Heterogeneous Beliefs, Bonds, and Interest Rates

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Abstract

We develop a general equilibrium model of interest rates in a continuous-time production economy populated by shareholders with heterogeneous beliefs. It allows us to study the impact of belief heterogeneity on the risk-free rate and bond characteristics. The model nests the CIR model with time-dependent parameters that depend endogenously on belief heterogeneity. Flight-to-safety induces an increase in the bond price when the average belief declines. In addition, the impact of belief dispersion increases with the bond maturity, and higher dispersion results in higher bond prices in economies where pessimistic and neutral shareholders hold most of the wealth. In optimistic economies, the relation reverses except for the case of highly dispersed beliefs and (very) long-term bonds. Lastly, heterogeneous beliefs increase bond yield volatility, helping to solve the excess bond yield volatility puzzle.

Keywords: Asset pricing, fixed income, heterogeneous beliefs, production economy

JEL classification: D51, D53, E43, G11, G12

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1 Introduction

Evidence shows that market participants have heterogeneous beliefs about future economic outcomes. Buraschi et al. (2022), for example, document a large, highly persistent, disagreement among investors about future government bond returns. These heterogeneous beliefs are reflected in portfolio allocations and trading behaviors (Giglio et al., 2021), thereby greatly impacting financial markets. Recent asset pricing models thus incorporate this important feature. It can, for instance, explain why investors trade so much (Banerjee and Kremer, 2010) or help to solve the excess stock volatility puzzle (Atmaz and Basak, 2018). However, most of these models consider pure exchange economies in which the production process is exogenously given and cannot be impacted by belief heterogeneity.[1] This setting can potentially undermine important effects as heterogeneous beliefs should also impact investment decisions and, consequently, the production process. In this paper, we therefore consider a production economy framework to provide a theoretical study of the full impact of heterogeneous beliefs on financial markets with a focus on bonds and interest rates.

More precisely, we consider a general equilibrium model based on a continuous-time production economy with a single firm owned by a large number of heterogeneous shareholders. These shareholders agree to disagree on the risk-return production function of the firm: optimistic shareholders overestimate the efficiency of the investment into capital transformation, and thus, for a given level of risk, they believe in higher expected returns than pessimistic shareholders. As they need to collectively determine the firm's optimal strategy, shareholders' heterogeneous beliefs directly impact the production dynamics. Specifically, in good times, when shareholders are more optimistic on average, they choose a riskier strategy in order to increase the firm's returns, leading to an activity expansion. Conversely, more pessimism leads to an activity contraction. In addition, shareholders have access to a risky stock and a risk-free asset, which can be associated with a stock market index and gov-

[1] Buss et al. (2016), Heyerdahl-Larsen and Walden (2022), and Jouini (forthcoming) are among the few exceptions.

ernment bonds, respectively. Because of their heterogeneous beliefs, their trading behaviors differ, and they individually hold different positions, implying additional belief heterogeneity impacts on market characteristics.

Our first set of results relates to the risk-free rate and the market price of risk. We find that the former increases with belief dispersion in good times, decreases with it in bad times, and is pro-cyclical. Conversely, the market price of risk decreases with belief dispersion in good times, increases with it in bad times, and is counter-cyclical. Therefore, our study shows that these effects, documented in an exchange economy framework in, e.g., Beddock and Jouini (2021), remain in a production economy. An additional appealing feature of our model is that the risk-free rate dynamics nests the CIR (Cox et al., 1985, Hull and White, 1990) model with time-dependent parameters as a special case. This special case is obtained by considering a specific risk-return production function, and the associated time-dependent parameters directly follow from the wealth-weighted average belief and belief dispersion instead of being exogenously given.

In addition, our production economy setting leads to stochastic bond prices. Specifically, we obtain that they decrease with the average belief for all maturities. A higher average belief is indeed equivalent to more optimism on average. Shareholders are thereby more willing, on average, to invest in stocks, which decreases the demand for bonds, leading to a decline in price. We also find that the impact of belief dispersion grows with the bond maturity. This is due to the fact that the bond price decreases with maturity and is thus pushed further away from its final value for long-term bonds, leaving more room for heterogeneous belief impacts. Moreover, higher belief dispersion enhances the price of both short-term and long-term bonds in economies where pessimistic and neutral shareholders hold most of the wealth. The underlying mechanism is related to flight-to-safety. In these situations, an increase in belief dispersion is in fact associated with more risk and encourages the shareholders to increase their demand for bonds. For optimistic economies, however, the sign of the relation is mostly negative and depends on the bond maturity and the level of belief dispersion. This is because, in such economies, keeping the share of optimists constant, an increase in belief dispersion implies an increase in the average belief, yielding a negative effect on the bond price. This effect reduces with the bond maturity as the share of optimists exhibits mean reversion, and it can eventually reverse for very long maturities or high levels of belief dispersion. The average belief and belief dispersion also affect the bond instantaneous expected return, and their impacts vary depending on whether optimistic, pessimistic, or neutral shareholders hold most of the wealth. These impacts directly follow from the implications on bond prices we just described. Looking at bond volatility, we further find that it increases with maturity and the average belief because of the widening gap between the bond's current and final values. Furthermore, except when the bond price is already close to its final value, the higher the dispersion, the more there are price fluctuations, yielding higher volatility.

Lastly, we study how belief heterogeneity affects bond yields and their dynamics. As for the risk-free rate, the yield curve level increases with belief dispersion in good times, decreases with it in bad times, and is pro-cyclical. Another important result is that, besides in pessimistic economies where shareholders have very heterogeneous beliefs, the bond yield volatility increases with belief dispersion. It is also always higher than in an homogeneous economy. This adds evidence that belief dispersion provides a rationale for the excessive bond yield volatility puzzle.

Regarding methodology, it is worth noticing that we consider a continuum of shareholders to model belief heterogeneity. Conveniently, it is a way of dealing with all potential beliefs. More importantly, it allows us to describe belief heterogeneity and its impacts in a parsimonious way using only two parameters, namely the wealth-weighted average belief and belief dispersion. Another implication of such a setting is that no shareholder dominates the economy in extreme states. This is similar to Atmaz and Basak (2018) and Jouini (forthcoming), who are the closest to our work. However, they both develop models where consumption only occurs at the end of a finite horizon economy. Conversely, our model features continuous consumption, which is crucial for studying bonds and interest rates. Additionally, like theirs, our model leads to closed-form expressions for all studied quantities. To the best of our knowledge, we thereby provide the first fully calculable general equilibrium model with production, belief heterogeneity, and continuous consumption.

Related literature

Our study echoes and complements the growing literature on the impact of belief heterogeneity on asset prices. While, e.g., Cvitanic et al. (2012) and Bhamra and Uppal (2014) look at stocks and bonds, most of this literature focuses only on stock characteristics (see, e.g., Miller, 1977, Scheinkman and Xiong, 2003, Jouini and Napp, 2007, Atmaz and Basak, 2018, Andrei et al., 2019, Jouini, forthcoming). Regarding bonds and interest rates, the paper is related to Xiong and Yan (2010), who develop a model where two groups of investors hold heterogeneous expectations about future interest rates. In particular, they show that belief heterogeneity can help to explain the excessive volatility of bond yields. Schneider (2022) proposes a continuous time model with investors with recursive preferences having different elasticity of intertemporal substitution and risk aversion and explains key features of the dynamics of nominal and real yields. Other papers of particular interest include Wang (1996), who studies the impact of preference heterogeneity on the term structure of interest rates, Jouini et al. (2010), who determine the equilibrium discount rate when heterogeneity in time preference rates is taken into account, and Vayanos and Vila (2021), who consider the interaction between investors with preferences for specific maturities and risk-averse arbitrageurs and analyze its impact on the yield curve.

Organization of the paper

Section 2 and Section 3 introduce and solve the theoretical model. We study the risk-free rate and the market price of risk in Section 4 and the bond characteristics in Section 5. We conclude in Section 6. Appendix A reports all proofs, and Appendix B shows an additional result on the yield curve.

2 Model

Let us consider a continuous-time model with a single firm owned by a large number of heterogeneous shareholders. It is an infinite horizon production economy in which there is a single source of risk, represented by a Brownian motion $(W_t)_{t\in\mathbb{R}^+}$, and we denote the associated Brownian filtration by $\left(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P}\right)$.

The firm—representative of the overall economy—produces an output y_t^θ that is partly consumed by the shareholders and partly reinvested in it. To model production, we follow Jouini (forthcoming) and develop a stochastic AK model with adjustment costs. More precisely, the firm faces two types of technology to produce consumption: one productive, risky, technology and one safe, less risky, technology. Both technologies follow a constant AK model with the same productivity parameter A that we set to one without loss of generality. The date t to $t+dt$ total production thus equals the sum of stock of capital in each technology and is given by $y_t^\theta dt$. We assume that a fixed share of the production is consumed by the shareholders at each date and the remaining one is reinvested in the production, with a proportion $\hat{\theta}_t$ being reinvested in the productive technology. The higher this proportion, the riskier the overall strategy, and the higher the firm's return. Formally, as shown in Jouini (forthcoming), the production dynamics is of the form

$$
\begin{aligned}
dy_t^\theta &= m(\theta_t)\, y_t^\theta dt + \theta_t y_t^\theta dW_t, \qquad (2.1)\\
y_0^\theta &= 1,
\end{aligned}
$$

where θ_t is a scaled version of $\hat{\theta}_t$ and represents the date-t macroeconomic volatility, and $m(\theta) = a + b\theta - c\theta^2$ is the risk-return production function with parameters a, b, and c whose values can be easily derived by identification. Note that this function is concave, meaning that it is increasingly costly in terms of additional risk to be borne to increase the returns.[2] In other words, there are decreasing returns to scale. At each date and in each state of the

[2] We refer to Jouini (forthcoming) for a more detailed explanation of the construction and analysis of the risk-return production function m. In particular, b and c are strictly positive.

world, the shareholders of the firm thus have to choose the optimal ratio $\hat{\theta}_t$ to invest in the productive technology to obtain a given risk-return combination for which the growth rate is a quadratic function of the level of exposure to economic shocks.[3] Because θ_t is a scaled version of this ratio, it is a control process for the firm, and we denote by Θ its set.

A continuum of heterogeneous shareholders own the firm and are indexed by their type δ. They all agree on the firm objective function described above, but they disagree on the risk-return production function m. In fact, at time t, Shareholder-δ believes that the risk-return production function is given by $m(\theta_t) + \delta\theta_t$. Formally, she is endowed with a subjective probability measure whose density with respect to the objective probability $\mathbb{P}$ is given by

$$M_t^\delta = \exp\left(-\frac{\delta^2}{2}t + \delta W_t\right), \tag{2.2}$$

where M_t^δ is considered as the time-t Shareholder-δ's belief and satisfies $dM_t^\delta = \delta M_t^\delta dW_t$. The shareholders thus have different beliefs about the distribution of the production and its dynamics. Importantly, we let δ take all possible real values to account for all possible beliefs. Because all shareholders having the same belief behave similarly, we can aggregate them and assume that there is only one shareholder of each type. For $\delta > 0$ (resp. $\delta < 0$, $\delta = 0$), we say that Shareholder-δ is optimistic (resp. pessimistic, rational), as she overestimates (resp. underestimates, correctly estimates) the efficiency of the investment into capital transformation. Following the empirical observations of Ma et al. (2020) and Buraschi et al. (2022), we further assume that the beliefs are persistent over time at the shareholders' level. In other words, the shareholders agree to disagree and do not learn.

Each shareholder is endowed with a given number of shares of the firm that depends on her belief, and the total number of shares is normalized to one without loss of generality. Specifically, we assume that the initial belief distribution, ν_δ, i.e., the distribution of the initial number of shares over the space of shareholders' types, which equivalently corresponds to

[3] Assuming that the shareholders are also in charge of choosing the production strategy of the firm is in line with Lewellen and Lewellen (2022) who, focusing on institutional investors, show that they have strong incentives to be engaged shareholders.

the initial wealth distribution over the shareholder space, is normal with parameters δ_0 and ω_0. It means that, at time 0, the wealth-weighted average shareholder is indexed by δ_0, which therefore represents the initial average belief. The parameter ω_0 represents the initial belief dispersion as it measures the standard deviation of the initial wealth-weighted belief distribution. In the specific case $\omega_0 = 0$, there is no belief dispersion and the economy is an homogeneous one where all shareholders are of type δ_0 and can be aggregated into a unique shareholder. We study this homogeneous case in Section 3.1.

The shareholders are assumed to be expected utility maximizers and extract utility from their stream of contingent allocation of capital. They all share a common time preference rate ρ and common logarithmic preferences. Finally, to deal with asset pricing issues, we assume that the shareholders can continuously trade in a risky stock and a risk-free asset.[4] These securities are described in more details in Section 5.

3 Equilibrium

In this production economy setting, the Arrow-Debreu equilibrium is reached when the shareholders collectively agree on the optimal strategy that maximizes the firm's value and individually determine their optimal consumption plan that maximizes their expected utility, given their belief and endowment. Moreover, the shareholders cannot consume more than what the firm produces. Formally, the equilibrium is characterized by an admissible strategy $\bar{\theta} \in \Theta$, a continuum of consumption plans $(\bar{c}_\delta)_{\delta \in \mathbb{R}}$, and a state price density $\bar{p}$, such that

$$y^{\bar{\theta}} = \underset{\theta \in \Theta}{\operatorname{argmax}}\ \mathbb{E}\left(\int_0^\infty \bar{p}_s y_s^\theta ds\right), \tag{3.1}$$

$$\bar{c}_\delta = \underset{\mathbb{E}\left(\int_0^\infty \bar{p}_s\left(c_{\delta,s} - \nu_{\delta,s} y_s^{\bar{\theta}}\right)ds\right) \leq 0}{\operatorname{argmax}}\ \mathbb{E}\left(\int_0^\infty \exp\left(-\rho t\right) M_t^\delta \ln\left(c_{\delta,t}\right) dt\right), \tag{3.2}$$

$$y^{\bar{\theta}} = \int \bar{c}_\delta d\delta. \tag{3.3}$$

[4] An Arrow-Debreu equilibrium can indeed be implemented by continuous trading of such long-lived securities (Duffie and Huang, 1985).

As shareholders have heterogeneous beliefs, an equilibrium wealth-weighted distribution of beliefs is also associated to the equilibrium. We can further characterize the equilibrium aggregate consumption share of the optimists.

3.1 Benchmark economies

Before looking at our main economy in Section 3.2, we focus on three benchmark economies. These benchmarks allow us to highlight the relevance of our general framework in order to study bonds and interest rates. First, we consider an economy populated by homogeneous shareholders, i.e., an economy in which $\omega_0 = 0$. Second, we analyze an economy with no decreasing returns to scale, i.e., an economy in which the risk-return production function is not concave. Formally, it is obtained by considering $c = 0$, which leads to a linear production function. Third, we examine an exchange economy, i.e., a production economy in which the production process is exogenously given. More precisely, we assume that it is given by a geometric Brownian motion with drift μ^* and volatility σ^* for all dates and states of the world. Proposition 1 gives the equilibrium strategy $\bar{\theta}_t$ prevailing in each of these economies. It also reports the equilibrium risk-free rate r_t^f, and the time-t price of a pure discount bond delivering one consumption unit at time h, denoted by $B_{t,h}$.

Proposition 1 (Benchmarks)**.** *In the benchmark economies, the equilibrium strategy, risk-free rate, and bond price are given respectively by*

1. *No belief dispersion economy:*

$$\bar{\theta}_t = \frac{b+\delta_0}{2c+1},\; r_t^f = \rho + a + c\left(\bar{\theta}_t\right)^2,\; B_{t,h} = exp\left(-\left(\rho + a + c\left(\bar{\theta}_t\right)^2\right)(h-t)\right).$$

2. *No decreasing returns to scale economy:*

$$\bar{\theta}_t = b + \frac{\delta_0 + \omega_0^2 W_t}{1 + t\omega_0^2},\; r_t^f = \rho + a,\; B_{t,h} = exp\left(-\left(\rho + a\right)(h-t)\right).$$

3. *Exchange economy:*

$$\bar{\theta}_t = \sigma^*,\; r_t^f = \rho + \mu^* + \delta_t\bar{\theta}_t - \left(\bar{\theta}_t\right)^2,\; B_{t,h} = exp\left(-\left(\rho + \mu^* + \delta_t\bar{\theta}_t - \frac{\omega_t^2 + \omega_h^2}{2\omega_h^2}\left(\bar{\theta}_t\right)^2\right)(h-t)\right),$$

$$\textit{with } \delta_t = \frac{\delta_0 + \omega_0^2 W_t}{1 + t\omega_0^2},\ \omega_t^2 = \frac{\omega_0^2}{1 + t\omega_0^2},\ \textit{and } \omega_h^2 = \frac{\omega_0^2}{1 + h\omega_0^2}.$$

We notice that in the no belief dispersion and exchange economies the shareholders agree on a constant strategy, independently of the state of the world. In addition, the risk-free rate is constant in the no belief dispersion and no decreasing returns to scale economies, but it is pro-cyclical in the exchange economy as in, e.g., Jouini and Napp (2011). In the two former benchmarks, the bond price is also non-stochastic and depends only on time to maturity. Thus, in these economies, the bond yield is constant and does not exhibit any volatility.

3.2 Main economy

We now turn to our general framework and consider a production economy with heterogeneous shareholders and decreasing returns to scale.

Proposition 2 (Model equilibrium and belief distribution)**.** *The equilibrium is characterized by*

$$\bar{\theta}_t = \frac{b + \delta_t}{2c + 1}, \tag{3.4}$$

$$\bar{c}_{\delta,t} = \exp\left(-\rho t\right) \lambda_\delta^{-1} M_t^\delta \bar{p}_t^{-1}, \tag{3.5}$$

$$\bar{p}_t = \sqrt{2\pi}\omega_t \exp\left(-\rho t + \frac{\delta_t^2}{2\omega_t^2}\right)\left(y_t^{\bar{\theta}}\right)^{-1}, \tag{3.6}$$

with

$$\lambda_\delta = \exp\left(\frac{\delta^2}{2\omega_0^2} - \frac{\delta_0 \delta}{\omega_0^2}\right). \tag{3.7}$$

The equilibrium wealth-weighted distribution of beliefs follows a normal distribution with pa-

rameters δ_t and ω_t, given respectively by

$$\delta_t = \frac{\delta_0 + \omega_0^2 W_t}{1 + t\omega_0^2}, \tag{3.8}$$

$$\omega_t = \frac{\omega_0}{\sqrt{1 + t\omega_0^2}}. \tag{3.9}$$

From (3.8), we see that the time-t wealth-weighted average belief, δ_t, is state-dependent. This is because, in good times, the beliefs of optimistic shareholders are realized. As their views are correct, optimists hold a higher fraction of the wealth, which induces a shift of the wealth-weighted distribution of beliefs toward optimism and thus increases the average belief. It is thereby equivalent to refer to an "economy in good times" or to an "optimistic economy." Additionally, the firm's optimal strategy, $\bar{\theta}_t$, is time- and state-dependent. More precisely, it depends positively on the average belief: when shareholders are more optimistic on average, they agree to adopt a riskier strategy in order to get higher production returns, leading to an activity expansion. In other words, shareholders collectively agree to reinvest a higher fraction of (unconsumed) production in the risky technology that yields higher returns.

3.3 Consumption share of the optimists

We can define equilibrium consumption shares from the equilibrium characteristics. Formally, we denote by $\tau_{\delta,t}$ the time-t consumption share of Shareholder-δ and by $\mathcal{T}_{\delta,t}$ the time-t aggregate consumption share of all shareholders who are less optimistic than Shareholder-δ. By definition, we have

$$\tau_{\delta,t} = \frac{\bar{c}_{\delta,t}}{y_t^{\bar{\theta}}}, \tag{3.10}$$

and

$$\mathcal{T}_{\delta,t} = \int_{-\infty}^{\delta} \tau_{\tilde{\delta},t} d\tilde{\delta}. \tag{3.11}$$

In particular, we can define the aggregate consumption share of the optimists, $\mathcal{T}_{opt,t} = 1 - \mathcal{T}_{0,t}$. In order to provide its explicit expression and dynamics, we introduce φ and Φ that are the probability density and cumulative distribution functions of the standard normal distribution, respectively.

Proposition 3 (Consumption share of the optimists)**.** *The equilibrium consumption share of the optimists is given by*

$$\mathcal{T}_{opt,t} = \Phi\left(\frac{\delta_t}{\omega_t}\right), \tag{3.12}$$

and follows

$$d\mathcal{T}_{opt,t} = -\omega_t^2 \varphi\left(\Phi^{-1}\left(\mathcal{T}_{opt,t}\right)\right)\Phi^{-1}\left(\mathcal{T}_{opt,t}\right)dt + \omega_t \varphi\left(\Phi^{-1}\left(\mathcal{T}_{opt,t}\right)\right)dW_t. \tag{3.13}$$

As the equilibrium consumption share of the optimists depends positively on the wealth-weighted average belief, it is pro-cyclical. Additionally, the drift in (3.13) is negative (resp. positive) when $\mathcal{T}_{opt,t} > 0.5$ (resp. $\mathcal{T}_{opt,t} < 0.5$). Thus, when the consumption share of all optimists is higher than the one of all pessimists, it tends to decrease, and vice versa, and the higher the belief dispersion, the stronger this effect. This observation complements Jouini and Napp (2011), who show in an exchange economy populated by two heterogeneous agents that the consumption share of each agent exhibits mean reversion. Importantly, it also implies that neither optimists nor pessimists persistently dominate the economy as time goes by.

4 Risk-free rate and market price of risk

Because it features continuous consumption, our model allows us to examine the instantaneous interest rate, i.e., the risk-free rate, and the market price of risk, and to study how the shareholders' heterogeneous beliefs affect them. Recall that the former is constant in homogeneous and no decreasing returns to scale economy settings.

Proposition 4 (Risk-free rate)**.** *The risk-free rate is given by*

$$r_t^f = \rho + a + x_t, \tag{4.1}$$

with $x_t = c\left(\bar{\theta}_t\right)^2$ *and*

$$dx_t = \kappa_t\left(\vartheta_t - x_t + \frac{b\sqrt{c}}{2c+1}\sqrt{x_t}\right)dt + \sigma_t^r\sqrt{x_t}dW_t, \tag{4.2}$$

with

$$\kappa_t = 2\omega_t^2, \tag{4.3}$$

$$\vartheta_t = \frac{c\omega_t^2}{2\left(2c+1\right)^2}, \tag{4.4}$$

$$\sigma_t^r = \frac{2\sqrt{c}\omega_t^2}{2c+1}. \tag{4.5}$$

A first important observation is that, if we make the assumption that $b = 0$ in the risk-return production function, the risk-free rate follows a shifted CIR model (CIR++, Brigo and Mercurio, 2016) with time-dependent parameters. If we further assume that $\rho + a = 0$, it satisfies a CIR model with time-dependent parameters (Cox et al., 1985, Hull and White, 1990). Our model thus nests these two popular models of interest rates as special cases. Interestingly, while in the CIR++ and CIR models the parameters are given exogenously, they endogenously derive from the heterogeneous belief distribution and the production function in ours.

To analyze the impact of belief heterogeneity, we provide graphs of the evolution of the risk-free rate in different economies. Specifically, we refer to the optimistic (resp. pessimistic, neutral) economy as the economy in which the time-t aggregate share of optimists equals 70% (resp. 30%, 50%). Similarly, we refer to the average (resp. high) belief dispersion economy as the economy in which the time-t belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%). The economy in which the belief dispersion is null is called the no belief dispersion economy. Within this framework, we study how the risk-free rate is affected in the optimistic, pessimistic, and neutral economies when the belief dispersion changes and how it is affected in the average, high, and no belief dispersion economies when the aggregate share of optimists changes. Note that the impact of the share of optimists determines the impact of the wealth-weighted average belief, as both quantities are positively related. Importantly, to allow for a relevant comparison across economies, we consider that their production process is the same, i.e., the values of $\bar{\theta}_t$ and $m\left(\bar{\theta}_t\right)$ are identical in each economy. It implies a comparison of economies in which firms have different risk-return production functions.[5] Unreported graphs obtained with a fixed production function, i.e., fixed values of b and c, but changing production processes lead to the same conclusions.

Insert Figure 1 here:

Evolution of the risk-free rate depending on belief dispersion and share of optimists.

In line with Beddock and Jouini (2021) in an exchange economy setting, the risk-free rate increases (resp. decreases) with belief dispersion in good (resp. bad) times (Panel A). In addition, it is pro-cyclical, as it is higher when the share of optimists—which is itself higher in good times—is higher (Panel B). Our model thus extends these results to a production economy framework.

Insert Figures 2 here:

Evolution of the diffusion of the risk-free rate depending on belief dispersion and share of optimists.

We further examine the evolution of the diffusion of the risk-free rate, $\sigma_t^{r^f} = \sigma_t^r \sqrt{x_t}$, depending on belief dispersion and share of optimists. First, we observe that it increases with belief

[5]This is because both the aggregate share of all optimists and the macroeconomic volatility depend on the wealth-weighted average belief, that depends itself on the belief dispersion and the state of the world. Formally, in order to keep the values of $\bar{\theta}_t$ and $m\left(\bar{\theta}_t\right)$ fixed, for a given ω_t (resp. $\mathcal{T}_{opt,t}$), the values of W_t, b, and c differ from an economy to another, and, for a given economy, they change as ω_t (resp. $\mathcal{T}_{opt,t}$) varies.

dispersion in the optimistic and neutral economies (Panel A) or, more generally, when the aggregate share of optimists is not too low (Panel B). However, the sign of the relation reverses in the pessimistic economy if the belief dispersion is high (Panel A) and in very pessimistic economies (Panel B). Second, for a given level of belief dispersion, the diffusion of the risk-free rate increases with the share of optimists (Panel B).

Proposition 5 (Market price of risk)**.** *The market price of risk is given by*

$$MPR_t = \bar{\theta}_t - \delta_t. \tag{4.6}$$

Insert Figure 3 here:

Evolution of the market price of risk depending on belief dispersion and share of optimists.

Using a similar methodology, we observe that the market price of risk decreases (resp. increases) with belief dispersion in good (resp. bad) times (Panel A), and is counter-cyclical (Panel B). We thereby further extends the results of Beddock and Jouini (2021) to a production economy.

5 Bonds

In order to deal with asset pricing issues, we assume that the shareholders of the firm can continuously trade in a risky stock, S, whose dividend process is given by the total production, and in a risk-free asset, S^0, whose dynamics is given by $dS_t^0 = r_t^f S_t^0 dt$.

Recall that given our logarithmic utility function setting and the fact that belief heterogeneity is the only type of heterogeneity we consider, our framework is not of particular interest to analyze the stock characteristics.[6] However, considering a model with continuous consumption allows us to relevantly study the price and characteristics of pure discount bonds

[6] More precisely, it is easy to show that our model leads to a time-t stock price that equals the production divided by the time preference rate, i.e., $S_t = y_t^{\bar{\theta}}/\rho$. Using Ito's lemma, we further obtain $dS_t = m\left(\bar{\theta}_t\right) S_t dt + \bar{\theta}_t S_t dW_t$, and observe that the time-t stock volatility is equal to the time-t macroeconomic volatility. The model is thus unable to generate excess volatility, as empirically observed. Conversely, Atmaz and Basak (2018) derive this feature in a model with constant relative risk aversion utility functions in an exchange

delivering one consumption unit at maturity which can be seen as government bonds. In this section, we thus examine bond prices and bond yields in our production framework.

5.1 Bond price

We let $B_{t,h}$ denote the time-t price of the pure discount bond delivering one consumption unit at time h.

Proposition 6 (Bond price)**.** *The bond price is given by*

$$B_{t,h}=\sqrt{\frac{2c+1}{2c+\frac{\omega_t^2}{\omega_h^2}}}\left(\frac{\omega_t}{\omega_h}\right)^{\frac{1}{2c+1}}\exp\left(-\left(\rho+a+\frac{2c+1}{2c+\frac{\omega_t^2}{\omega_h^2}}c\left(\bar{\theta}_t\right)^2\right)(h-t)\right). \tag{5.1}$$

The bond instantaneous expected return is given by

$$\mu_t^{B,h}=\frac{c}{2c+\frac{\omega_t^2}{\omega_h^2}}\left(\frac{\left(\omega_h^2-\omega_t^2\right)^2}{\omega_h^2\left(2c+1\right)}+\left(2\delta_t\bar{\theta}_t-\frac{\omega_t^2}{2c+1}+\frac{\left(\bar{\theta}_t\right)^2}{2c+\frac{\omega_t^2}{\omega_h^2}}\left(\left(2c+1\right)\left(1-\frac{\omega_t^2}{\omega_h^2}\right)+2\omega_t^2c\left(h-t\right)\right)\right)\omega_t^2\left(h-t\right)\right), \tag{5.2}$$

and the bond volatility is given by

$$\sigma_t^{B,h}=\frac{2\omega_t^2c\bar{\theta}_t\left(h-t\right)}{2c+\frac{\omega_t^2}{\omega_h^2}}. \tag{5.3}$$

In order to study the effects of belief heterogeneity on bond prices, we first look at how they evolve depending on maturity in each of the economies defined in Section 4. Considering a short-term ($h=1$) and a long-term ($h=30$) bond, we then examine how their price is impacted in the optimistic, pessimistic, and neutral (resp. average, high, and no belief dispersion) economies when the belief dispersion (resp. aggregate share of optimists) varies.

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economy setting. Considering correlated belief and time preference heterogeneities Beddock and Jouini (2021) derive excess stock volatility in a log setting.

Insert Figure 4 here:

Evolution of the bond price depending on maturity.

Insert Figure 5 here:

Evolution of the bond price depending on belief dispersion and share of optimists.

First, in Figure 4, we see that, in all economies, bond prices decrease with maturity. Second, from Panels B and C (in which belief dispersion is fixed), we observe that, for all maturities, they also decrease with the share of optimists—and thus with the average belief. The more optimistic shareholders are on average, the more inclined they are, on average, to take risks and therefore prefer to hold stocks rather than bonds. As some are more optimistic and others more pessimistic, trading can take place, and, on average, demand for bonds declines, which in turn leads to lower prices. Conversely, when there is no belief dispersion, i.e., everyone has the same beliefs, no trading occurs and thus the average belief has no impact on bond prices (Panel A). These results are confirmed both for short-term and long-term bonds when looking at the right column of Figure 5. Third, looking at the top-left graph of Figure 5, which focuses on the impact of belief dispersion on the price of the short-term bond while keeping the share of optimists constant for each curve, we notice a slight increase in the neutral economy. In this case, there are as many optimists as pessimists, and increasing belief dispersion increases the level of risk, inducing flight-to-safety that increases the demand of bonds and thus their price. The effect is more pronounced in the pessimistic economy where pessimists hold most of the wealth. This is because in this economy increasing belief dispersion while keeping the share of optimists constant additionally induces a negative shift in the average belief, i.e., a decrease in δ_t.[7] Shareholders thereby become more pessimistic on average which amplifies flight-to-safety and thus strengthens the positive relation between belief dispersion and short-

[7]The choice of keeping the share of optimists fixed to study the impact of belief dispersion—implying a fluctuating average belief bias—is motivated by the fact that most data sources used to assess shareholders'/investors' beliefs tend to be more qualitative than quantitative. It is thus more tedious to precisely estimate the belief bias, while the share of optimists can still be approximated. For instance, in the commonly used University of Michigan Surveys of Consumers, most questions have categorical answers rather than numerical.

term bond price. Conversely, in the optimistic economy, an increase in belief dispersion for a fixed share of optimists leads to an increase in δ_t. Because shareholders are more optimistic on average, they are willing to take more risk and their demand of stocks increases. Thus, the relation is reversed and we observe that the higher the belief dispersion, the lower the short-term bond price. Finally, looking at the bottom-left figure, we observe that the impact of belief dispersion on bond prices increases with the maturity of the bond. The reason is that the bond price decreases with maturity and is thus pushed further away from its final value, leaving room for higher effects of belief heterogeneity. In addition, the relation between long-term bond price and belief dispersion appears to be more convex. The higher is the maturity, the more prominent is this convexity effect and, for high levels of belief dispersion, an increase in belief dispersion can even lead to an increase in the (very) long-term bond price in the optimistic economy. As stated in Section 3.3, this is due to the fact that the share of optimists exhibits mean reversion and that the higher the belief dispersion, the quicker it mean reverts. For long-term bonds, shareholders thus expect that the share of optimists will decrease in the future, thereby reducing their willingness to take risk and weakening the negative relation. Overall, for optimistic economies, the sign of the relation between bond price and belief dispersion depends both on the bond maturity and the level of belief dispersion. Because of the mean reversion of $\mathcal{T}_{opt,t}$, there exists a level above which the relation becomes positive, and the longer is the bond maturity, the lower is this critical level.

Insert Figure 6 here:

Evolution of the bond instantaneous expected return depending on maturity.

Insert Figure 7 here:

Evolution of the bond instantaneous expected return depending on belief dispersion and share of optimists.

Using a similar approach, we analyze the impact of heterogeneous beliefs on the bond instantaneous expected return. Unlike in the no belief dispersion economy, this return is stochastic when there is belief dispersion. From Figure 6, we notice that it increases with maturity in the optimistic economy and decreases with it in the neutral and pessimistic ones. This is in line with the shareholders' expectations, and a longer maturity increases price fluctuations, thereby reinforcing the effect. From Figure 7, we further observe that the impacts of belief dispersion and the share of optimists are similar for short-term and long-term bonds, but varies for different economies. Specifically, the bond instantaneous expected return increases with belief dispersion in the optimistic economy and, except for high levels of belief dispersion, tends to decrease in the neutral and pessimistic ones. Apart for very pessimistic economies, i.e., economies in which the share of optimists is below 30%, it also increases with the share of optimists. These observations are consistent with the bond price fluctuations observed in Figure 5.

Insert Figure 8 here:

Evolution of the bond volatility depending on maturity.

Insert Figure 9 here:

Evolution of the bond volatility depending on belief dispersion and share of optimists.

Turning to bond price volatility, we see in Figure 8 that it increases with the bond maturity. It also increases with the aggregate share of optimists (right column of Figure 9). This is because the bond price decreases with the share of optimists, pushing the price further away from its (known) value at maturity. The fact that the gap between the bond's current and final values widens induces higher volatility. Finally, in most cases, bond volatility increases with belief dispersion (left column of Figure 9). This effect is documented for stocks in, e.g., Atmaz and Basak (2018), and is due to the fact that the higher is the dispersion, the more there are price fluctuations, yielding higher volatility. In line with the observations of Figure 5, the effect is stronger for the long-term bond than for the short-term one. However, when the share of optimists is very low (see the right column of Figure 9), or when it is low and the belief dispersion is sufficiently high (left column), the sign of the relation reverses and the bond volatility decreases with belief dispersion. This is because, in these situations, the bond price tends to be relatively close to its final value, thereby decreasing price fluctuations and thus volatility.

5.2 Bond yield

The bond yield between time t and time h is defined by

$$R_{t,h} = -\frac{1}{h-t}\ln\left(B_{t,h}\right), \tag{5.4}$$

and allows us to study the equilibrium yield curve.

Proposition 7 (Bond yield)**.** *The bond yield is given by*

$$R_{t,h} = \rho + a + \frac{2c+1}{2c+\frac{\omega_t^2}{\omega_h^2}}c\left(\bar{\theta}_t\right)^2 - \frac{1}{2\left(h-t\right)}\left(ln\left(\frac{2c+1}{2c+\frac{\omega_t^2}{\omega_h^2}}\right) + \frac{1}{2c+1}ln\left(\frac{\omega_t^2}{\omega_h^2}\right)\right). \tag{5.5}$$

The bond yield volatility is given by

$$\sigma_t^{R,h} = \frac{2\omega_t^2 c\bar{\theta}_t}{2c+\frac{\omega_t^2}{\omega_h^2}}. \tag{5.6}$$

Insert Figure 10 here:

Evolution of the bond yield depending on maturity.

Insert Figure 11 here:

Evolution of the bond yield depending on belief dispersion and share of optimists.

Without belief dispersion, the bond yield is constant and equals $\rho+a+c\left(\bar{\theta}_t\right)^2$, implying a flat yield curve. In the presence of heterogeneous beliefs, however, the bond yield is stochastic. We observe in Figure 10 that it results in a slightly decreasing yield curve.[8] From Figure 11,

[8]Figure 10 shows decreasing yield curves obtained with parameters set to match standard market conditions (i.e., a drift of 5%, a volatility of 20%, a share of optimists of 30%, 50%, or 70%, and a belief dispersion

we further derive that, both for the short-term and the long-term bonds, the level of the bond yield is pro-cyclical as the higher is the share of optimists, the higher it is. As for the risk-free rate, it also increases with belief dispersion in the optimistic economy, while it decreases in the neutral and pessimistic ones.

Insert Figure 12: Evolution of the bond yield volatility depending on maturity.

Insert Figure 13: Evolution of the bond yield volatility depending on belief dispersion and share of optimists.

Regarding bond yield volatility, several observations are in order. First we see in Figure 12 that it slightly decreases with maturity in all economies. This is in line with the empirical observation of Vayanos and Vila (2021). Second, as can be observed from the right column in Figure 13, bond yield volatility is pro-cyclical. Third, we see from the left column in Figure 13 that, except in pessimistic economies in which shareholders have very heterogeneous beliefs, it also increases with belief dispersion. As the bond yield volatility is always higher than in the case with homogeneous beliefs, it highlights the role of belief heterogeneity to explain the excess bond yield volatility puzzle. This is in line with Xiong and Yan (2010), and our study thereby extends their result to a production economy setting populated by a large number of shareholders.

6 Conclusion

This paper develops a general equilibrium model of interest rates in a production economy with heterogeneous shareholders and continuous consumption. It nests the CIR model as a special case. More broadly, our setting provides a relevant framework to study the impact of belief heterogeneity on bonds and interest rates. In particular, we find that the effects of

of 0%, 3.23%, or 3.61%). Nevertheless, it is worth noticing that our model can generate a rich class of shapes for the term structure of interest rates. We refer to Appendix B for a detailed analysis of a condition leading to increasing or hump-shaped yield curves.

heterogeneous beliefs increase with the bond maturity. Because of flight-to-safety, bonds are more expensive in neutral and pessimistic economies when belief dispersion is higher, but cheaper in optimistic economies. Similarly, there is a negative link between the average belief and the bond price. We further derive that both the risk-free rate and bond yields increase with belief dispersion in good times and decrease with it in bad times. Belief heterogeneity also increases bond yield volatility, helping to solve the excess bond yield volatility puzzle. Overall, our model yields numerous testable implications that we would like to test empirically in future research.

A Proofs

Proposition 1

The results correspond to specific cases of Proposition 2, Proposition 4, and Proposition 6. Specifically, the results for the homogeneous economy are obtained by taking $\omega_0 = 0$, which leads to $\delta_t = \delta_0$ and $\omega_t/\omega_h = 1$, and it suffices to take $c = 0$ to obtain the results of the no decreasing returns to scale economy.

Regarding the exchange economy, it corresponds to a production economy setting where, for given μ^* and σ^*, we have $a = \mu^* - c(\sigma^*)^2$, $b = 2c\sigma^*$, and $c \to +\infty$.[9] With these values of a, b, and c, the optimal choice of θ_t is indeed necessarily σ^* at each date and in every state of the world. This is because these values lead to $m(\sigma^*) = \mu^*$ for all c and $\lim_{c\to+\infty} m(\theta) = -\infty$ for $\theta \neq \sigma^*$. It thus corresponds to the exogenous production setting where

$$
\begin{aligned}
&dy_t^{\sigma^*} = \mu^* y_t^{\sigma^*} dt + \sigma^* y_t^{\sigma^*} dW_t, \\
&y_0^{\sigma^*} = 1.
\end{aligned}
$$

It then suffices to replace a, b, and c by those values in the corresponding formulas.

Proposition 2

Considering (3.2), the first order condition is

$$
\exp(-\rho t) M_t^{\delta} \bar{c}_{\delta,t}^{-1} - \lambda_\delta \bar{p}_t = 0,
$$

where λ_δ is the Lagrange multiplier. Rearranging the terms gives (3.5), and, plugging (3.6) and (3.7), it is easy to show that the market clearing condition (3.3) is verified.

Let us check that (3.4) solves (3.1).

[9] μ^* takes into account that the shareholders consume a fixed share of the production at each date, i.e., denoting this fixed share by C, μ^* is of the form $\mu^* = \tilde{\mu}^* - C$ for a given $\tilde{\mu}^*$.

To do so, let us show that $y^{\bar{\theta}}$ maximizes $\mathbb{E}\left(\int_0^{\infty} \bar{p}_s y_s^{\theta} ds\right)$ over Θ.
It is equivalent to show that $\bar{\theta}$ maximizes

$$\mathbb{E}\left(\int_0^{\infty} \exp\left(-\rho s\right) q_s z_s^{\theta} ds\right),$$

with $q_t = \dfrac{\omega_0 \sqrt{2\pi}}{\sqrt{1+t\omega_0^2}} \exp\left(\dfrac{\left(\delta_0 + \omega_0^2 W_t\right)^2}{2\omega_0^2\left(1+t\omega_0^2\right)}\right)$ and $z_t^{\theta} = y_t^{\theta}\left(y_t^{\bar{\theta}}\right)^{-1}$.
By Ito's lemma, we have

$$\begin{aligned} dq_t &= \frac{\delta_0 + \omega_0^2 W_t}{1+t\omega_0^2} q_t dW_t = \sigma_t^q q_t dW_t, \\ dz_t^{\theta} &= \left(m\left(\theta_t\right) - m\left(\bar{\theta}_t\right) + \bar{\theta}_t^2 - \theta_t \bar{\theta}_t\right) z_t^{\theta} dt + \left(\theta_t - \bar{\theta}_t\right) z_t^{\theta} dW_t \\ &= \mu_t^{\theta} z_t^{\theta} dt + \sigma_t^{\theta} z_t^{\theta} dW_t. \end{aligned}$$

Let us denote by $\mathcal{V}$ the function defined by

$$\mathcal{V}\left(t, q_t, z_t, W_t\right) = \max_{\theta \in \Theta} \mathbb{E}\left(\int_0^{\infty} \exp\left(-\rho s\right) q_s z_s^{\theta} ds\right).$$

This is an optimal control problem whose associated HJB equation is given by (see Theorem 3.5.3. in Pham, 2009)

$$\rho \mathcal{V} - \mathcal{L}^{\bar{\theta}} \mathcal{V} - q z^{\bar{\theta}} = 0, \tag{A.1}$$

with

$$\mathcal{L}^{\bar{\theta}} \mathcal{V} = \mu_t^{\bar{\theta}} z_t^{\bar{\theta}} \mathcal{V}_z + \frac{\left(\sigma_t^q q_t\right)^2 \mathcal{V}_{qq} + \left(\sigma_t^{\bar{\theta}} z_t^{\bar{\theta}}\right)^2 \mathcal{V}_{zz} + \mathcal{V}_{ww}}{2} + \sigma_t^{\bar{\theta}} z_t^{\bar{\theta}} \left(\sigma_t^q q_t \mathcal{V}_{qz} + \mathcal{V}_{zw}\right) + \sigma_t^q q_t \mathcal{V}_{qw}.$$

To solve this problem, we need to find a control $\bar{\theta}$ and a function $\mathcal{V}$ such that (A.1) is verified.

Positing that $\mathcal{V}$ is of the form $\mathcal{V}(t,q,z,w) = qzF(t,w)$, we get

$$\rho F - \frac{F_{ww}}{2} - \sigma_t^q F_w - 1 = 0, \tag{A.2}$$
$$\frac{\partial}{\partial\theta}\left(\mu_t^\theta F + \sigma_t^\theta \sigma_t^q F + \sigma_t^\theta F_w\right)\Big|_{\theta=\bar\theta} = 0. \tag{A.3}$$

$F(t,w) = \dfrac{1}{\rho}$ solves (A.2), and (A.3) becomes $\left.\dfrac{\partial\left(\mu_t^\theta + \sigma_t^\theta\sigma_t^q\right)}{\partial\theta}\right|_{\theta=\bar\theta} = 0$, leading to (3.4).

It remains to show that (3.5) solves (3.2).

To do so, we need to show that $\mathbb{E}\left(\int_0^\infty \bar{p}_s \bar{c}_{\delta,s} ds\right)$ is proportional to ν_δ.

Explicit computations give

$$\mathbb{E}\left(\int_0^\infty \bar{p}_s\bar{c}_{\delta,s}ds\right) = \frac{1}{\rho}\exp\left(-\frac{\delta^2}{2\omega_0^2} + \frac{\delta_0\delta}{\omega_0^2}\right) = \frac{\omega_0\sqrt{2\pi}}{\rho}\exp\left(\frac{\delta_0^2}{2\omega_0^2}\right)\times\nu_\delta.$$

In addition, the uniqueness of the equilibrium is easily obtained following Dana (1995).

Lastly, the time-t wealth-weighted distribution of beliefs is obtained by considering a similar economy that starts at time t with an initial average belief δ_t and an initial belief dispersion ω_t^2 and adapting the beginning of the proof to this economy.

Plugging (3.8) and (3.9) leads to the expressions reported in the proposition.

The explicit expressions of $y_t^{\bar\theta}$ and $\bar{p}_t$ are further given by

$$y_t^{\bar\theta} = \left(1+t\omega_0^2\right)^{-\frac{1}{2(2c+1)}}\exp\left(at + \frac{b\left(bt+2W_t\right)}{2\left(2c+1\right)} + \frac{1}{2\left(2c+1\right)}\left(\frac{\delta_t^2}{\omega_t^2} - \frac{\delta_0^2}{\omega_0^2}\right)\right), \tag{A.4}$$
$$\bar{p}_t = \frac{\omega_0\sqrt{2\pi}}{\left(1+t\omega_0^2\right)^{\frac{c}{2c+1}}}\exp\left(-\left(\rho+a\right)t - \frac{b\left(bt+2W_t\right)}{2\left(2c+1\right)} + \frac{1}{2\left(2c+1\right)}\left(2c\frac{\delta_t^2}{\omega_t^2} + \frac{\delta_0^2}{\omega_0^2}\right)\right). \tag{A.5}$$

To show this, let us define f such that $f(t,W_t) = y_t^{\bar\theta}$. We check that $\dfrac{\partial f(t,W_t)}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 f(t,W_t)}{\partial x^2} = m\left(\bar\theta_t\right)y_t^{\bar\theta}$, $\dfrac{\partial f(t,W_t)}{\partial x} = \bar\theta_t y_t^{\bar\theta}$, and $f(0,0)=1$. (A.4) is then easily obtained using Ito's lemma, and (A.5) follows from (3.6).

Proposition 3

Direct computations using (3.10) and (3.11) lead to

$$\tau_{\delta,t} = \frac{1}{\omega_t}\varphi\left(\frac{\delta - \delta_t}{\omega_t}\right),$$

and

$$\mathcal{T}_{\delta,t} = \Phi\left(\frac{\delta - \delta_t}{\omega_t}\right).$$

We then have $\mathcal{T}_{opt,t} = 1 - \mathcal{T}_{0,t} = 1 - \Phi\left(-\frac{\delta_t}{\omega_t}\right) = \Phi\left(\frac{\delta_t}{\omega_t}\right)$.
Applying Ito's lemma we further get

$$d\mathcal{T}_{\delta,t} = -\omega_t\varphi\left(\frac{\delta - \delta_t}{\omega_t}\right)\left(-\delta_t dt + dW_t\right).$$

Inverting (3.12) to get $\frac{\delta_t}{\omega_t} = \Phi^{-1}\left(\mathcal{T}_{opt,t}\right)$ and the fact that $\mathcal{T}_{opt,t} = 1 - \mathcal{T}_{0,t}$ yield (3.13).
Using the fact that $-\omega_t^2\varphi\left(\Phi^{-1}\left(\mathcal{T}_{opt,t}\right)\right) \leq 0$ and that $\Phi^{-1}(x) > 0$ for $x > 0.5$ and $\Phi^{-1}(x) < 0$ for $x < 0.5$, we get that the drift in (3.13) is negative (resp. positive) when $\mathcal{T}_{opt,t} > 0.5$ (resp. $\mathcal{T}_{opt,t} < 0.5$).

Proposition 4

By definition, the risk-free rate is given by $r_t^f = -\mu_t^{\bar{p}}$. Using (A.5) and Ito's lemma, we get

$$d\bar{p}_t = -\left(\rho + a + \frac{c\left(b + \delta_t\right)^2}{\left(2c+1\right)^2}\right)\bar{p}_t dt + \frac{2c\delta_t - b}{2c+1}\bar{p}_t dW_t, \tag{A.6}$$

which gives

$$r_t^f = \rho + a + c\frac{\left(b+\delta_t\right)^2}{\left(2c+1\right)^2}.$$

Using $x_t = c\left(\bar{\theta}_t\right)^2$, (4.3), (4.4), and (4.5), explicit computations lead to $\frac{\partial x_t}{\partial t} + \frac{1}{2}\frac{\partial^2 x_t}{\partial x^2} = \kappa_t\Big(\vartheta_t - x_t + \frac{b\sqrt{c}}{2c+1}\sqrt{x_t}\Big)$ and $\frac{\partial x_t}{\partial x} = \sigma_t^r \sqrt{x_t}$. Ito's lemma then leads to (4.2).

Proposition 5

By definition, the market price of risk is given by $MPR_t = -\sigma_t^{\bar{p}}$. Using (A.6) gives

$$MPR_t = \frac{b - 2c\delta_t}{2c+1} = \frac{b + \delta_t}{2c+1} - \delta_t.$$

Proposition 6

By definition, we have $B_{t,h} = \frac{1}{\bar{p}_t}\mathbb{E}_t\left(\bar{p}_h\right)$, and its dynamics is given by

$$dB_{t,h} = \mu_t^{B,h} B_{t,h} dt + \tilde{\sigma}_t^{B,h} B_{t,h} dW_t.$$

From there, (5.1), (5.2), and (5.3) are obtained through explicit computations and using Ito's lemma, with $\sigma_t^{B,h} = |\tilde{\sigma}_t^{B,h}|$.

Proposition 7

(5.5) directly follows from (5.4). (5.6) is obtained from Ito's lemma with

$$dR_{t,h} = \mu_t^{R,h} dt + \sigma_t^{R,h} dW_t.$$

B An additional result on the yield curve

Lemma 1. *The initial slope of the yield curve is positive if and only if* $\omega_0^2 > \dfrac{2\left(b+\delta_0\right)^2}{2c+1}$.

Proof: Let us consider $t=0$ and $W_t=0$. Using Equation 5.5, we have

$$R_{0,h} = \rho + a + \frac{c\left(b+\delta_0\right)^2}{\left(2c+1\right)\left(2c+1+h\omega_0^2\right)} - \frac{1}{2h}\left(\ln\left(\frac{2c+1}{2c+1+h\omega_0^2}\right) + \frac{1}{2c+1}\ln\left(1+h\omega_0^2\right)\right).$$

Direct computations yield

$$\begin{aligned}\frac{\partial R_{0,h}}{\partial h} &= \frac{c\omega_0^2}{\left(2c+1\right)\left(2c+1+h\omega_0^2\right)}\left(\frac{\omega_0^2}{1+h\omega_0^2} - \frac{\left(b+\delta_0\right)^2}{2c+1+h\omega_0^2}\right) \\ &\quad + \frac{1}{2h^2}\left(\ln\left(\frac{2c+1}{2c+1+h\omega_0^2}\right) + \frac{1}{2c+1}\ln\left(1+h\omega_0^2\right)\right).\end{aligned}$$

Using the fact that

$$\frac{1}{2h^2}\left(\ln\left(\frac{2c+1}{2c+1+h\omega_0^2}\right) + \frac{1}{2c+1}\ln\left(1+h\omega_0^2\right)\right) = -\frac{c\omega_0^4}{2\left(2c+1\right)^2} + \mathcal{O}\left(h\right),$$

we obtain that $\lim_{h\to 0}\dfrac{\partial R_{0,h}}{\partial h} = \dfrac{c\omega_0^2}{2\left(2c+1\right)^3}\left(\omega_0^2\left(2c+1\right) - 2\left(b+\delta_0\right)^2\right)$.

This limit is strictly positive if and only if $\omega_0^2\left(2c+1\right) - 2\left(b+\delta_0\right)^2 > 0$, i.e., $\omega_0^2 > \dfrac{2\left(b+\delta_0\right)^2}{2c+1}$.

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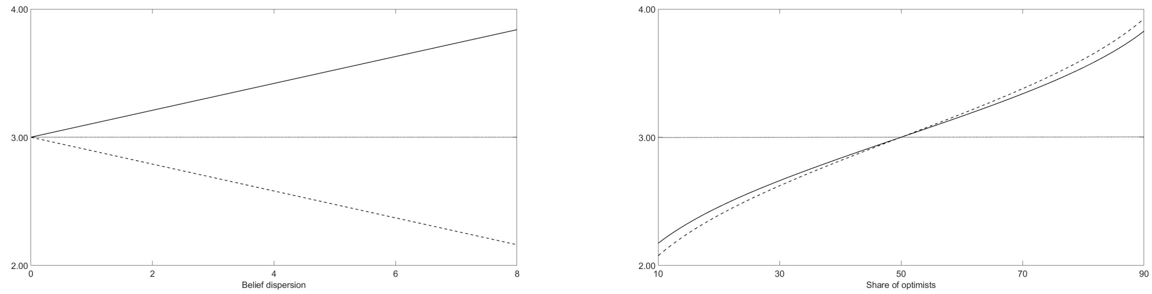
Figure 1: Evolution of the risk-free rate depending on...

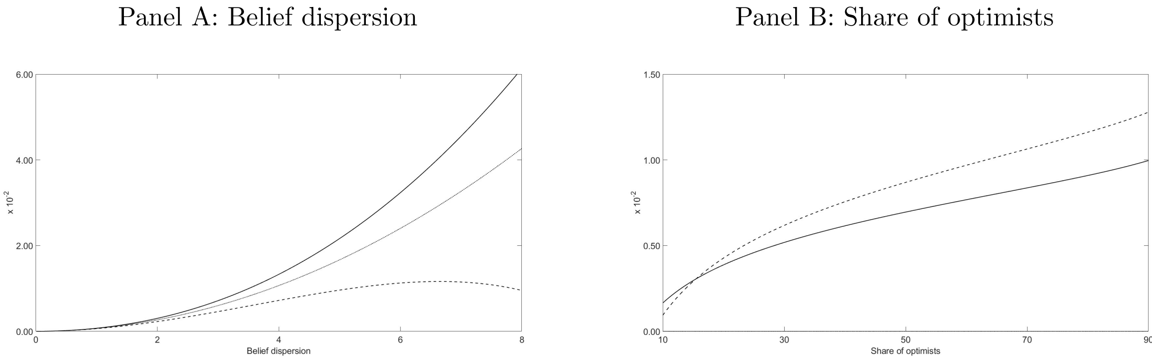
Panel A: Belief dispersion

Panel B: Share of optimists



The figure presents the evolution of the risk-free rate, r_t^f, depending on levels of current belief dispersion, ω_t, (Panel A) and share of optimists, $\mathcal{T}_{opt,t}$, (Panel B). All values are displayed in percentages. In Panel A, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy, i.e., the economy in which the current share of optimists equals 70% (resp. 30%, 50%). In Panel B, the solid (resp. dashed) line represents the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%), and the dotted line represents the no belief dispersion economy. The parameters are chosen such that, for all economies and all levels of current belief dispersion or share of optimists, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). Thus, in Panel A (resp. Panel B), for a given ω_t (resp. $\mathcal{T}_{opt,t}$), the values of W_t, b, and c differ from an economy to another, and, for a given economy, they change as ω_t (resp. $\mathcal{T}_{opt,t}$) varies. We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

Figure 2: Evolution of the diffusion of the risk-free rate depending on...

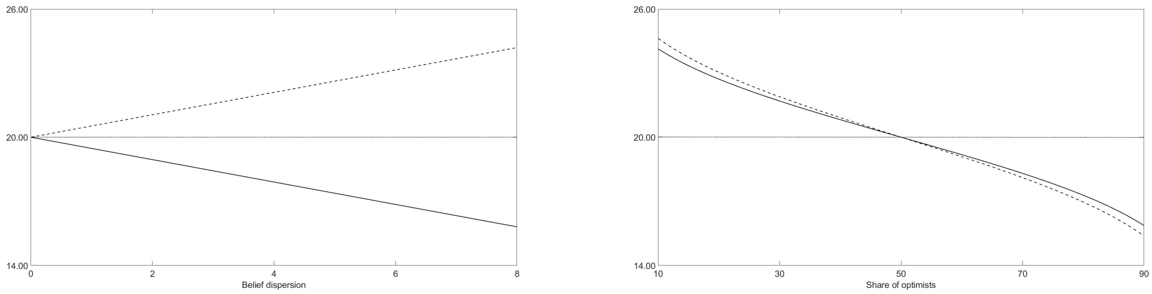


The figure presents the evolution of the diffusion of the risk-free rate, $\sigma_t^{rf} = \sigma_t^r \sqrt{x_t}$, depending on levels of current belief dispersion, ω_t, (Panel A) and share of optimists, $\mathcal{T}_{opt,t}$, (Panel B). All values are displayed in percentages. In Panel A, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy, i.e., the economy in which the current share of optimists equals 70% (resp. 30%, 50%). In Panel B, the solid (resp. dashed) line represents the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%), and the dotted line represents the no belief dispersion economy. The parameters are chosen such that, for all economies and all levels of current belief dispersion or share of optimists, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). Thus, in Panel A (resp. Panel B), for a given ω_t (resp. $\mathcal{T}_{opt,t}$), the values of W_t, b, and c differ from an economy to another, and, for a given economy, they change as ω_t (resp. $\mathcal{T}_{opt,t}$) varies. We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

Figure 3: Evolution of the market price of risk depending on...

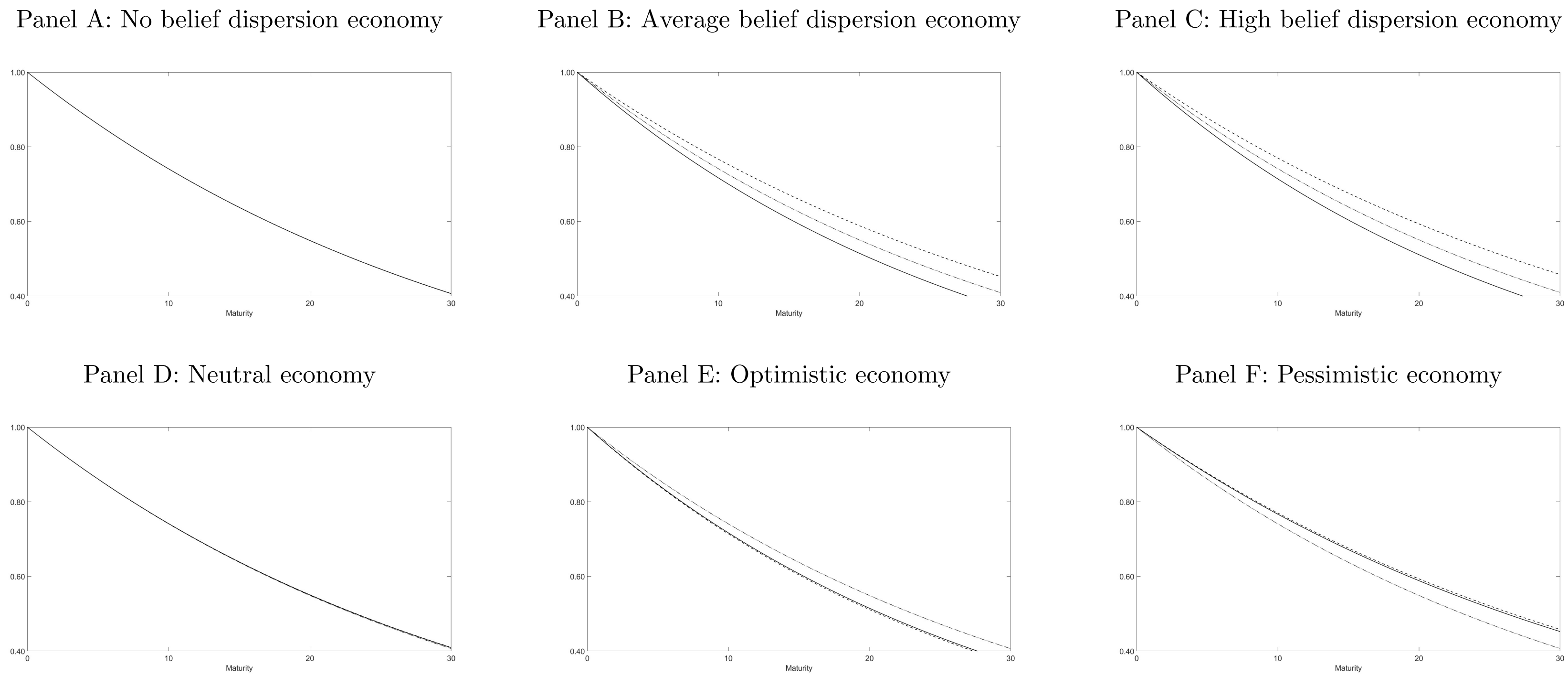
Panel A: Belief dispersion

Panel B: Share of optimists



The figure presents the evolution of the market price of risk, MPR_t, depending on levels of current belief dispersion, ω_t, (Panel A) and share of optimists, $\mathcal{T}_{opt,t}$, (Panel B). All values are displayed in percentages. In Panel A, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy, i.e., the economy in which the current share of optimists equals 70% (resp. 30%, 50%). In Panel B, the solid (resp. dashed) line represents the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%), and the dotted line represents the no belief dispersion economy. The parameters are chosen such that, for all economies and all levels of current belief dispersion or share of optimists, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). Thus, in Panel A (resp. Panel B), for a given ω_t (resp. $\mathcal{T}_{opt,t}$), the values of W_t, b, and c differ from an economy to another, and, for a given economy, they change as ω_t (resp. $\mathcal{T}_{opt,t}$) varies. We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

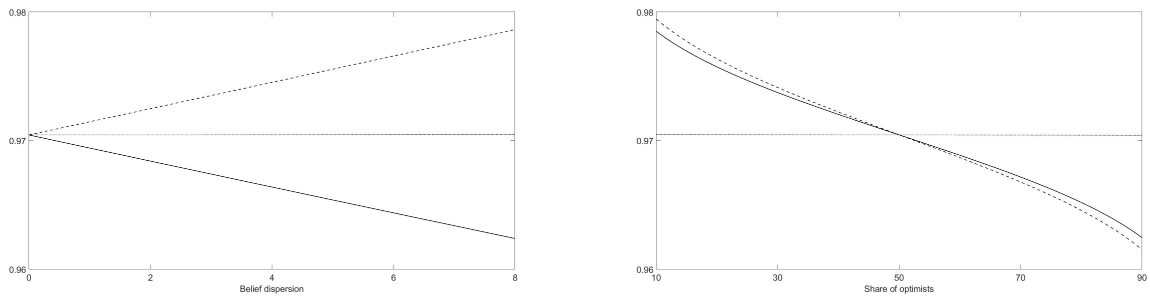
Figure 4: Evolution of the bond price depending on maturity in a...



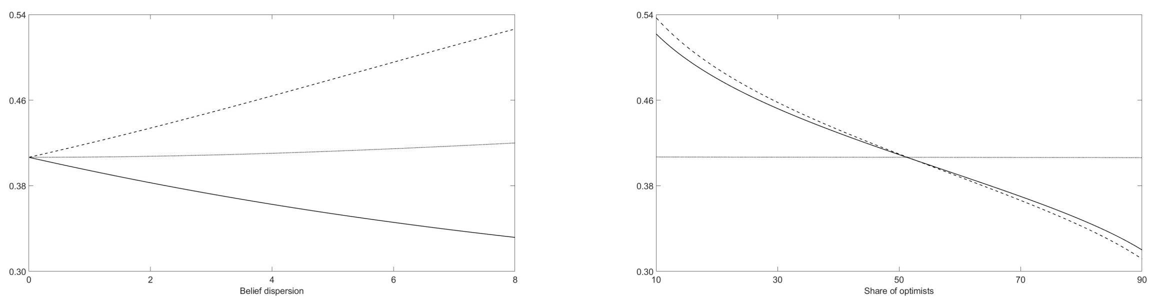
The figure presents the evolution of the bond price, $B_{t,h}$, depending on the bond maturity, h. Panel A shows the no belief dispersion economy. Panel B (resp. C) shows the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%). Panel D (resp. E, F) shows the neutral (resp. optimistic, pessimistic) economy , i.e., the economy in which the current share of optimists equals 50% (resp. 70%, 30%). In Panels A to C, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy. In Panels D to E, the solid (resp. dashed, dotted) line represents the average (resp. high, no) belief dispersion economy. The parameters are chosen such that, for all economies, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

Figure 5: Evolution of the bond price depending on belief dispersion and share of optimists

Panel A: Short-term bond ($h = 1$)

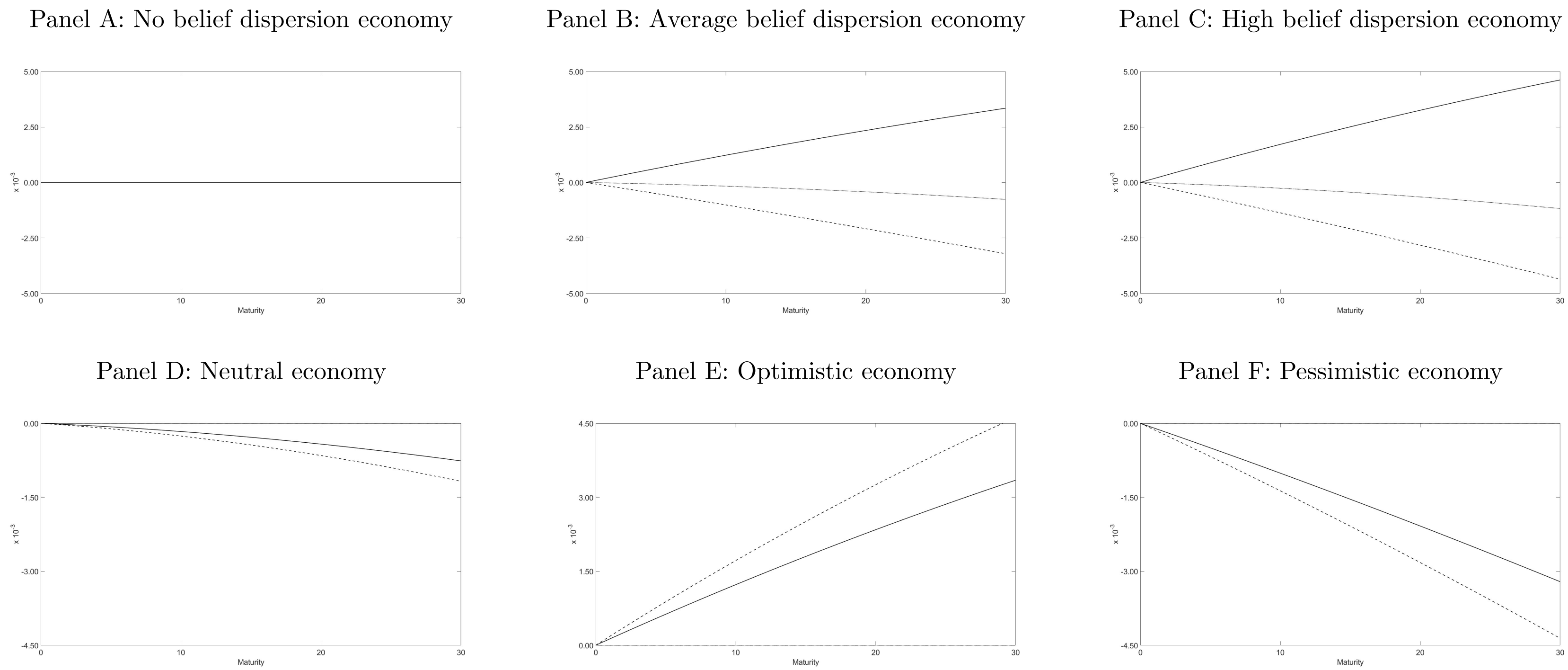


Panel B: Long-term bond ($h = 30$)



The figure presents the evolution of the price of bonds, $B_{t,h}$, depending on levels of current belief dispersion, ω_t, (left column) and share of optimists, $\mathcal{T}_{opt,t}$, (right column). ω_t and $\mathcal{T}_{opt,t}$ are displayed in percentages. Panel A (resp. B) shows the price of a short-term (resp. long-term) bond, i.e., a bond with a maturity of $h = 1$ (resp. 30). In the left column, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy, i.e., the economy in which the current share of optimists equals 70% (resp. 30%, 50%). In the right column, the solid (resp. dashed) line represents the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%), and the dotted line represents the no belief dispersion economy. The parameters are chosen such that, for all economies and all levels of current belief dispersion or share of optimists, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). Thus, in the left (resp. right) column, for a given ω_t (resp. $\mathcal{T}_{opt,t}$), the values of W_t, b, and c differ from an economy to another, and, for a given economy, they change as ω_t (resp. $\mathcal{T}_{opt,t}$) varies. We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

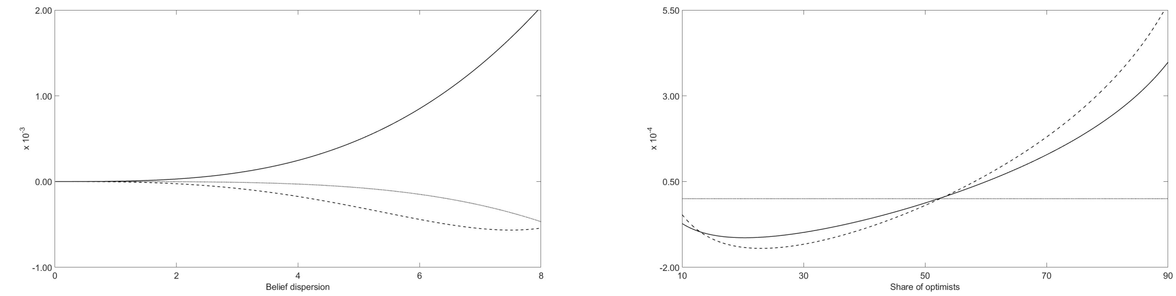
Figure 6: Evolution of the bond instantaneous expected return depending on maturity in a...



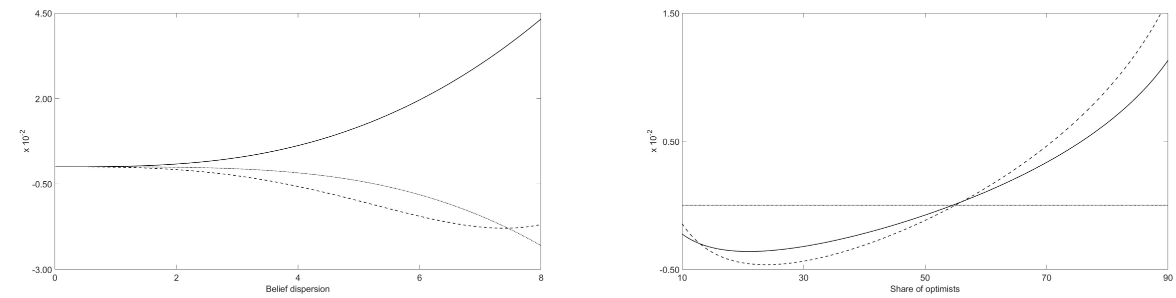
The figure presents the evolution of the bond instantaneous expected return, $\mu_t^{B,h}$, depending on the bond maturity, h. $\mu_t^{B,h}$ is displayed in percentages. Panel A shows the no belief dispersion economy. Panel B (resp. C) shows the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%). Panel D (resp. E, F) shows the neutral (resp. optimistic, pessimistic) economy , i.e., the economy in which the current share of optimists equals 50% (resp. 70%, 30%). In Panels A to C, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy. In Panels D to E, the solid (resp. dashed, dotted) line represents the average (resp. high, no) belief dispersion economy. The parameters are chosen such that, for all economies, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

Figure 7: Evolution of the bond instantaneous expected return depending on belief dispersion and share of optimists

Panel A: Short-term bond ($h = 1$)

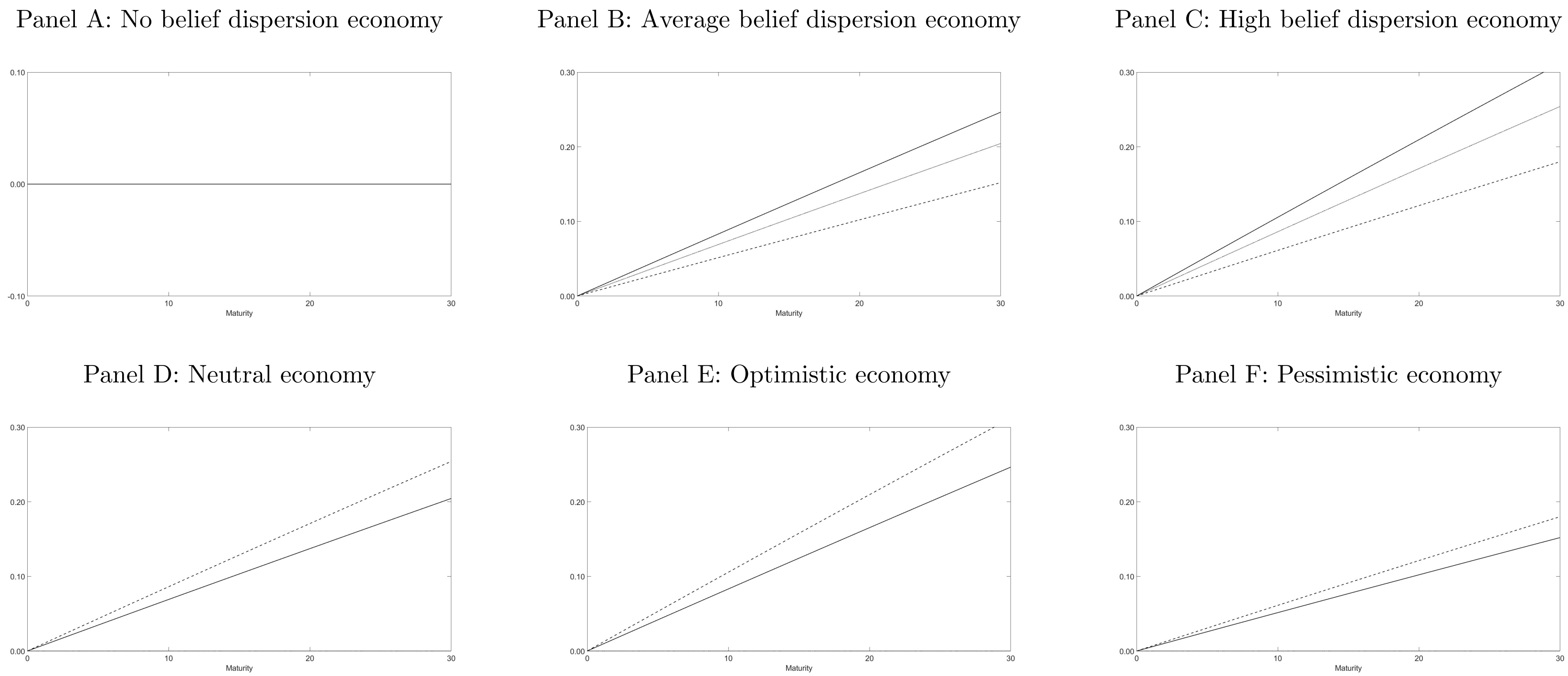


Panel B: Long-term bond ($h = 30$)



The figure presents the evolution of the bond instantaneous expected return, $\mu_t^{B,h}$, depending on levels of current belief dispersion, ω_t, (left column) and share of optimists, $\mathcal{T}_{opt,t}$, (right column). All values are displayed in percentages. Panel A (resp. B) shows the instantaneous expected return of a short-term (resp. long-term) bond, i.e., a bond with a maturity of $h = 1$ (resp. 30). In the left column, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy, i.e., the economy in which the current share of optimists equals 70% (resp. 30%, 50%). In the right column, the solid (resp. dashed) line represents the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%), and the dotted line represents the no belief dispersion economy. The parameters are chosen such that, for all economies and all levels of current belief dispersion or share of optimists, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). Thus, in the left (resp. right) column, for a given ω_t (resp. $\mathcal{T}_{opt,t}$), the values of W_t, b, and c differ from an economy to another, and, for a given economy, they change as ω_t (resp. $\mathcal{T}_{opt,t}$) varies. We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

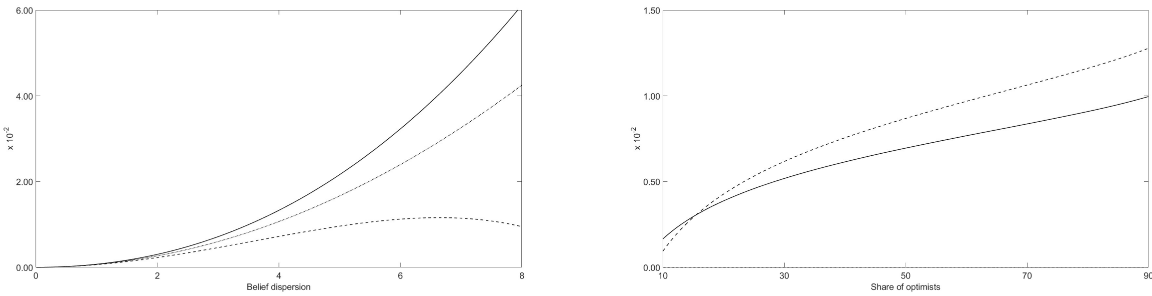
Figure 8: Evolution of the bond volatility depending on maturity in a...



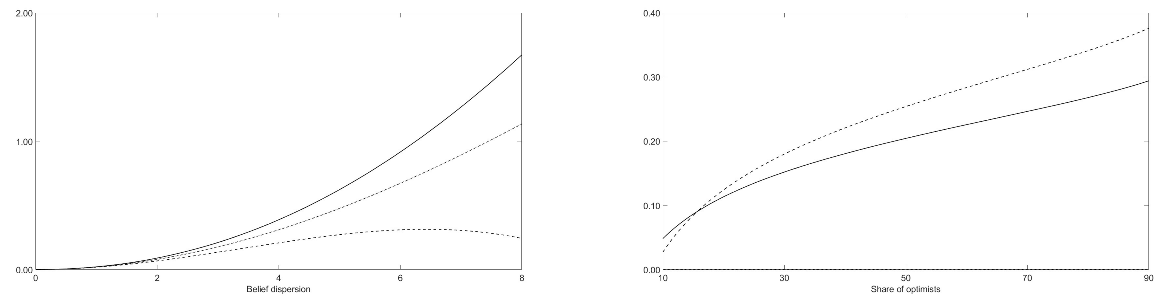
The figure presents the evolution of the bond volatility, $\sigma_t^{B,h}$, depending on the bond maturity, h. $\sigma_t^{B,h}$ is displayed in percentages. Panel A shows the no belief dispersion economy. Panel B (resp. C) shows the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%). Panel D (resp. E, F) shows the neutral (resp. optimistic, pessimistic) economy , i.e., the economy in which the current share of optimists equals 50% (resp. 70%, 30%). In Panels A to C, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy. In Panels D to E, the solid (resp. dashed, dotted) line represents the average (resp. high, no) belief dispersion economy. The parameters are chosen such that, for all economies, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

Figure 9: Evolution of the bond volatility depending on belief dispersion and share of optimists

Panel A: Short-term bond ($h = 1$)

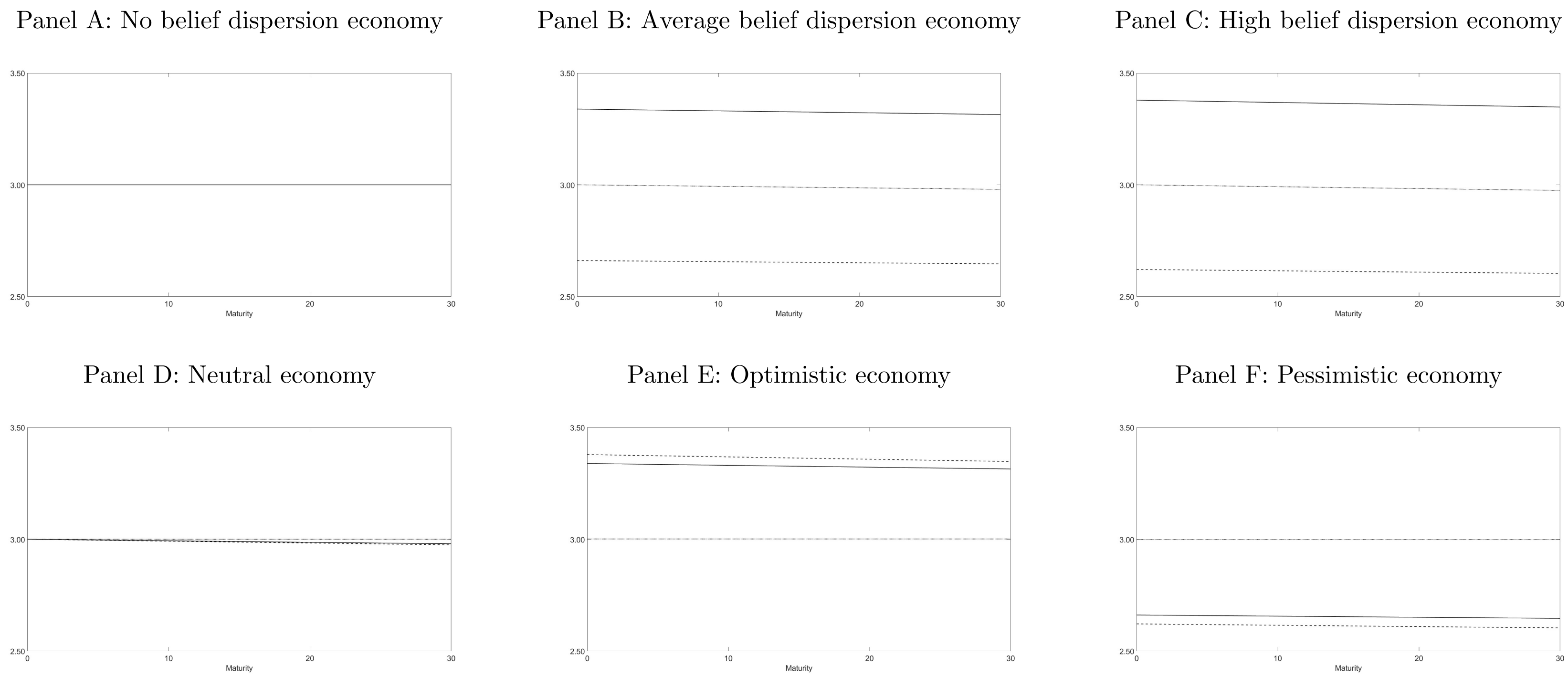


Panel B: Long-term bond ($h = 30$)



The figure presents the evolution of the bond volatility, $\sigma_t^{B,h}$, depending on levels of current belief dispersion, ω_t, (left column) and share of optimists, $\mathcal{T}_{opt,t}$, (right column). All values are displayed in percentages. Panel A (resp. B) shows the volatility of a short-term (resp. long-term) bond, i.e., a bond with a maturity of $h = 1$ (resp. 30). In the left column, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy, i.e., the economy in which the current share of optimists equals 70% (resp. 30%, 50%). In the right column, the solid (resp. dashed) line represents the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%), and the dotted line represents the no belief dispersion economy. The parameters are chosen such that, for all economies and all levels of current belief dispersion or share of optimists, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). Thus, in the left (resp. right) column, for a given ω_t (resp. $\mathcal{T}_{opt,t}$), the values of W_t, b, and c differ from an economy to another, and, for a given economy, they change as ω_t (resp. $\mathcal{T}_{opt,t}$) varies. We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

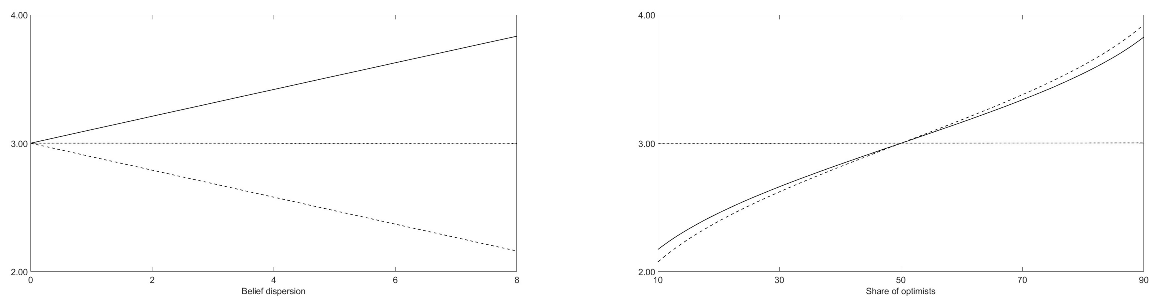
Figure 10: Evolution of the bond yield depending on maturity in a...



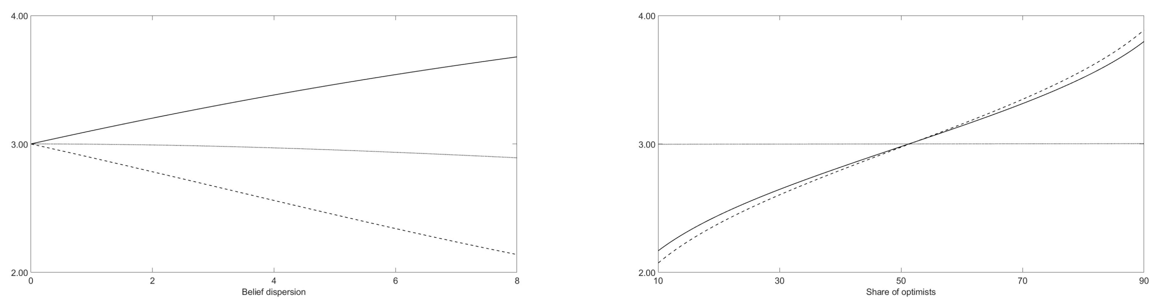
The figure presents the evolution of the bond yield, $R_{t,h}$, depending on the bond maturity, h. $R_{t,h}$ is displayed in percentages. Panel A shows the no belief dispersion economy. Panel B (resp. C) shows the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%). Panel D (resp. E, F) shows the neutral (resp. optimistic, pessimistic) economy , i.e., the economy in which the current share of optimists equals 50% (resp. 70%, 30%). In Panels A to C, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy. In Panels D to E, the solid (resp. dashed, dotted) line represents the average (resp. high, no) belief dispersion economy. The parameters are chosen such that, for all economies, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

Figure 11: Evolution of the bond yield depending on belief dispersion and share of optimists

Panel A: Short-term bond ($h = 1$)

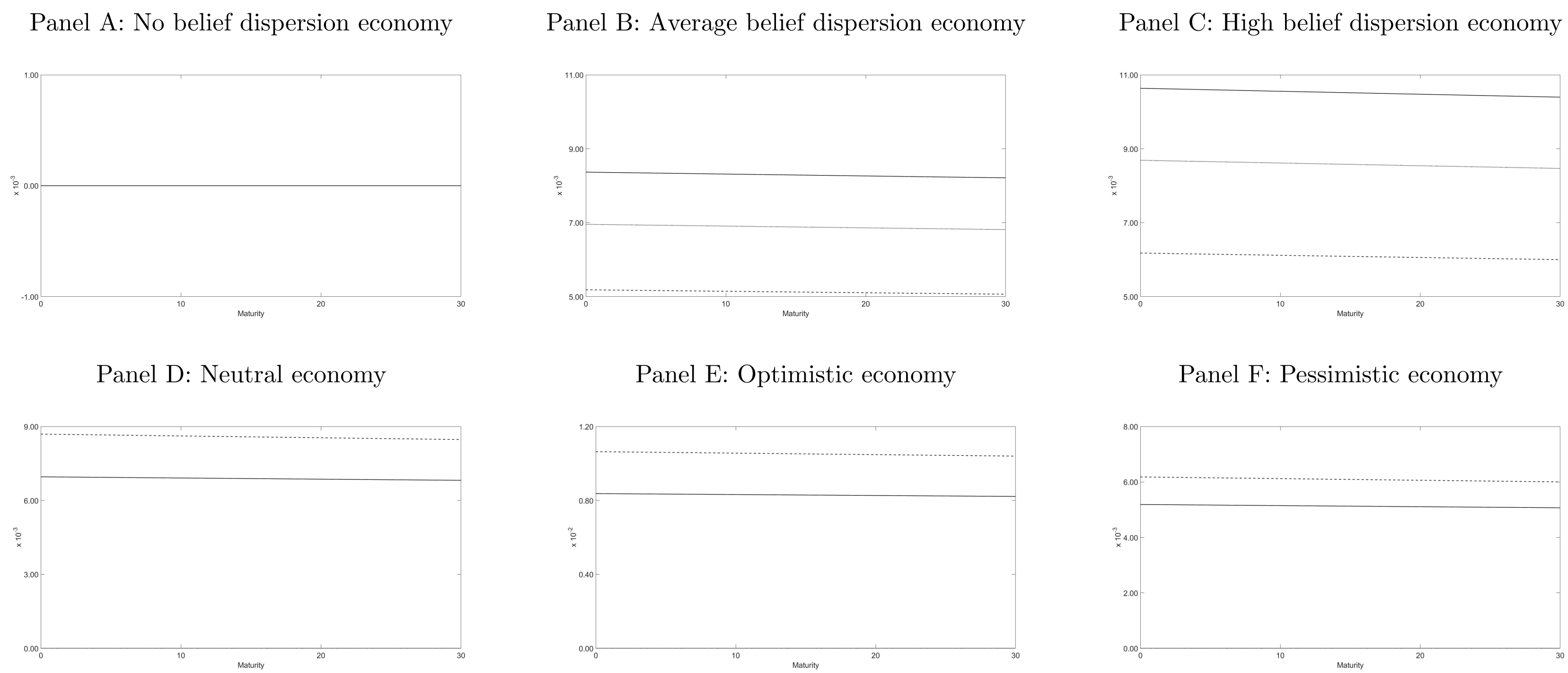


Panel B: Long-term bond ($h = 30$)



The figure presents the evolution of the bond yield, $R_{t,h}$, depending on levels of current belief dispersion, ω_t, (left column) and share of optimists, $\mathcal{T}_{opt,t}$, (right column). All values are displayed in percentages. Panel A (resp. B) shows the bond yield of a short-term (resp. long-term) bond, i.e., a bond with a maturity of $h = 1$ (resp. 30). In the left column, the solid (resp. dashed, dotted) line represents the optimistic (resp. pessimistic, neutral) economy, i.e., the economy in which the current share of optimists equals 70% (resp. 30%, 50%). In the right column, the solid (resp. dashed) line represents the average (resp. high) belief dispersion economy, i.e., the economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% given in Yu (2011) (with standard deviation of 0.38%), and the dotted line represents the no belief dispersion economy. The parameters are chosen such that, for all economies and all levels of current belief dispersion or share of optimists, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). Thus, in the left (resp. right) column, for a given ω_t (resp. $\mathcal{T}_{opt,t}$), the values of W_t, b, and c differ from an economy to another, and, for a given economy, they change as ω_t (resp. $\mathcal{T}_{opt,t}$) varies. We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

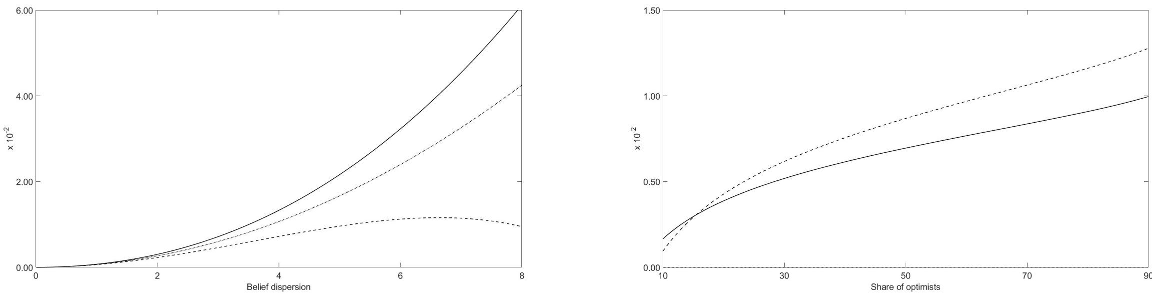
Figure 12: Evolution of the bond yield volatility depending on maturity in a...



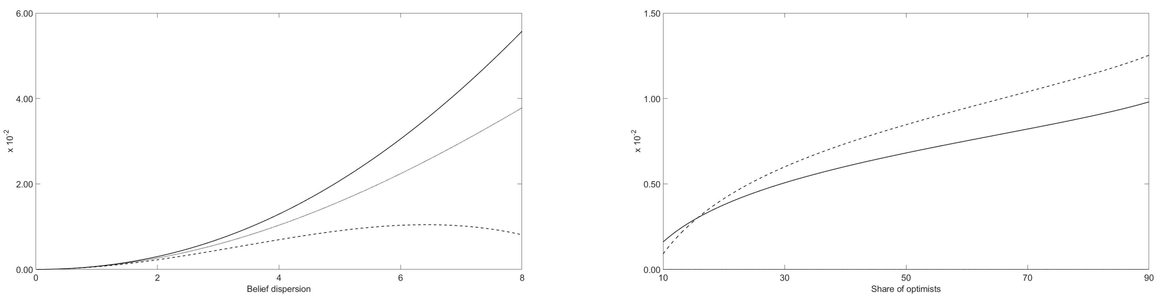
The figure presents the evolution of the bond yield volatility, $\sigma_t^{R,h}$, depending on the bond maturity, h. $\sigma_t^{R,h}$ is displayed in percentages. Panel A shows an homogeneous economy. Panel B (resp. C) shows an average (resp. high) belief dispersion economy, i.e., an economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% (0.38%) given in Yu (2011). Panel D (resp. E, F) shows a neutral (resp. optimistic, pessimistic) economy , i.e., an economy in which the current share of optimists equals 50% (resp. 70%, 30%). In Panels A to C, the solid (resp. dashed, dotted) line represents an optimistic (resp. pessimistic, neutral) economy. In Panels D to E, the solid (resp. dashed, dotted) line represents an average (resp. high, no) belief dispersion economy. The parameters are chosen such that, for all economies, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.

Figure 13: Evolution of the bond yield volatility depending on belief dispersion and share of optimists

Panel A: Short-term bond ($h = 1$)



Panel B: Long-term bond ($h = 30$)



The figure presents the evolution of the bond yield volatility, $\sigma_t^{R,h}$, depending on levels of current belief dispersion, ω_t, (left column) and share of optimists, $\mathcal{T}_{opt,t}$, (right column). All values are displayed in percentages. Panel A (resp. B) shows the diffusion of the bond yield volatility of a short-term (resp. long-term) bond, i.e., a bond with a maturity of $h = 1$ (resp. 30). In the left column, the solid (resp. dashed, dotted) line represents an optimistic (resp. pessimistic, neutral) economy, i.e., an economy in which the current share of optimists equals 70% (resp. 30%, 50%). In the right column, the solid (resp. dashed) line represents an average (resp. high) belief dispersion economy, i.e., an economy in which the current belief dispersion is equal to (resp. one standard deviation higher than) the average empirical belief dispersion of 3.23% (0.38%) given in Yu (2011), and the dotted line represents an homogeneous economy. The parameters are chosen such that, for all economies and all levels of current belief dispersion or share of optimists, $m\left(\bar{\theta}_t\right) = 5.00\%$ and $\bar{\theta}_t = 20.00\%$, as in Wang (1996). Thus, in the left (resp. right) column, for a given ω_t (resp. $\mathcal{T}_{opt,t}$), the values of W_t, b, and c differ from an economy to another, and, for a given economy, they change as ω_t (resp. $\mathcal{T}_{opt,t}$) varies. We take $\rho = 2\%$ (Wang, 1996) and $\delta_0 = 0$ (Atmaz and Basak, 2018), and we set $t = 0$ and $a = 0$ without loss of generality.