the price function jumps at $d = \pm 1$. Indeed, when $d > 1$, the market maker knows for sure that the insider has placed a positive order. But the IT only does so when $v > \alpha$. By contrast, if $d = 1^-$, $X(v) = 0$ remains possible, so we can only deduce that $v > -\alpha(= -0.3)$.

3.4.3 Constant cost on trades of magnitude larger than x_0

Absent penalties, the IT picks $X(v) = v$. Hence, if she is sanctioned only for trades of magnitude larger than x_0, she will not change her demand as long a $|v| \leq x_0$: this corresponds to the increasing linear section in the middle of Figure 5. For intermediate values of v, the IT prefers to block her demand at the value x_0 (or $-x_0$) in order to avoid the penalty: this corresponds to the flat sections in Figure 5. When v becomes large enough ($|v| > \sqrt{2K}(\approx 0.63)$), the penalty is recouped in expectation by using the strategy that prevails in the absence of costs: it appears as a sunk cost and the IT selects again the demand $X(v) = v$. This corresponds to the increasing linear sections at the left and right of Figure 5.

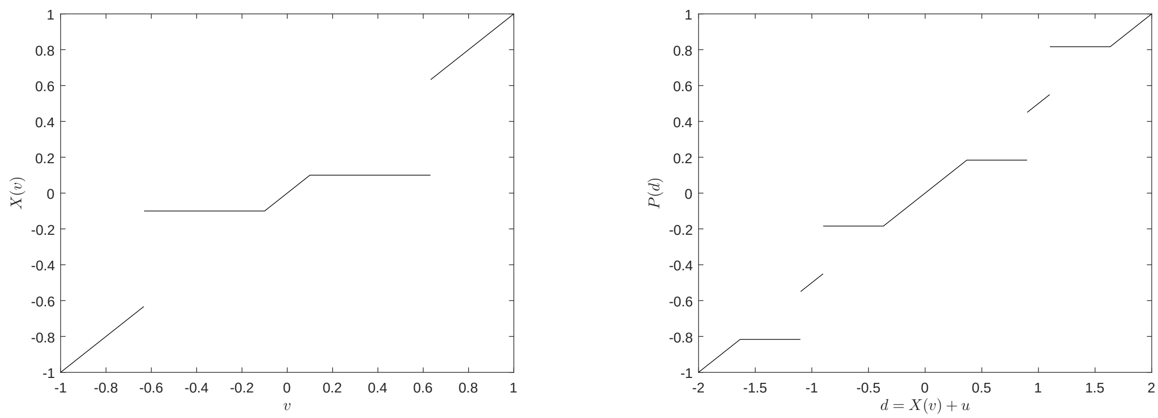


Figure 5: Insider's demand and pricing under constant penalty on large trades

$$C(x) = K\mathbb{I}_{|x|>x_0},\ K = 0.2,\ x_0 = 0.1.$$

Left panel: IT demand X. Right panel: price function P.

The price function jumps at $d = \pm(1 - x_0)(= \pm 0.9)$ and $d = \pm(1 + x_0)(= \pm 1.1)$. The intuition is as in the linear penalty case. When d exceeds $1 - x_0$, the MM knows that the demand of the IT was larger than $-x_0$ which rules out all values of v at the left of $-\sqrt{2K}$, the left jump of X. Similarly, when d exceeds $1 + x_0$, the MM knows that the demand of the IT was larger than x_0, which rules out all values of v at the left of $\sqrt{2K}$, the right jump of X.

A robustness exercise in the case of Gaussian noise is conducted in Appendix B.1 and shows that most of the effects described above qualitatively subsist. We do not prove formally existence of an equilibrium in the case of Gaussian noise; instead we run a fixed-point algorithm on Equations (5) and (6).

4 Application: insider trading regulation

We now wish to apply our main theorem to regulation issues. To do so and as mentioned before, it will be useful to interpret the function C as a product $C = \alpha\tilde{C}$. α is the exogenous probability that an investigation starts and succeeds, and $\tilde{C}$ is the penalty imposed upon the insider trader conditional on investigation success.

4.1 The regulator's problem

The regulator we consider is concerned about:

(i) the post-trade standard deviation of the fundamental, $\sigma(v|d)$,

(ii) the P& L of the uninformed traders:

$$g(u,v) = u(v - P(X(v)+u)). \tag{14}$$

(In Section 4.3.2, the regulator additionally needs to take care of the expected fine it collects for budget reasons.)

Quantity (i) matters because one would like to have informative prices: when (i) is small, the residual uncertainty about v is also small. Quantity (ii) captures the willingness of the regulator to have liquid markets. In a liquid market, agents who have to trade for non-fundamental reasons do not experience high losses. This corresponds to a situation where g is not too negative. The core issue is that improving upon criterion (i) generally causes criterion (ii) to worsen.
Let

$$S = \mathbb{E}[\sigma(v|d)] \tag{15}$$

be the expectation of the post-trade standard deviation of v and

$$G = \mathbb{E}[g(u,v)] \tag{16}$$

denote the expected P& L of the NT.

The objective of the regulator can now be stated as the characterisation of the *efficient frontier*, with the following definition:

Definition 5 *(i) A point (G,S) is implementable if it is the outcome of an equilibrium of $\mathcal{K}(C)$ for some admissible penalty C.*

(ii) An implementable point (G,S) is dominated by (G',S') if (G',S') is implementable and $G' \geq G$, $S' \leq S$ with at least one strict inequality.

(iii) The set of implementable non-dominated points is called the efficient frontier.

In Section 4.3.2, we will need the following refinement of (ii):

(ii') An implementable point (G, S) belonging to some subset of the plane H is dominated in H by (G', S') if (G', S') is implementable, $G' \geq G$, $S' \leq S$ with at least one strict inequality and $(G', S') \in H$.

Points outside the efficient frontier are irrelevant from the regulator's perspective, as it can improve upon one of its objectives without harming the other one. By contrast, any point belonging to the efficient frontier could be picked by a regulator for a suitable weighing of the objectives.[4] Our goal is to characterise the efficient frontier and the penalties that implement it. We do so in three different settings: without a budget constraint (Section 4.2), under a budget constraint with non-pecuniary (Section 4.3.1) and pecuniary (Section 4.3.2) penalties. First, we introduce some useful notation.

Let

$$\pi^N(v) := \psi_C(X(v), v) \tag{17}$$

be the expected net profit of the insider trader in state v,

$$\Pi^N := \mathbb{E}_v[\pi^N(v)] \tag{18}$$

be the overall expected net profit (after fine, if any), and

$$F := \mathbb{E}[C(X(v))] \tag{19}$$

be the expected penalty that the insider undergoes.

Observe that we can write

$$|G| = \int_0^1 X(v)\left(v - \frac{X(v)}{2}\right) \mathrm{d}v = \underbrace{\int_0^1 \frac{v^2}{2}\,\mathrm{d}v}_{1/6} - \frac{1}{2}\int_0^1 (v - X(v))^2\,\mathrm{d}v. \tag{20}$$

This way of seeing the expected losses of the uninformed traders as (an affine transformation of) the L^2 distance between X and the identity will be useful in Section 4.3.1.

4.2 Efficient frontier without a budget constraint

Theorem 2 *When the regulator does not face a budget constraint, the equation of the efficient frontier is*

$$S = \frac{1}{\sqrt{3}}(1 + 2G), \quad -\frac{1}{6} \leq G \leq 0.$$

[4] Our approach has the advantage that we take no stance on what the preferences of the regulator are.

The set of penalties that implements the efficient frontier is exactly the class $\mathcal{O}$ defined as

$$\mathcal{O} = \Big\{ C \in \mathcal{C},\, \exists K \in [0, 1/2]\,, \qquad C(x) \geq x\left(\sqrt{2K} - \frac{x}{2}\right) \quad \textit{for } 0 \leq x \leq \sqrt{2K}, \\ C(x) = K \qquad \textit{for } \sqrt{2K} < x \leq 1 \Big\}. \tag{21}$$

When $C \in \mathcal{O}$, the demand of the insider writes

$$X_K(v) = \begin{cases} 0 & |v| \leq \sqrt{2K}, \\ v & |v| > \sqrt{2K}, \end{cases}$$

for the $K \in [0, 1/2]$ associated with C.

Figure 6 gives a graphical representation of functions in $\mathcal{O}$.

If two penalties in $\mathcal{O}$ are associated with the same K, they implement the same demand schedule X_K. Moreover, it is easy to see that any point in the efficient frontier is implemented by X_K for exactly one value of K.[5] Therefore, K parametrises the efficient frontier. Points associated with a small (resp. large) K are selected by a regulator who puts more weight on information incorporation (resp. on dampening the uninformed traders' losses).

Any regulator that puts nonzero weight on both objectives must at least somewhat reduce insider trading, but not totally. As we shall detail later, the optimal solution is to allow some large trades for large realisations of $|v|$, because they incorporate a lot of information; more precisely, the regulator wants to implement $X(v) = v$ for large values of $|v|$. The cutoff point $\sqrt{2K}$ in the schedule X_K then appears as the solution to the equation $\frac{v^2}{2} = K$. (Recall that $\frac{v^2}{2}$ is the profit of the IT when there is no penalty). This characterises the magnitude of v above which the penalty appears as a sunk cost to the insider, who then effectively optimises as if there was no penalty and selects the mimicking demand $X(v) = v$.

[5] The penalty $C(x) = K\mathbb{I}_{x \neq 0}$ belongs to $\mathcal{O}$ and implements X_K. A direct calculation shows that the P& L G of the uninformed traders under the demand X_K is $-\frac{1}{6}\left(1 - (2K)^{3/2}\right)$. Hence, the value of K that implements the point (G, S) of the efficient frontier is the solution to $G = -\frac{1}{6}\left(1 - (2K)^{3/2}\right)$.

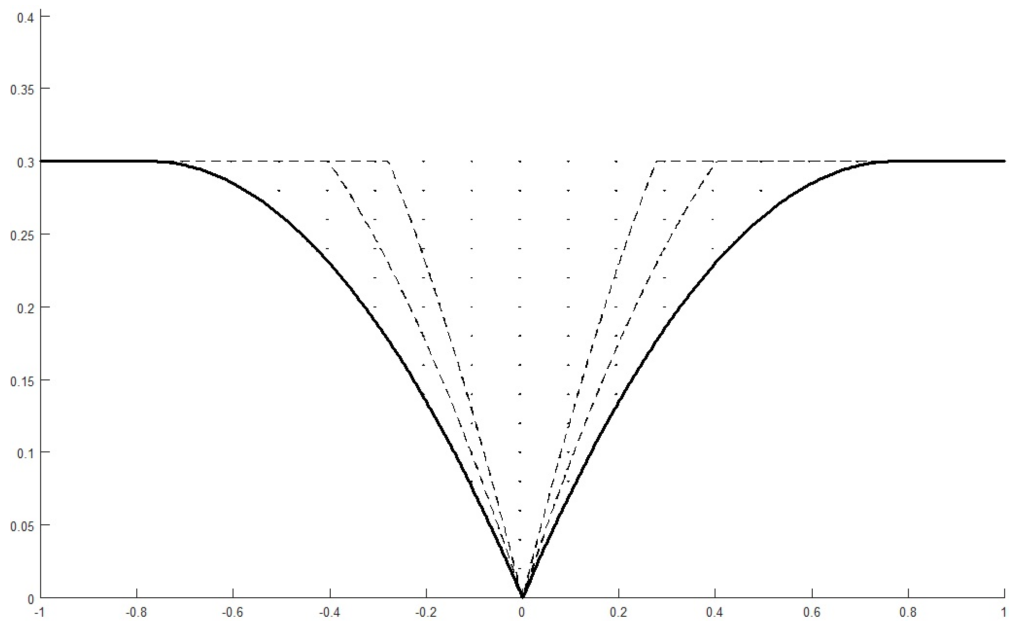


Figure 6: Some penalty functions in $\mathcal{O}$.

In Figure 6, the thick line represents the lower bound in the definition of $\mathcal{O}$ when $K = 0.3$ (then, $\sqrt{2K} \approx 0.77$) : any penalty in $\mathcal{O}$ must be above this line. Given that a penalty is symmetrical and non-decreasing over $[0, 1]$, the graph of a function in $\mathcal{O}$ must be included in the dotted area. The two dashed lines represent two such functions.

4.2.1 Preliminary results on the regulator's objective

Lemma 5 *In equilibrium, the net profits satisfy*

$$\pi^N(v) = \int_0^v X(s)\,\mathrm{d}s, \tag{22}$$

$$\Pi^N = \int_0^1 (1-v)X(v)\,\mathrm{d}v. \tag{23}$$

Proof. Consider the parametrised objective function

$$\psi_C : [0,1] \times [0,1] \to \mathbb{R}$$

defined in (11). Notice that (i) $\psi_C(x, .)$ is linear in v and therefore absolutely continuous, (ii) $|\partial_v \psi_C(x, v)| = |x| \leq 1$. (i) and (ii) guarantee that the assumptions of Theorem 2 in Milgrom and

Segal (2002) are satisfied. In the present case, this theorem tells us that we can write:

$$\begin{aligned}\pi^N(v) &= \pi^N(0) + \int_0^v \partial_2 \psi_C(X(s), s)\,\mathrm{d}s \\ &= \int_0^v X(s)\,\mathrm{d}s,\end{aligned}$$

since the insider does not make any profit when the fundamental v is 0. Finally,

$$\begin{aligned}\Pi^N = \frac{1}{2}\int_{-1}^{1} \pi^N(v)\,\mathrm{d}v &= \frac{1}{2}\int_{-1}^{1}\int_0^v X(y)\,\mathrm{d}y\,\mathrm{d}v \\ &= \int_0^1\int_0^v X(y)\,\mathrm{d}y\,\mathrm{d}v \\ &= \int_0^1 (1-v)X(v)\,\mathrm{d}v.\end{aligned}$$

■

Lemma 6 expresses the expected post-trade standard deviation as a function of the demand profile X.

Lemma 6 *The expected post-trade standard deviation satisfies*

$$S = \frac{1}{\sqrt{3}}\left(1 - \int_0^1 vX(v)\,\mathrm{d}v\right). \tag{24}$$

Proof. By the proof of Lemma 1, $v|d$ is uniform over

$$I_X(d) \equiv [(X_\ell^{-1}((d-1)\vee(-x_M))\,;X_r^{-1}((d+1)\wedge x_M))].$$

Since the standard deviation of a uniform variable over $[a;b]$ equals $\frac{1}{2\sqrt{3}}(b-a)$, Lemma 6 is an immediate consequence of the following result: *if X is an odd non-decreasing function from $[-1;1]$ to $[-x_M;x_M]$, then the expected length of the interval $I_X(X(v)+u)$ equals* $2\left(1-\int_0^1 vX(v)\,\mathrm{d}v\right)$, which we must now prove.

For $v \in [-1;1]$, define

$$\begin{aligned}Y_v &= X_r^{-1}((X(v)+u+1)\wedge x_M) \\ Z_v &= X_l^{-1}((X(v)+u-1)\vee(-x_M)).\end{aligned}$$

What we need to prove is that $\mathbb{E}_{v,u}[Y_v - Z_v] = 2\left(1-\int_0^1 vX(v)\,\mathrm{d}v\right)$. By symmetry, $\mathbb{E}_{v,u}[Z_v] =$

$-\mathbb{E}_{v,u}[Y_v]$, thus, it remains to prove that:

$$\mathbb{E}_{v,u}[Y_v] = 1 - \int_0^1 vX(v)\,\mathrm{d}v.$$

Let us consider v fixed. The random variable Y_v takes values in $[-1,1]$: using Fubini theorem,

$$\mathbb{E}[Y_v] = \mathbb{E}\left[\int_{-1}^1 \mathbb{I}_{-1\le y\le Y_v}\,\mathrm{d}y\right] - 1 = \int_{-1}^1 \mathbb{P}\left(y \le Y_v\right)\,\mathrm{d}y - 1.$$

By definition of X_r^{-1}, if $X(y) \le (X(v)+u+1)\wedge x_M$ then $y \le Y_v$. Besides, if $y < Y_v$, then using the fact that X is non decreasing, $X(y) \le (X(v)+u+1)\wedge x_M$. Thus:

$$\{y \le Y_v\} \setminus \{X(y) \le (X(v)+u+1)\wedge x_M\} \subset \{y = Y_v\}.$$

Let us remark that $Y_v = y$ can hold for two different values of u if and only if X is discontinuous at y or $y = 1$. In particular,

$$\{y \ne 1 | \mathbb{P}(y = Y_v) > 0\} \subset \{y | X(y^-) \ne X(y^+)\}.$$

It follows from this discussion that :

$$\left|\mathbb{E}[Y_v] - \int_{-1}^1 \mathbb{P}(X(y) \le X(v)+u+1)\,\mathrm{d}y + 1\right| \le \int_{-1}^1 \mathbb{P}(Y_v = y)\,\mathrm{d}y \le \mu\left(\{y|X(y^-) \ne X(y^+)\}\right),$$

where μ is the Lebesgue measure on $[-1,1]$. Since X is non-decreasing, it has a countable number of discontinuity points. In particular $\mu\left(\{y|X(y^-) \ne X(y^+)\}\right) = 0$ and:

$$\mathbb{E}[Y_v] = \int_{-1}^1 \mathbb{P}(X(y) \le X(v)+u+1)\,\mathrm{d}y - 1.$$

Now,

$$\begin{aligned} \mathbb{P}(X(y) \le X(v)+u+1) &= \mathbb{P}(u \ge X(y) - X(v) - 1) \\ &= 1 + \left(\frac{1}{2}(X(v) - X(y)) \wedge 0\right). \end{aligned}$$

Going back to the expression of $\mathbb{E}[Y_v]$, we obtain

$$\mathbb{E}[Y_v] = 1 - \frac{1}{2}\int_v^1 X(y)\,\mathrm{d}y + \frac{1}{2}(1-v)X(v).$$

Integrating over v:

$$\begin{aligned}\mathbb{E}_{v,u}[Y_v] &= 1-\frac{1}{4}\int_{-1}^{1}\int_{v}^{1}X(y)\,\mathrm{d}y\,\mathrm{d}v+\frac{1}{4}\int_{-1}^{1}(1-v)X(v)\,\mathrm{d}v\\ &= 1-\frac{1}{4}\int_{-1}^{1}(v+1)X(v)\,\mathrm{d}v+\frac{1}{4}\int_{-1}^{1}(1-v)X(v)\,\mathrm{d}v\\ &= 1-\frac{1}{2}\int_{-1}^{1}vX(v)\,\mathrm{d}v\\ &= 1-\int_{0}^{1}vX(v)\,\mathrm{d}v,\end{aligned}$$

where in line 3, we used the fact that X is odd. This concludes the proof. ■

One consequence of Lemma 6 is that large orders associated with large values of the fundamental are the ones that contribute the most to incorporating information into prices. Indeed, the values of v such that the product $vX(v)$ is large have the strongest negative impact on S, as can be seen from (24).

4.2.2 Shape of the efficient frontier and efficient demand functions

In this section, we give the shape of the efficient frontier and explain what demand schedules are compatible with it. We call these schedules *efficient demand functions*.

Lemma 7 *Let C be a penalty function in $\mathcal{C}$. In the equilibrium of $\mathcal{K}(C)$,*

$$S \geq \frac{1}{\sqrt{3}}(1+2G)$$

with equality if and only if there is $v^* \in [0,1]$ such that $X(v)=0$ for $|v|< v^*$ and $X(v)=v$ for $|v|> v^*$.

Proof. Due to Lemma 6, what we need to show is that

$$-\int_0^1 vX(v)\,\mathrm{d}v \geq -2\int_0^1 X(v)\left(v-\frac{X(v)}{2}\right)\,\mathrm{d}v.$$

This is equivalent to

$$\int_0^1 vX(v)dv \geq \int_0^1 X(v)^2\,\mathrm{d}v,$$

or

$$\int_0^1 X(v)(v-X(v))\,\mathrm{d}v \geq 0 \tag{25}$$

which holds because $0 \leq X(v) \leq v$ for $v \in [0;1]$.

For the equality to hold, it is necessary and sufficient to have $X(v) = 0$ or $X(v) = v$ almost everywhere. Since X is non-decreasing, it is equivalent to $X(v) = 0$ for $|v| < v^*$ and $X(v) = v$ for $|v| > v^*$, where $v^* = \sup\{v,\ X(v) = 0\}$. ■

Equation (25) is particularly convenient because it immediately indicates what type of demand function is needed to implement the efficient frontier. Of course, X is an endogenous outcome: what remains to be seen is what regulations implement the efficient demand functions.

4.2.3 Implementation of the efficient demand functions

Lemma 8 *The efficient demand functions derived in Lemma 7 are implemented exactly by the penalties $C \in \mathcal{O}$.*

Proof. See Appendix A.3. ■

By construction, penalties in $\mathcal{O}$ increase quickly as $|v|$ departs from 0 and are flat for large values of $|v|$ (see Figure 6). Intuitively, this is what is required to implement the efficient demand functions. Indeed, when $|v|$ realises at a small value, the marginal impact of increasing demand on the expected penalty is large, and the IT prefers to refrain from trading. For $|v|$ large, however, the penalty schedule being flat on large demands, a large order allows to cover the expected fine, which appears as a sunk cost. The IT then optimises as in the linear mimicking equilibrium and demands $X(v) = v$.

The proof of Theorem 2 is complete: Lemma 7 characterises the efficient frontier and due to Lemma 8, achieving the efficient frontier can only be done by selecting a cost $C \in \mathcal{O}$, characterised by a $K \in [0, {}^1\!/\!_2]$.

4.2.4 Illustrations and discussion

Varying K between 0 and ${}^1\!/\!_2$ allows to cover the entire efficient frontier. As K increases, the losses $(-G)$ of the uninformed traders decrease from $\int_0^1 \frac{v^2}{2}\,\mathrm{d}v = \frac{1}{6} \approx 0.167$ to 0, while the expected post-trade standard deviation increases from $\frac{1}{\sqrt{3}}(1 - 2/6) = \frac{2}{3\sqrt{3}} \approx 0.385$ to $\frac{1}{\sqrt{3}} \approx 0.577$.

Each point of Figure 7 corresponds to a penalty function C; it represents the outcomes $(S, -G)$ in the unique equilibrium of $\mathcal{K}(C)$. For a fixed y-coordinate (a fixed S) the preferred option of the regulator is to select a point with the smallest x-coordinate (that minimises

$-G$). Consistent with Theorem 2, penalties in $\mathcal{O}$ achieve the efficient frontier, which is linear.

Outcomes $(S, -G)$ corresponding to quadratic and linear penalties ($C(x) = \alpha x^2$, $C(x) = \alpha|x|$ for varying $\alpha \geq 0$) are also reported in Figure 7. As one can see, they perform significantly worse than penalties $C \in \mathcal{O}$. This is also the case of penalties with no cost on small trades and big costs on large trades, $C(x) = K^H \mathbb{I}_{|x|>x_0}$. Here K^H is a constant large enough so that the insider never chooses to trade more than x_0. The fact that these particular penalty functions perform poorly compared to penalties in $\mathcal{O}$ is consistent with the intuition given below Lemma 6. Indeed, they imply that $X(v) = v$ for $|v|$ small and $|X|$ stops growing for $|v|$ large (the opposite of the demand functions implied by $C \in \mathcal{O}$), so that the reduction of the expected standard deviation, measured by the term $\int_0^1 vX(v)\,\mathrm{d}v$, is low.

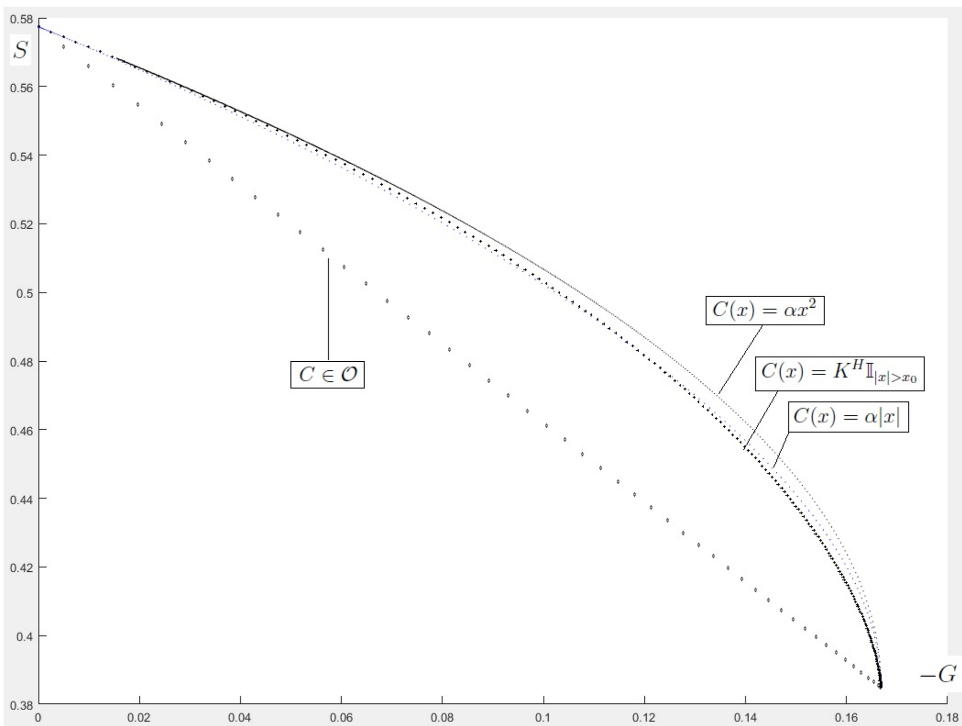


Figure 7: Locus of $(S, -G)$ for some penalty functions.

Figure 7 shows that quadratic costs are the most inefficient among the considered costs. In fact, they have the worst performance among *all penalty functions*:

Proposition 1 *Quadratic penalties implement the upper frontier of the locus of outcomes (S, G) generated by all penalty functions in $\mathcal{C}$, i.e. they induce the highest possible expected post-trade standard deviation for a given P&L of the uninformed traders.*

Proof. See Appendix A.2. ■

In Appendix B.2, we repeat numerically the construction of Figure 7 in the case of Gaussian noise and obtain similar results.

4.3 Efficient frontiers under a budget constraint

So far, our analysis assumes away a real-world constraint on the regulator: investigation costs. Conducting investigations requires time, financial and human resources. How do the regulator's efficient policies change in that case?

In Section 4.3.1, we consider the case of non-pecuniary penalties: the regulator cannot use fines to soften the budget constraint. In our model, this caps the maximal expected penalty that can be imposed on insider trades. Hence, in Section 4.3.1, the regulator faces the same problem as before—maximising the efficiency of prices for a given level of uninformed traders' losses—yet its set of admissible strategies is narrower.

In Section 4.3.2, we study pecuniary fines. We shall assume that the regulator's initial budget is insufficient to cover its investigation expenditures: it needs to levy fines to break even. Hence, in Section 4.3.2, the regulator faces a richer problem than before—it now needs to consider a third criterion—yet, this time, there is no restriction *a priori* on its set of admissible strategies.

4.3.1 Non-pecuniary penalties

Let $B > 0$ be the budget allotted to the regulator and $\kappa > 0$ denote the investigation cost. We work under the assumption that the investigation probability α is constant—leaving the relaxation of this hypothesis for future research. The regulator operates under the constraint

$$\alpha\kappa \leq B. \tag{26}$$

Recall that the insider trader optimises under an expected penalty schedule $C = \alpha\tilde{C}$, where $\tilde{C}$ is the actual sanction conditional on investigation success.

First, consider the case where there is no constraint on $\tilde{C}$. Then, the regulator can trivially get around its budget constraint (26) by reducing α and increasing $\tilde{C}$. In that case, we are back to the unconstrained problem solved in Section 4.2.

We now consider the more interesting case where there is an upper bound $\tilde{C}^M$ on $\tilde{C}$.

The existence of $\tilde{C}^M$ can be justified on several grounds: (i) stronger sanctions involve a larger burden of proof to be enforced. Newkirk and Robertson (1998) report that in the

Netherlands, a law was passed at the end of the XXth century, presented as the "toughest in the world" against insider trading. The result was that in the following years, virtually no conviction was possible—not because there were no cases of insider trading suspicions, but because evidence was almost never strong enough "to satisfy the heavy burden of proof that must be met to support a criminal conviction". Hence, $\tilde{C}^M$ can be seen as a threshold penalty above which the sanction is effectively not enforceable. (ii) Alternatively, $\tilde{C}^M$ can be seen as the maximal disutility that a non-pecuniary sanction (such as lifetime imprisonment) can impose upon a human being.

From (26), the insider trader faces an expected penalty

$$C = \alpha\tilde{C} \leq K := \frac{B}{\kappa}\tilde{C}^M. \tag{27}$$

The constraint (27) means that we have restricted the regulator's set of admissible penalties:[6]

Definition 6 *In the non-pecuniary case, the set of admissible penalties with a budget constraint is*

$$\mathcal{C}_K = \{C \in \mathcal{C},\, C(1) \leq K\}.$$

Note that $C(1) \leq K$ is equivalent to (27) because any penalty in $\mathcal{C}$ is symmetrical and non-decreasing over $[0,1]$. Moreover, the budget constraint is an actual constraint for $K \in [0, 1/2)$; for $K \geq 1/2$, $\mathcal{C}_K = \mathcal{C}$.

What happens when one restricts the set of admissible penalties? First, some previously efficient points may no longer be feasible. Second, some points that were not previously efficient may no longer be dominated by any point still implementable under the budget constraint. We recast Definition 5 in this new setting:

Definition 7 *In the non-pecuniary case, the efficient frontier under a budget constraint is the set of points (G, S) implementable by a penalty in $\mathcal{C}_K$ that are not dominated by any point implementable by a penalty in $\mathcal{C}_K$.*

Denote

$$\mathcal{O}_K = \mathcal{O} \cap \mathcal{C}_K.$$

[6] Of course, we could obtain the same constraint by ignoring investigation costs, setting $\alpha = 1$ and assuming that the cap $\tilde{C}^M$ on $\tilde{C} = C$ is below $1/2$. The idea here is that if investigation was systematic, the bound $\tilde{C}^M$ would likely be non-binding. It only becomes binding because investigation is costly, which reduces the expected penalty that the IT faces. The extent to which it binds depends on the budget-relevant parameters B and κ: see (27).

$\mathcal{O}_K$ is the set of efficient penalties derived in Section 4.2.3, that are still feasible under the budget constraint. These penalties are still efficient under the budget constraint. Moreover, by direct computation, we obtain that as C varies in $\mathcal{O}_K$, $|G|$ describes the interval $\left[|G|_{\min}(K), \frac{1}{6}\right]$, where

$$|G|_{\min}(K) := \frac{1}{6}\left(1 - (2K)^{3/2}\right). \tag{28}$$

The truncature of the previously efficient frontier at the right (in the $(|G|, S)$ plane) of $|G|_{\min}(K)$ is part of the efficient frontier under the budget constraint. The key question is to know what happens at the left of $|G|_{\min}(K)$. Theorem 4 shows that no penalty in $\mathcal{C}_K$ can implement $|G| < |G|_{\min}(K)$. This immediately implies the characterisation of the constrained efficient frontier:

Theorem 3 *The efficient frontier under the constraint $C \leq K$ is the truncature $|G| \geq |G|_{min}(K)$ of the efficient frontier of Theorem 2 and is implemented exactly by penalties in $\mathcal{O}_K$.*

As explained above, Theorem 3 is a consequence of:

Theorem 4 *Let $K \leq 1/2$. Under the constraint $C \leq K$, the expected losses of the uninformed traders are at least*

$$|G| \geq |G|_{min}(K).$$

Proof. See Appendix A.4, where we provide the proof as well as several intuitions and graphical interpretations. ■

One consequence of Theorem 3 is that it is not possible to infer from a regulator's choice of penalty whether it is constrained or not. In the non-pecuniary case, a regulator subject to a binding budget constraint effectively behaves like an unconstrained regulator that would assign less weight to curtailing the losses of the uninformed traders. In the next section, we study the case of pecuniary penalties and show that, by contrast to the previous result, the introduction of the constraint creates new efficient points.

4.3.2 Pecuniary penalties

We now consider pecuniary penalties, collected by the regulator. We maintain the assumption of a constant α. We suppose that the regulator must have a balanced budget in expectation. The budget constraint (26) transforms into

$$\alpha\kappa \leq B + \underbrace{\mathbb{E}[C(X(v))]}_{F}. \tag{29}$$

If $B \geq \alpha\kappa$, we are back to the case studied in Section 4.2. We now consider the case $B < \alpha\kappa$ and aim at characterising the new efficient frontier under the constraint (29). This frontier will obtain by projection once we determine the *efficient surface*:

Definition 8 *The efficient surface Σ is the locus of points (G, S, F) generated by any $C \in \mathcal{C}$ such that no $C' \in \mathcal{C}$ can weakly (i) increase G, (ii) decrease S, (iii) increase F with at least one among (i), (ii) or (iii) being strict.*

Define the set of indices $J := \{(x, y),\, 0 \leq y/(1+y) \leq x \leq y \leq 1\}$.

Theorem 5 *The efficient surface Σ in the space (G, S, F) is constructed exactly by the demand schedules $(X_{v_1,v_2})_{(v_1,v_2)\in J}$, that are implemented by the penalties $(C_{v_1,v_2})_{(v_1,v_2)\in J}$:*

$$X_{v_1,v_2}(v) = \begin{cases} 0 & v \in [0, v_1] \\ \frac{v_2}{v_2 - v_1}(v - v_1) & v \in (v_1, v_2] \\ v & v \in (v_2, 1] \\ -X_{v_1,v_2}(-v) & v < 0. \end{cases} \quad ; \quad C_{v_1,v_2}(x) = \begin{cases} v_1|x| - \frac{v_1}{2v_2}x^2 & |x| \leq v_2 \\ \frac{v_1 v_2}{2} & |x| > v_2. \end{cases}$$

Proof. See Appendix A.5. ■

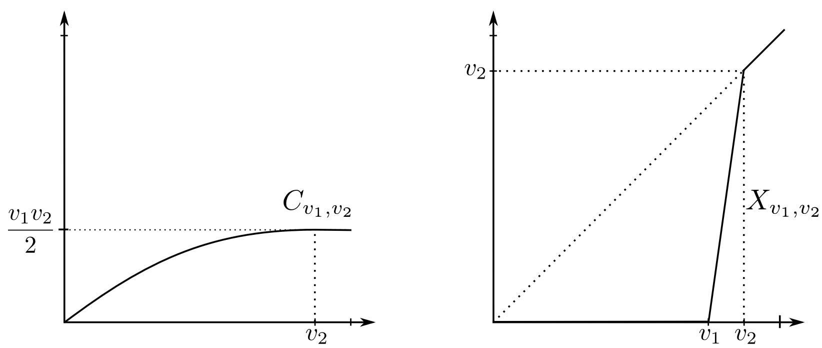


Figure 8: Efficient demand schedule and penalty function under a budget constraint with pecuniary fines.

There is a key difference between proving Theorem 4 and Theorem 5. Here, the proof can be reduced to a routine pointwise minimisation, because the pointwise minimiser of the weighted regulatory objective (see the proof in Appendix A.5) turns out to be an implementable demand schedule. Reducing the problem to a pointwise minimisation exercise was not possible to prove Theorem 4.

We can now turn to the characterisation of the efficient (G, S) frontiers for various regulator's budgets. The budget constraint can be written as $F = \mathbb{E}[C(X(v))] \geq F_{\min} := \alpha\kappa - B$. Hence, we can make the following definition.

Definition 9 *The F_{min}-efficient frontier is the set of non-dominated points in*

$$\mathcal{F}(F_{min}) := \{(G(X), S(X)),\ X \text{ implemented by some } C \in \mathcal{C} \text{ with } \mathbb{E}[C(X(v))] \geq F_{min}\}\,.$$

We can now obtain the $F_{\min}$-efficient frontiers from the efficient surface Σ by projection. Denote $\pi_{GS} : (G, S, F) \mapsto (G, S)$ the projection on the (G, S)-plane: the $F_{\min}$-efficient frontier is the set of points of $\pi_{GS}(\Sigma \cap \{F \geq F_{\min}\})$ that are not dominated in $\pi_{GS}(\Sigma \cap \{F \geq F_{\min}\})$.

Of course, the penalties in $\mathcal{O}$ that implement $F \geq F_{\min}$ are still part of the $F_{\min}$-efficient frontier, but new constrained efficient points emerge (dotted arcs in Figure 9a), which are associated with penalty functions and demand schedules that were not previously optimal. The $F_{\min}$-efficient frontier does not even intersect the unconstrained frontier for $F_{\min}$ very large.[7]

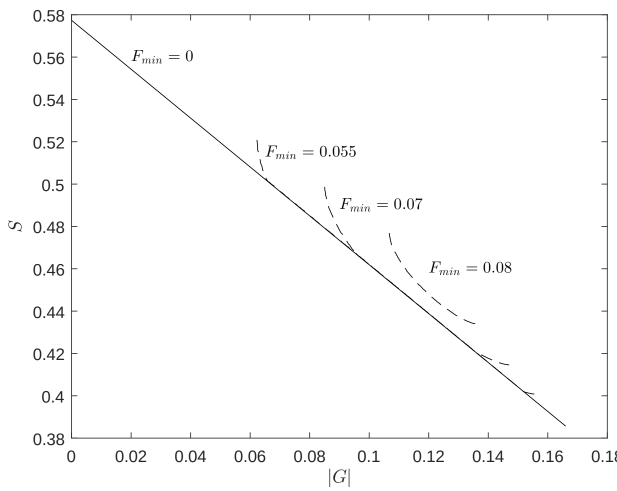
A lax regulation of insider trading obviously does not allow to levy a lot of fines. But a very stringent regulation does not either, because it deters insiders from trading and reduces the rate of conviction. Hence, the regulator must pick some "intermediate" level of regulation when it operates under a tight budget constraint. This is illustrated by the dotted frontiers in Figure 9a, which correspond to "intermediate" levels of $|G|$ and S. One takeaway is that if regulators do need fines to balance their budget, they claiming that their goal is to completely rule out insider trading may not be credible.

Points of the $F_{\min}$-efficient frontier correspond to points in Σ, which means that they are associated with demand schedules of the form X_{v_1,v_2} defined in Theorem 5. To understand how the budget constraint $F \geq F_{\min}$ modifies the nature of the optimal strategies, Figure 9b plots the (v_1, v_2) used on the $F_{\min}$-efficient frontier for various values of $F_{\min}$. For example, the red dot, $(v_1, v_2) \approx (0.48, 0.61)$ represents $X_{0.48,0.61}$ and indicates that this demand schedule implements one point of the efficient frontier when the budget constraint of the regulator is such that $F_{\min} = 0.07$.

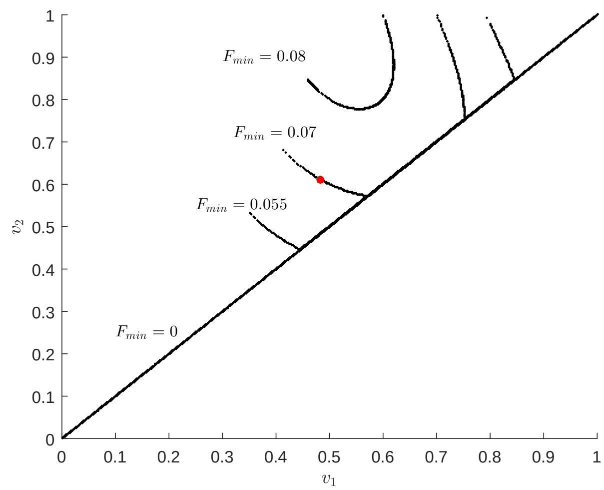
When $F_{\min} = 0$, we obtain the line $v_2 = v_1$, in which case X_{v_1,v_2} is implemented by a penalty $C \in \mathcal{O}$, consistent with Section 4.2.3. We observe that as $F_{\min}$ increases, one needs to widen the gap $v_2 - v_1$. The intuition is that the linear section over $[v_1, v_2]$ of the demand schedule X_{v_1,v_2} best resolves the trade off between large fines and large trade volumes of the insider trader and allows to collect a relatively high amount of fines in expectation. As an example, recall that the demand schedule that implies the highest expected fine (1/12) had $v_2 - v_1 = \frac{1}{2}$.

Price functions are also modified by the introduction of a budget constraint with pecu-

[7] Indeed, it is easy to show that the maximal expected fine under a penalty in $\mathcal{O}$ is 2/27, so if $F_{\min} > \frac{2}{27}$, no penalty in $\mathcal{O}$ allows to balance the regulator's budget. Using penalties in $\mathcal{C}$, the regulator can levy up to 1/12 in expectation, e.g. using $C_{\frac{1}{2},1}$. If $F_{\min} > \frac{1}{12}$, the regulator cannot balance its budget.



(a) Efficient frontiers



(b) Indices (v_1, v_2)

Figure 9: Efficient $(|G|, S)$ frontiers and indices (v_1, v_2) of the efficient demand functions X_{v_1,v_2}, for various constraints $F \geq F_{\mathbf{min}}$

niary fines.

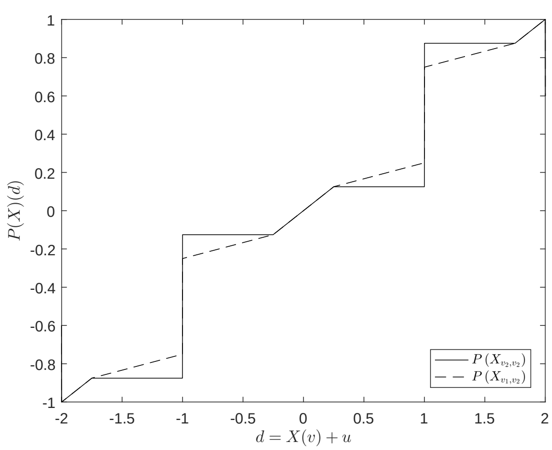


Figure 10: New price functions.
$v_1 = 0.5$ and $v_2 = 0.75$.

Figure 10 compares the price functions implied by a demand schedule efficient absent a budget constraint, X_{v_2,v_2} and a constrained efficient demand schedule X_{v_1,v_2}. Contrary to $P(X_{v_2,v_2})$, $P(X_{v_1,v_2})$ has no flat sections and is everywhere increasing. In particular, in the unconstrained case, the random price is partly discrete: with positive probability, it will be equal to one of the ordinates of the flat sections of $P(X_{v_2,v_2})$. Conversly, in the case of a strong budget constraint, the random price has a continuous density.

5 Conclusion

We have shown how one can obtain an existence and uniqueness result in a Kyle model of insider trading by assuming uniformly—instead of normally—distributed noise. One advantage of our approach is that the proofs bear an intuitive interpretation and the resulting

equilibrium is tractable. But the key appeal of our techniques is that they apply indifferently to the case where insider trading comes at no cost other than the price impact, and to the case where there is an extra cost. Therefore, our results can be readily used as a building block in a more general model featuring transactions costs, or strategic—but illegal—trading of superiorly informed agents.

We use our existence and uniqueness theorem to contribute to the long-standing debate on insider trading regulation. We solve the problem of a regulator aiming at maximising price efficiency for a given level of noise traders' losses in three cases: unconstrained, constrained with non-pecuniary penalties and constrained with pecuniary penalties.

Two extensions of the paper seem interesting. First, one could attempt to endogeneise the investigation probability α. Second, one may want to consider a penalty schedule which does not depend solely on the magnitude of the insider's order, but also on other observable quantities, such as the realised profit.

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Appendix

A Additional Proofs

A.1 Lemma 3

First, let us show that $\mathcal{X}_C(v)$ is never empty. Let $v \in [-1, 1]$, the function $\psi_C(., v)$ has a finite upper bound as $C \geq 0$. Let $M = \sup_x \psi_C(x, v) < \infty$ and (x_n) such that $\psi_C(x_n, v) \to M$. There is an extraction of (x_n), still denoted (x_n), such that x_n converges to x and either (i) (x_n) is increasing or (ii) (x_n) is decreasing. By symmetry, we can assume without loss of generality that $x > 0$ or $x = 0$ and the case (ii) holds. Let us first consider case (i). Since C is left-continuous and $x \mapsto x\left(v - \frac{x}{2}\right)$ is continuous, $\psi_C(x_n, v)$ converges to $\psi_C(x, v)$: therefore $\psi_C(x, v) = M$ and $x \in \mathcal{X}_C(v)$. Let us now consider case (ii). Since C is non decreasing, it has a right limit at x denoted by $C(x^+)$ which is greater than $C(x)$. Taking the limit in the definition of $\psi_C(x_n, v)$, the value of $\psi_C(x_n, v)$ converges to $x\left(v - \frac{x}{2}\right) - C(x^+) \leq x\left(v - \frac{x}{2}\right) - C(x)$. Using the fact that $\psi_C(x_n, v)$ converges to M, we conclude that $C(x^+) = C(x)$ and $\psi_C(x, v) = M$.

Now, let us show that $\mathcal{X}_C$ is a non-decreasing correspondence. Let $v_1 < v_2$ in $[-1; 1]$ and $x_1^* \in \mathcal{X}_C(v_1)$ and $x_2^* \in \mathcal{X}_C(v_2)$. For any $x \in [-1, 1]$:

$$\psi_C(x, v_2) = \psi_C(x, v_1) + (v_2 - v_1)x.$$

Using the fact that $x_1^* \in \mathcal{X}_C(v_1)$ and $v_1 < v_2$, for any $x < x_1^*$,

$$\psi_C(x, v_2) < \psi_C(x_1^*, v_1) + (v_2 - v_1)x_1^* = \psi_C(x_1^*, v_2).$$

By definition, $\psi_C(x_2^*, v_2) \geq \psi_C(x_1^*, v_2)$, thus $x_2^* \geq x_1^*$. Since this inequality holds for any $x_1^* \in \mathcal{X}_C(v_1)$ and $x_2^* \in \mathcal{X}_C(v_2)$, we get that $\sup \mathcal{X}_C(v_1) \leq \inf \mathcal{X}_C(v_2)$: the correspondence $\mathcal{X}_C$ is non-decreasing.

A.2 Proposition 1

Using Lemma 6, we can write

$$\begin{aligned}
-G &= \int_0^1 X(v)\left(v - \frac{X(v)}{2}\right) \mathrm{d}v \\
&= 1 - \sqrt{3}S - \frac{1}{2}\int_0^1 X(v)^2 \, \mathrm{d}v. \qquad (30)
\end{aligned}$$

By Cauchy-Schwarz inequality

$$\begin{aligned}
\left(\int_0^1 vX(v)dv\right)^2 &\leq \int_0^1 v^2 dv \int_0^1 X(v)^2 \, \mathrm{d}v \qquad (31) \\
&\leq \frac{1}{3}\int_0^1 X(v)^2 \, \mathrm{d}v \\
-\frac{1}{2}\int_0^1 X(v)^2 \, \mathrm{d}v &\leq -\frac{3}{2}\left(\int_0^1 vX(v)\right)^2 \mathrm{d}v = -\frac{3}{2}(1-\sqrt{3}S)^2.
\end{aligned}$$

Plugging this into (30), we obtain

$$G \geq \sqrt{3}S - 1 + \frac{3}{2}(1-\sqrt{3}S)^2. \qquad (32)$$

This inequality determines the highest possible S given G. But there is equality in (32) if and only if there is equality in the Cauchy-Schwarz bound (31). This is the case if and only if the two functions in the left-hand side are colinear, i.e. if $X(v)$ is proportional to v: $X(v) = \beta v$. Since $0 \leq X(v) \leq 1$ for $0 \leq v \leq 1$, $\beta \in [0;1]$. We conclude by noting that if $\beta \in [0;1]$ and $\gamma \in [0;\infty]$ is defined by $\gamma = \frac{1}{2\beta} - \frac{1}{2}$, the quadratic penalty $C(x) = \gamma x^2$ implements $X(v) = \beta v$.

A.3 Lemma 8

We first need to introduce some definitions:

Let f be a function defined over $[0,1]$ and $x \in [0,1]$. We define:

$$\begin{aligned}
\overline{D}^- f(x) &= \limsup_{x' \nearrow x} \frac{f(x') - f(x)}{x' - x}, \\
\underline{D}^- f(x) &= \liminf_{x' \nearrow x} \frac{f(x') - f(x)}{x' - x},
\end{aligned}$$

One can define similarly $\overline{D}^+ f(x)$ and $\underline{D}^+ f(x)$. Let us recall the first order conditions satisfied by a function at a local maximum.

If x^* is a local maximum of f, then:

$$\begin{aligned}
&\overline{D}^+ f(x^*) \leq 0, \\
&\underline{D}^- f(x^*) \geq 0
\end{aligned}$$

We will also use the following real analysis result:

Lemma 9 *Any continuous function f on $]0,1]$ with a null left derivative is contant.*

Let C be a penalty function such that the strategy of the IT satisfies that for any $v \in [0,1]$, $X(v)$ is either 0 or v. Since the strategy of the IT is non-decreasing, there exists v_0 such that $X(v) = 0$ for any $v \in [0, v_0[$ and $X(v) = v$ for any $v \in]v_0, 1]$.

Besides, the penalty function C must be continuous on $]v_0, 1]$. Indeed, if $v' > v \geq v_0$, using the fact that $X(v') = v'$,

$$v\left(v' - \frac{v}{2}\right) - C(v) \leq v'\left(v' - \frac{v'}{2}\right) - C(v'),$$

thus, since C is non-decreasing,

$$0 \leq C(v') - C(v) \leq v'\left(v' - \frac{v'}{2}\right) - v\left(v' - \frac{v}{2}\right).$$

Taking the limit as v' goes to v, we see that C is right continuous at v. Since by hypothesis it is left continuous on $[0,1]$, the penalty function C is continuous on $]v_0, 1]$.

Let us show that C has a null left derivative on $]v_0, 1]$. If $v \in]v_0, 1]$, we know that v is a profit maximiser at v: $v \in \arg\max_x f_v(x)$:. Using the first order condition for the lower left derivative $\underline{D}^-$ recalled above, at v, $\underline{D}^- f_v(v) \geq 0$. Since $\underline{D}^- f_v(v) = -\overline{D}^- C(v)$, we obtain $\overline{D}^- C(v) \leq 0$. Yet, C is increasing, so the lower and upper left derivatives must be positive : $0 \leq \underline{D}^- C(v) \leq \overline{D}^- C(v)$. Thus:

$$\underline{D}^- C(v) = \overline{D}^- C(v) = 0.$$

This means that the cost function C admits a left derivative at any $v \in]v_0, 1]$, and the value of this left derivative is zero.

Thus C is continuous and has a null left derivative on $]v_0, 1]$. Using Lemma 9, we obtain that C is constant on $]v_0, 1]$. Let us denote by K the value of C on this interval.

The IT does not trade for $v \in [0, v_0)$. In that case, since we know that $0 \leq X(v) \leq v$, we must have

$$\forall\, x \in [0, v], \quad x\left(v - \frac{x}{2}\right) \leq C(x).$$

By continuity of the left-hand term and the fact that the right-hand term is non-decreasing, we obtain

$$\forall\, x \in [0, v_0], \quad x\left(v_0 - \frac{x}{2}\right) \leq C(x).$$

There must be equality for $x = v_0$, because otherwise it would not be optimal to select $X(v) = v$ on the right neighborhood of v_0. For the same reason, C can not jump at v_0. This implies that $v_0\left(v_0 - \frac{v_0}{2}\right) = K$, or $v_0 = \sqrt{2K}$ and therefore C must belong to $\mathcal{O}$.

Assume conversely that $C \in \mathcal{O}$. Then for $0 \leq v < v_0$, the insider trader will make negative expected profits if she trades, so that $X(v) = 0$. For $v > v_0$, there are two cases to consider. (i) The IT plays $x \geq v_0$. In that case, K appears as a sunk cost and the best choice is $x = v$, leading to a net profit of $\frac{v^2}{2} - K$. (ii) The IT

plays $x \in [0, v)$. The net profit is then

$$\begin{aligned} x\left(v-\frac{x}{2}\right)-C(x) &= x\left(v_0-\frac{x}{2}\right)-C(x)+x(v-v_0) \\ &\leq x(v-v_0) \\ &\leq v_0(v-v_0) \end{aligned}$$

where the second line uses the fact that $C \in \mathcal{O}$. Since

$$\begin{aligned} \frac{v^2}{2}-K &= \frac{v^2}{2}-\frac{v_0^2}{2} \\ &= \frac{1}{2}(v+v_0)(v-v_0) \\ &> v_0(v-v_0), \end{aligned}$$

choice (i) is always preferred. Hence, if $C \in \mathcal{O}$, $X(v) = 0$ for $|v| < v_0$ and $X(v) = v$ for $|v| > v_0$, which concludes the proof.

A.4 Theorem 4

First, define for $0 \leq \alpha \leq 1 - \sqrt{2K}$:

$$X_\alpha(v) = \begin{cases} v & 0 \leq v \leq \alpha \\ \alpha & \alpha < v \leq \alpha + \sqrt{2K} \\ v & v > \alpha + \sqrt{2K} \\ -X_\alpha(-v) & v < 0. \end{cases}$$

We will see that the X_α for $0 \leq \alpha \leq 1 - \sqrt{2K}$ are exactly the demand schedules that implement the lower bound in Theorem 4. (Note that these demand schedules are implemented by the penalties C_α where $C_\alpha(x) = K\mathbb{I}_{|x|>\alpha}$.)

Step 1: transformation of the problem into a constrained problem of L^2 distance maximisation.

Recall Equation (20):

$$|G| = \frac{1}{6} - \frac{1}{2}\int_0^1 (v - X(v))^2 \,\mathrm{d}v.$$

This means that obtaining the bound of the Theorem is equivalent to showing

$$\max_{C \in \mathcal{C}_K} \int_0^1 (v - X(v))^2 \,\mathrm{d}v = \frac{(2K)^{3/2}}{3}, \tag{33}$$

subject to the constraint that $X(v)$ maximises the net profit $\psi_C(., v)$.

Let $g(v) = v - X(v)$, so that we are looking for an upper bound of $\int_0^1 g^2$. By Lemma 5 and under the constraint $C \leq K$, we obtain:

$$\begin{aligned}\int_0^1 g &= \int_0^1 v\,\mathrm{d}v - \int_0^1 X(v)\,\mathrm{d}v \\ &= \frac{1}{2} - \pi^N(1) \\ &\leq K. \end{aligned} \tag{34}$$

This is because, when $v = 1$, the IT can achieve at least a net profit of $\frac{1}{2} - C(1) \geq \frac{1}{2} - K$. Therefore, the maximum in (33) is less or equal to

$$\sup \int_0^1 g^2$$

subject to the constraints (i) $\int_0^1 g \leq K$, and (ii) $g(0) = 0 \leq g(v)$ and $v \mapsto v - g(v)$ is non-decreasing. (i) comes from (34), and (ii) is an immediate consequence of the properties of an optimal demand schedule X.

Notice how crucial Lemma 5 is, and therefore how effective the result of Milgrom and Segal (2002) is. Once noted that $C(1) \leq K$ implies a lower bound on the net profit at 1, Lemma 5 allows (i) to incorporate the constraint the X is a maximiser in a parsimonious way, (ii) to reduce the two constraints–$C \leq K$ and X must maximise ψ_C–into a single condition, $\int g \leq K$, which is particularly convenient, as it is a L^1 bound in a L^2 maximisation problem.

Absent the fact that X must be non-decreasing, which translates into the fact that $v \mapsto v - g(v)$ is non-decreasing, the maximisation of $\int g^2$ subject to $\int g = K$ (and $0 \leq g(v) \leq v$) would be standard: to "spread mass as unenvenly as possible", one would pick $g(v) = v\mathbb{I}_{v \geq v^*}$ with $\int_{v^*}^1 v\,\mathrm{d}v = K$. This is not feasible, however, because it violates the monotonicity constraint. The $g_\alpha : v \mapsto v - X_\alpha(v)$ are then natural candidate maximisers, as they are constructed in a similar spirit of variance maximisation, but respect the monotonicity constraint.

The g_α all have the same L^2 norm, but are away from zero over different intervals. This hints at the fact that for a general function g, when trying to find a bound on $\int g^2$, we will have no way to know where g must be small or large, and therefore little grip on g. The idea is then to consider the repartition function φ of g, because (i) one can reconstruct the moments of g with those of φ (see Step 3) and (ii) it does not matter *where* g is large, only how often it is large. In fact, all the g_α have the same repartition function, which suggests that this is the correct perspective to adopt.

For any function f and $x \neq y$, let

$$\tau_{x,y} f = \frac{f(y) - f(x)}{y - x}.$$

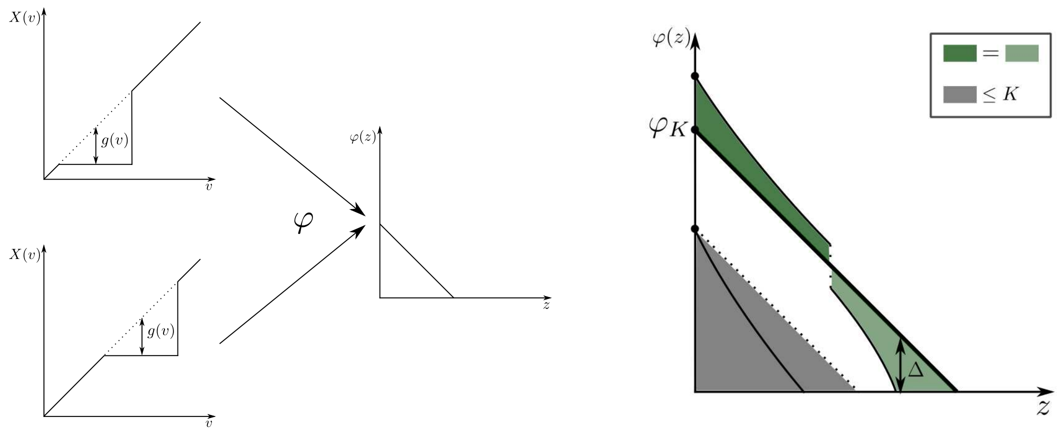
Since X is non-decreasing, we have

$$\tau_{x,y} g \leq 1 \tag{35}$$

for all $x \neq y$. Now, define

$$\varphi(z) = \mu\left(\{x,\, g(x) \geq z\}\right).$$

Figure 11a provides a graphical representation.



(a) Using the repartition function to transform g

(b) The transform φ of a maximiser g must be φ_K.

Figure 11: Illustrating some steps of the proof

Step 2: (35) implies

$$\tau_{x,y}\varphi \leq -1 \tag{36}$$

for all $x < y$ such that $\varphi(y) > 0$.

g is subject to a monotonicity constraint (namely $v \mapsto v - g(v)$ must be non-decreasing), which we need to transform into a constraint for φ. Clearly, if g increases at speed 1, φ decreases at speed 1. What we show here is that if g increases at speed less than 1 then φ decreases at speed larger than 1.

Since $\varphi(y) > 0$, the set $\{u,\, g(u) \geq y\}$ is nonempty, so we can consider

$$u^+ = \inf\{u,\, g(u) \geq y\}.$$

Since $g(0) = 0 \leq x$ we can also define

$$u^- = \sup\{u \leq u^+,\, g(u) \leq x\}.$$

Because of (35), the function g can not jump upwards, hence $g(u^-) = x$ and $g(u^+) = y$. By construction of u^- and u^+, we have:

$$[u^-, u^+) \subset \{u,\, g(u) \in [x, y)\}. \tag{37}$$

Since $\tau_{u^-,u^+} g \leq 1$, we have:

$$u^+ - u^- \geq g(u^+) - g(u^-) = y - x, \tag{38}$$

We can now obtain (36):

$$\begin{aligned}
\tau_{x,y}\varphi &= \frac{\mu\left(\{u,\, g(u) \geq y\}\right) - \mu\left(\{u,\, g(u) \geq x\}\right)}{y-x} \\
&= -\frac{\mu\left(\{u,\, g(u) \in [x,y)\}\right)}{y-x} \\
&\leq -\frac{\mu\left([u^-, u^+)\right)}{y-x} \\
&\leq -1.
\end{aligned}$$

Line 3 uses (37) and Line 4 is a consequence of (38).

Step 3: expression of the moments of g as a function of the moments of φ.

Recall that

$$\begin{aligned}
\int_0^1 g &= \int_0^1 \varphi \\
\int_0^1 g^2 &= 2\int_0^1 y\varphi(y)\,\mathrm{d}y.
\end{aligned} \tag{39}$$

Indeed,

$$\begin{aligned}
\int_0^1 g^2(y)\,\mathrm{d}y &= \int_0^1\int_0^1 \mathbb{I}_{0\leq s\leq g^2(y)}\,\mathrm{d}s\,\mathrm{d}y \\
&= \int_0^1 \mu\left(\left\{u,\, g^2(u)\geq s\right\}\right)\,\mathrm{d}s \\
&= \int_0^1 \mu\left(\left\{u,\, g(u)\geq \sqrt{s}\right\}\right)\,\mathrm{d}s \\
&= 2\int_0^1 y\varphi(y)\,\mathrm{d}y,
\end{aligned}$$

by using the change of variable $y=\sqrt{s}$. The other equality in (39) is proven similarly.

Step 4: translation into a functional maximisation problem with respect to the transform φ.

Using the previous discussion,

$$\begin{aligned}
\sup_{C\in\mathcal{C}_K}\int_0^1 (v-X(v))^2\,\mathrm{d}v &\leq 2\sup_{\varphi\in\Phi_K^{\leq}}\int_0^1 y\varphi(y)\,\mathrm{d}y \\
&\leq 2\sup_{\varphi\in\Phi_K}\int_0^1 y\varphi(y)\,\mathrm{d}y
\end{aligned} \tag{40}$$

where $\Phi_K^{\leq}$ is the set of measurable functions $\left\{\varphi:[0,1]\to[0,1],\, \sup_{x,y}\tau_{x,y}\varphi\leq -1,\, \int_0^1\varphi(y)\,\mathrm{d}y\leq K\right\}$ and $\Phi_K = \{\varphi\in\Phi_K^{\leq}, \int_0^1\varphi = K\}$. Clearly, in (40) the right-hand-side of Line 1 equals the term in Line 2.

Define $\varphi_K(z) = \max\left\{\sqrt{2K} - z, 0\right\}$ for $0 \le z \le 1$. Note that $\varphi_K \in \Phi_K$. If $\varphi \in \Phi_K$, $\varphi(0) \ge \varphi_K(0)$. Otherwise, using the fact that $\tau_{0,y}\varphi \le -1$,

$$\varphi(y) \le \varphi(0) - y < \varphi_K(0) - y \le \varphi_K(y).$$

Hence, $\int_0^1 \varphi(y)\,\mathrm{d}y$ would be strictly less than $K = \int_0^1 \phi_K(y)\,\mathrm{d}y$.
Define $\Delta = \varphi - \varphi_K$: we proved that $\Delta(0) > 0$. Besides, by construction, $\int_0^1 \Delta(y)\,\mathrm{d}y = 0$. Define

$$y_0 = \inf\left\{y,\, \Delta(y) \le 0\right\}.$$

Because $\tau_{y_0,y}\varphi \le -1$, we have $\Delta(y) \le 0$ for $y > y_0$ and $\Delta(y) \ge 0$ for $y < y_0$. Hence:

$$\begin{aligned}
\int_0^1 y\varphi(y)\,\mathrm{d}y - \int_0^1 y\varphi_K(y)\,\mathrm{d}y &= \int_0^1 y\Delta(y)\,\mathrm{d}y \\
&= \int_0^{y_0} y\Delta(y)\,\mathrm{d}y + \int_{y_0}^1 y\Delta(y)\,\mathrm{d}y \\
&\le y_0 \int_0^{y_0} \Delta(y)\,\mathrm{d}y + y_0 \int_{y_0}^1 \Delta(y)\,\mathrm{d}y \le 0.
\end{aligned}$$

Thus, the supremum in (40) is attained only by the function φ_K and equal to

$$\begin{aligned}
2\int_0^1 y\varphi_K(y)\,\mathrm{d}y &= \int_0^{\sqrt{2K}} y(\sqrt{2K} - y)\,\mathrm{d}y \\
&= \frac{(2K)^{3/2}}{3},
\end{aligned}$$

which establishes the bound of the Theorem.

Figure 11b provides some intuition: (i) starting from a point $\varphi(0) < \varphi_K(0)$ (lowest thick dot on the y-axis), φ (solid black curved line) remains below the dotted line and its integral is therefore smaller than the area of the grey region, itself below K. (ii) After crossing φ_K, φ must remain below φ_K. Here, the crossing occurs through a downwards jump of φ.
Step 5: The maximum in (33) is attained exclusively by the demand schedules $(X_\alpha)_{\alpha\in[0,1-\sqrt{2K}]}$ defined in the Theorem.

First, it is easy to see that these demand schedules achieve the maximum in (33). It remains to show that they are the only one to do so. Let X be a demand schedule obtained under a penalty $C \in \mathcal{C}$, $C \le K$. Let us suppose that it achieves the maximum in (33). Consider, as in step 2, the function φ associated with $g(v) = v - X(v)$. The function φ is then a supremum of (40) and by step 3, $\varphi = \varphi_K$. Since

$$\sup_x g(x) \ge \sup\{x,\, \varphi(x) > 0\} = \sup\{x,\, \varphi_K(x) > 0\} = \sqrt{2K},$$

the supremum of $g(v)$ is at least $\sqrt{2K}$. Let us remark that:

$$\sup_v g(v) = \sup_v \sup_{s\in[0,v]} g(s).$$

Since $\tau_{\cdot,\cdot,\cdot}g \leq 1$, the function $\overline{g}(v) = \sup_{s\in[0,v]} g(s)$ is continuous: the supremum of $\overline{g}(v)$ and thus of $g(v)$ is attained at a point v_0. Since $\tau_{\cdot,\cdot,\cdot}g \leq 1$, $v_0 \geq \sqrt{2K}$ and for $v \in [v_0 - \sqrt{2K}, v_0]$, $g(v) \geq v - v_0 + \sqrt{2K}$. Since $g \geq 0$, we obtain

$$\begin{aligned} \int_0^1 g &\geq \int_{v_0-\sqrt{2K}}^{v_0} g \\ &\geq \int_{v_0-\sqrt{2K}}^{v_0} (v - v_0 + \sqrt{2K})\,\mathrm{d}v \\ &\geq K \end{aligned}$$

with equality if and only if $g = 0$ outside $[v_0 - \sqrt{2K}, v_0]$ and $g(v) = v - v_0 + \sqrt{2K}$ over $[v_0 - \sqrt{2K}, v_0]$. But there must be equality because $g \in \Phi_K$. Hence g has the above form, and the demand function X, given by $X(v) = v - g(v)$, is equal to X_α as stated in the Theorem, with $\alpha = v_0 - \sqrt{2K}$.

A.5 Theorem 5

As a consequence of Lemma 5, in equilibrium the expected fine satisfies

$$\mathbb{E}[C(X(v))] = \int_0^1 X(v)\left(v - \frac{X(v)}{2}\right)\mathrm{d}v - \int_0^1 (1-v)X(v)\,\mathrm{d}v,$$

and we are working under a constraint $\mathbb{E}[C(X(v))] \geq K_1$.

By Lemma 6, an upper bound constraint on the expected post-trade standard deviation translates into a constraint

$$\int_0^1 vX(v)dv \geq K_2.$$

This leads us to consider the following minimisation problem:

$$\begin{aligned} \min_X \quad & \int_0^1 X(v)\left(v - \frac{X(v)}{2}\right)\mathrm{d}v + \gamma\left(K_1 - \int_0^1 X(v)\left(v - \frac{X(v)}{2}\right)\mathrm{d}v + \int_0^1 (1-v)X(v)\,\mathrm{d}v\right) \\ + \quad & \eta\left(K_2 - \int_0^1 vX(v)\,\mathrm{d}v\right), \end{aligned}$$

for some weights $\gamma, \eta \geq 0$. Gathering terms, we obtain that this program is equivalent to

$$\min_X \int_0^1 X(v)\left(\gamma + (1-2\gamma-\eta)v + \frac{\gamma-1}{2}X(v)\right)\mathrm{d}v \tag{41}$$

For $0 \leq v \leq 1$, define

$$\begin{aligned} P_v : [0,v] &\to \mathbb{R} \\ x &\mapsto x\left(\gamma + (1-2\gamma-\eta)v + \frac{\gamma-1}{2}x\right) \end{aligned}$$

Case 1: $\gamma > 1$. P_v is the restriction to $[0,v]$ of a second-order polynomial with positive leading coefficient. Therefore it reaches its minimum at either 0, v, or when the first order condition is satisfied, say at $x_0(v)$,

and $x_0(v)$ achieves the minimum as soon as $0 \leq x_0(v) \leq v$. Given that

$$x_0(v) = \frac{(2\gamma + \eta - 1)v - \gamma}{\gamma - 1},$$

algebra shows that

$$\arg\max P_v = \begin{cases} 0 & v \leq \frac{\gamma}{2\gamma+\eta-1} \\ x_0(v) & \frac{\gamma}{2\gamma+\eta-1} \leq v \leq \frac{\gamma}{\gamma+\eta} \\ v & v > \frac{\gamma}{\gamma+\eta}. \end{cases}$$

Let $v_1 = \frac{\gamma}{2\gamma+\eta-1}$ and $v_2 = \frac{\gamma}{\gamma+\eta}$. We have obtained that with the function X_{v_1,v_2} given in the Theorem, the equality

$$\arg\max P_v = X_{v_1,v_2}(v)$$

holds. Direct calculations show that X_{v_1,v_2} is implemented by C_{v_1,v_2}. This means that we have found an implementable demand schedule that maximises the integral in (41) pointwise, which implies that X_{v_1,v_2} is a minimiser of the program (41), and it is the only one because the pointwise minimisation of the integral in (41) has a unique solution.

Case 2: $\gamma \leq 1$. P_v is now either linear or with a negative leading coefficient, meaning that its minimum is attained either at 0 or v. Algebra shows that $\arg\max P_v = v$ (for $0 \leq v \leq 1$) if and only if

$$\gamma + 2\eta \geq 1 \tag{42}$$

and

$$v \geq v^* := \frac{\gamma}{\eta + \frac{3\gamma}{2} - \frac{1}{2}},$$

where, by condition (42), $v^* \in [0,1]$. With $v_1 = v_2 = v^*$ we conclude as before that X_{v_1,v_2} is the unique minimiser of (41). Finally, if (42) is not satisfied, the minimiser of (41) is identically zero, which corresponds to $X_{1,1}$ defined in the Theorem.

Finally, it is easy to see that the (v_1, v_2) constructed above describe the set J as $\gamma, \eta \geq 0$ vary, and J is the family of indices specified in the Theorem. So any index in J corresponds to an efficient demand function. This shows that $(X_{v_1,v_2})_{(v_1,v_2)\in J}$ is the family of efficient demand functions.

B Robustness checks: the case of Gaussian noise

B.1 Shape of X and P under Gaussian noise

Which effects of Section 3.4 are peculiar to uniform noise and which effects are robust when we revert to a normality assumption?

The qualitative behaviour of the demand function X does not depend on the distribution of the noise. Consider for instance a cost $C(x) = K\mathbb{I}_{|x|>x_0}$ with $K, x_0 > 0$. When the magnitude of the optimal demand absent penalties is below x_0, it remains optimal under the penalty C. The IT then blocks its demand at x_0 in order to avoid the expected penalty K, as long as trading does not allow to recoup K on average. For

v sufficiently large ($|v| > v_0$ for some $v_0 > 0$), the IT switches back to trading. This creates a jump in the demand function at $\pm v_0$. All these effects are independent of the assumptions on the noise.

The qualitative behaviour of the price function is robust as far as non-linearity is concerned. Flat sections in the demand schedule X induce steep sections in the price function P. Indeed, when X increases slowly as a function of v, the information that X is likely to have increased a little (obtained through the observation of $d = X(v) + u$) implies that v is likely to have increased a lot. Similarly, steep sections of X induce flat sections of P. Since the introduction of penalties produces steep and flat sections for X, it produces flat and steep sections for P.

What does not hold in general is the fact that P has discontinuities. Those are due to the fact that the uniform distribution has a discontinuous density $\frac{1}{2}\mathbb{I}_{[-1,1]}$. In general, one must have discontinuities in the density of the noise to obtain discontinuities in the price function. With a continuous noise density, jumps are replaced with sections where P increases fast.

To support these arguments, we report the equilibrium (X, P) for the model with Gaussian noise ($u, v \sim N(0,1)$) and penalty C. We consider the same penalties C as above: quadratic, linear and constant on large trades.

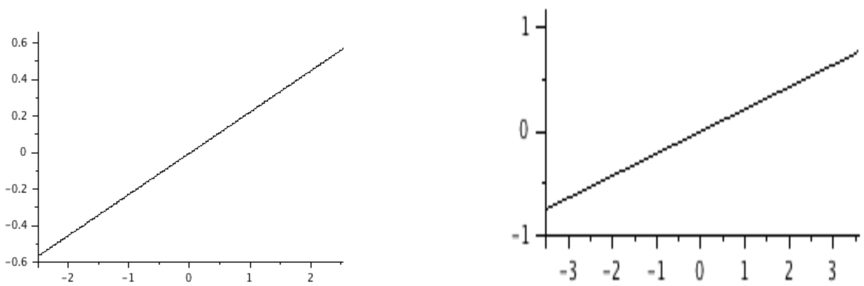


Figure 12: IT demand and pricing under quadratic penalty, Gaussian case
$C(x) = \alpha x^2$, $\alpha = 2$. Left panel: IT demand X. Right panel: price function P.

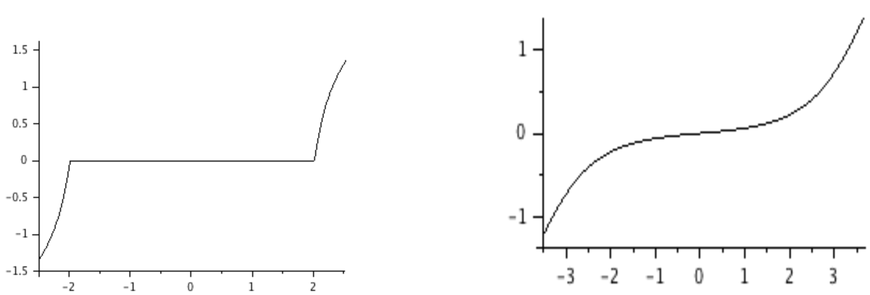


Figure 13: IT demand and pricing under linear penalty, Gaussian case
$C(x) = \alpha|x|$, $\alpha = 2$. Left panel: IT demand X. Right panel: price function P.

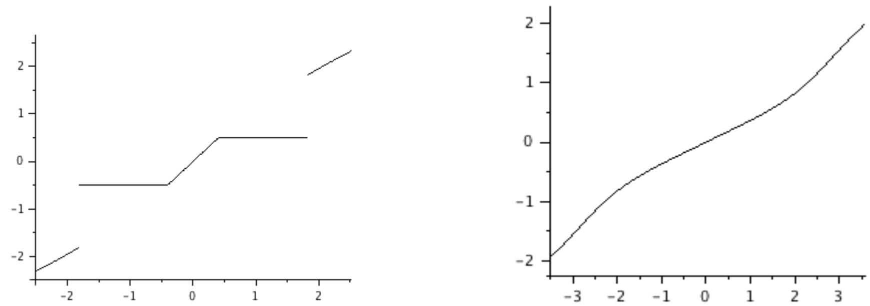


Figure 14: IT demand and pricing under constant penalty, Gaussian case
$C(x) = K\mathbb{I}_{|x|>x_0}$, $K = 1$, $x_0 = 0.5$.
Left panel: IT demand X. Right panel: price function P.

B.2 Figure 7 under Gaussian noise

We repeat the construction of Figure 7 by assuming Gaussian noise: $u, v \sim N(0; 1)$. We obtain Figure 15. The constant costs upon nonzero trades $C(x) = K\mathbb{I}_{x\neq 0}$ are doing best among the penalty functions considered. This is consistent with the results in the uniform noise case. Other penalties are suboptimal, as before, and the locus of points $(S, -G)$ they generate is very similar in shape.

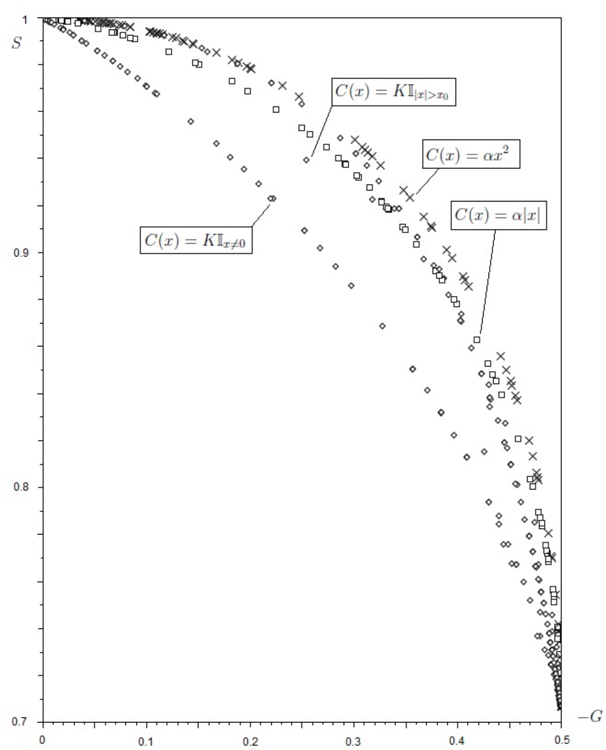


Figure 15: Locus of $(S, -G)$ for different penalty functions - Gaussian noise.