Optimal Liquidity and Asset Bubbles

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Abstract

This paper examines how the interplay between arbitrageurs, stockholders and liquidity providers fuels bubble creation in a dynamic financial market with endogenous entry. Credit lines contribute to the formation of bubbles and amplify the impact of shocks, negatively impacting existing stockholders, but allow arbitrageurs to enter the market at low cost, thereby rehabilitating the risk concentration channel of limited participation models. Optimal liquidity balances arbitrage benefits and costs; however, we demonstrate that excessive liquidity can destabilize markets. This is particularly relevant in the current financial landscape, where the rise of cryptocurrencies poses unique regulatory challenges.

Keywords: Bubble formation, Credit lines, Entry, Pecuniary externality, Welfare analysis.

JEL classification: D51, D52, G11, G12.

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1 Introduction

Over the past few decades, specialized investors, such as hedge funds and other financial intermediaries, have experienced significant growth in both activity and size. By the fourth quarter of 2024, the global hedge fund industry's assets under management reached approximately 5.3 trillion, a substantial increase from 1.4 trillion in the fourth quarter of 2009. Notably, funds specializing in arbitrage strategies, including convertible arbitrage, equity long/short, and merger arbitrage, represented nearly 5.5% of the 2024 total, amounting to 265 billion.[1] Given that hedge funds frequently utilize leverage, liquidity providers have been crucial in financing this expansion.

To analyze the dynamics of these observed market developments, particularly the role of arbitrageurs and liquidity providers, we propose a theoretical framework. Specifically, this paper examines the incentives for their emergence within a dynamic competitive equilibrium that features asset bubbles. We develop a general equilibrium model with endogenous participation and funding liquidity to investigate how regulations and associated market structures affect equilibrium prices, strategies, and investor welfare.

We build upon the restricted market participation framework established in Basak and Cuoco (1998) and Hugonnier and Prieto (2015), where investors are exogenously classified as participants and non-participants. However, our model distinguishes itself by endogenizing segmentation and entry. We retain the same market primitives: a riskless asset in zero net supply and a dividend-paying risky asset in positive supply. In these models, asset price dynamics are driven by a risk concentration channel. Specifically, the interest rate decreases, and the Sharpe ratio increases relative to a frictionless economy. This occurs because liquidity providers, i.e., non-participants, generate an excess supply of the riskless asset, which lowers the interest rate and consequently fuels the leveraged positions (concentration) of market participants (stockholders).

A key contribution of this paper is to explain an often overlooked, yet critical, feature of equilibria with limited participation: the potential for both traded securities to contain bubbles.[2] Specifically, we demonstrate how the emergence of these bubbles is linked to the economy's primitives, thereby establishing this family of models as a suitable framework for microfounding the emergence of assets with no intrinsic value, such as

[1] See e.g. the estimates compiled by BarclayHedge.com in Hedge Fund Industry Assets Under Management.

[2] A bubble on the price of a security is the difference between the market price of the security and its fundamental value, defined as the minimal amount of capital that an unconstrained agent needs to hold to replicate the security's cash flows while maintaining nonnegative wealth (Santos and Woodford, 1997, Loewenstein and Willard, 2000a).

cryptocurrencies, given that the riskless asset in our economy effectively functions as fiat money.

We categorize market participants into three distinct types in our baseline model: *Regular stockholders* are unconstrained in their portfolio choice and are subject to a standard non-negativity constraint on wealth. *Arbitrageurs* differ from regular stockholders along two key aspects. First, they could initially not hold any capital prior to entry. Second, they can pay to access a credit line that guarantees funding, even during transitory periods of negative wealth, thus enabling them to execute more complex investment strategies. *Liquidity providers* may be former stockholders willing to change roles and establish credit facilities for which they receive compensation from arbitrageurs. Critically, they do not participate directly in the stock market but instead provide riskless funding to other market participants.

Our main contributions can be summarized as follows:

First, we investigate the link between stock market participation, investor heterogeneity and the formation of bubbles within an extension of Basak and Cuoco (1998). We derive the equilibrium in closed-form in terms of aggregate consumption and an endogenous state variable that measures the consumption share of liquidity providers, providing rare explicit existence results through pathwise comparison arguments. Our analysis reveals that heterogeneous risk aversion influences both the direction of liquidity needs (i.e., the supply of funds) and the allocation of risk, thereby determining the conditions for bubble emergence. Specifically, we demonstrate that bubbles arise if and only if stockholders are equally or more risk-averse than liquidity providers, that is, they emerge when stockholders must be levered on the stock, absorbing all of the market risk, to encourage them to hold a position that is compatible with market clearing and that is, a priori, a position that is opposite to their preferences absent the participation friction. Conversely, when stockholders are less risk-averse than liquidity providers, both securities are bubble-free. This finding provides a novel microfoundation for equilibrium asset price bubbles, linking funding liquidity and arbitrage strategies to fundamental economic parameters.

Second, we show how credit lines play a central role in the formation of bubbles and the amplification of fundamental shocks. Liquidity providers cause the formation of asset bubbles, as their emergence modifies portfolio strategies across investors so that bubbles, both on the risky and riskless assets, are necessary in equilibrium. Simultaneously, arbitrageurs tap the credit line to exploit mispriced assets and thus actively participate in price correction but generate a pecuniary externality as their trades increase stock price volatility, limiting the ability of regular stockholders to exploit asset bubbles due to their more stringent wealth constraints. This novel feedback loop has significant quantitative implications. Specifically, we rehabilitate the risk concentration channel of limited participation models, challenged by Khorrami (2022). His analysis of an extended Basak and Cuoco (1998) model with entry suggests that if limited participation drives large and volatile risk premia, implied participation costs would need to be exceptionally high, approximately 90% of wealth, to replicate empirically observed asset prices. In our equilibrium, where existing market participants can transition into liquidity providers (non-participants), the effective cost of entry is reduced to approximately 10% of wealth. This role transition is incentive-compatible within our framework. Our model retains the desirable asset pricing dynamics consistent with empirical evidence, including substantial countercyclical excess volatility, risk premia, and low interest rates, which initially made the concentration channel a compelling mechanism.

Third, we endogenize the liquidity constraint by considering a utilitarian regulator who maximizes a social welfare function, taking into account *all* agents. Optimal liquidity balances the marginal benefits and costs of arbitrage for all agents and is consistent with the existence of bubbles in equity and bond markets. Stock bubbles are only arbitraged away when liquidity is maximal, which occurs when the marginal benefit of arbitrage profits is low relative to the starting cash in the economy.

Fourth, we demonstrate that the economy's fragility increases when liquidity is at its maximum. Specifically, we show that unanticipated liquidity shocks, which can trigger the insolvency of arbitrageurs, pose a risk of substantial losses for liquidity providers. The proliferation of liquidity within markets may destabilize financial systems, particularly given the rapid expansion of non-traditional channels such as crypto-lending and decentralized finance (DeFi) platforms in recent years (OECD, 2022). These platforms may present significant regulatory challenges as they achieve greater systemic importance (Azar et al., 2022).[3]

[3]In January 2024, the U.S. Securities and Exchange Commission approved the first spot cryptocurrency exchange-traded funds, marking a significant milestone. U.S. retail and institutional investors, including pension funds, now have direct exposure to cryptocurrencies through regulated products. The top five cryptocurrency ETFs have collectively surpassed 100 billion in assets under management since their inception (iShares Bitcoin Trust ETF, Grayscale Bitcoin Trust (GBTC), Fidelity Wise Origin Bitcoin Fund, ARK 21Shares Bitcoin ETF, Bitwise Bitcoin ETF) and are poised to exceed the assets under management of arbitrage strategy hedge funds.

Related literature

Rational bubbles in continuous-time models were introduced by Loewenstein and Willard (2000a).[4] Most papers in this literature have studied the impact but not the origin of bubbles in partial equilibrium. For example, Cox and Hobson (2005) and Heston et al. (2007) analyze bubbles in the context of option pricing. Jarrow et al. (2010) examine general semi-martingale models with bubbles and Jarrow (2015) provides a recent survey. Equilibrium papers are rare, with notable exceptions being Hugonnier (2012) and Hugonnier and Prieto (2015). More recently, Weston (2022) shows that prices are bubble free in an economy with limited participation, CARA investors, and a financial market comprised of a stock and an annuity, both in positive supply. We show that *both* riskless borrowing and lending and wealth effects play a crucial role in bubble formation. Khorrami (2022) extends Basak and Cuoco (1998) to show that new cohorts paying an entry cost to trade in the stock market prevent participants' wealth from approaching zero, avoiding the explosive behavior of local risk prices observed with bubbles. We provide some nuance by incorporating heterogeneity in preferences and participation frictions to show how liquidity needs and the allocation of risk induced by heterogeneity determine the emergence of bubbles. Even models with explosive state prices may be bubble free. Our results, thus, connect the literature of continuous time bubbles with theories in macro-finance where risk absorption and asset supply/shortages are the channels behind mispricing (Caballero, 2006, Caballero and Krishnamurthy, 2009, Caballero and Simsek, 2016).

Our paper is naturally related to the large body of theoretical literature that studies the impact of frictions in the amplification and propagation of aggregate shocks, see e.g., Bernanke and Gertler (1989), Kiyotaki and Moore (1997) and Bernanke et al. (1999). Importantly, financial constraints and incomplete markets arise endogenously in our setup,[5] and we focus on the risk concentration channel. Earlier contributions include Mankiw and Zeldes (1991), Gomes and Michaelides (2008), Parker and Vissing-Jørgensen

[4]The literature on speculative bubbles, see e.g., Miller (1977), Harrison and Kreps (1978) and Scheinkman and Xiong (2003), uses a different definition of the fundamental value that is not based on any cash flow replication considerations and, therefore, cannot connect bubbles to the existence of arbitrage opportunities. Furthermore, these models are in general set in partial equilibrium as they assume the existence of a riskless technology in infinitely elastic supply. See Brunnermeier and Oehmke (2013) and Simsek (2021) for recent reviews that contrast different theories of bubbles.

[5]See e.g., Detemple and Murthy (1997), Kogan and Uppal (2001), Pavlova and Rigobon (2008), Gârleanu and Pedersen (2011), Bhamra and Uppal (2009), Dumas and Lyasoff (2012), Buss et al. (2016), Chabakauri (2013), He and Krishnamurthy (2013), Rytchkov (2014), Chabakauri (2015), Brunnermeier and Sannikov (2016). A recent survey by Panageas (2020) provides a nice summary of this extensive literature.

(2009), Malloy et al. (2009), Guvenen (2009). More recently, and in addition to Khorrami (2022), Gârleanu et al. (2020) study a model with endogenous entry with a continuum of investors, assets and financial intermediaries. Agents are prevented from trading in all assets due to participation costs and collateral constraints. This implies that diversifiable risk is priced and exposes riskless arbitrage opportunities that cannot be eliminated due to participation costs. By contrast, we show that costly credit lines allow for limited participation and study the feedback effects of various equilibrium configurations with asset bubbles. In Haddad (2014) agents choose dynamically whether to be levered in the stock, bearing more aggregate risk, and entry is free. There are neither risky arbitrage opportunities nor excess volatility as in our model.

Our welfare analysis is related to the large literature on inefficiencies and pecuniary externalities in models with financial market imperfections, going back to Geanakoplos and Polemarchakis (1986). The pecuniary externality that matters in our model is distributive, see e.g. Dávila and Korinek (2017), and comes through an unconventional credit line channel. Our findings are related to those of Gromb and Vayanos (2002, 2008) who investigate the welfare implications of financially constrained arbitrage in an exogenously segmented market with zero net supply securities and an exogenous interest rate. In their model arbitrageurs exploit the riskless arbitrage opportunities that arise across markets and, thereby, allow Pareto improving trade to occur. In contrast, arbitrageurs in our model compensate liquidity providers and their trading activity hinders the welfare of regular stockholders. Relatedly, Guembel and Sussman (2015) and Caballero and Simsek (2016) show that segmentation generally raises volatility and reduces investor welfare. In contrast, we provide a model with *endogenous* incomplete markets and risky arbitrage opportunities that persist in equilibrium despite the presence of unconstrained market participants in the economy. Hébert (2022) explores an equilibrium model with financial intermediaries and limited participation, where short-lived arbitrage opportunities exist but are not fully exploitable due to regulation of intermediaries and limited participation of other agents. We build on a similar theme, but importantly, we show how arbitrage opportunities emerge endogenously from entry.

The remainder of the paper is organized as follows. In Section 2 we present assumptions about the economy and solve two benchmark economies with complete and incomplete markets. In Section 3 we extend the model to include heterogeneous risk aversion and explain the emergence of bubbles. In Section 4 we motivate the credit facilities and derive equilibrium in the economy. We provide existence results, and analyze the main properties and welfare implications. In Section 5 we solve the optimal liquidity problem and discuss the impact of unanticipated liquidity shocks. We conclude in Section 6. All proofs are gathered in Appendix A. Additional results can be found in the Online Appendix.

2 Bubbles in equilibrium

We review a complete market economy with three agents who trade on a stock and a riskless asset, then a similar yet incomplete market economy with an exogenous participation constraint. These economies are variations of the two-agent economies in Basak and Cuoco (1998). Bubbles on both the stock and the riskless asset are necessary for markets to clear (Hugonnier, 2012).

We present novel computations for welfare functions in the incomplete market case, which serves as benchmark for our economy with endogenous stock market participation in Section 4.

2.1 Securities markets and bubbles

Primitive securities. We consider a continuous-time economy on an infinite horizon. Uncertainty is represented by a probability space carrying a Brownian motion Z_t. In what follows, we assume that all random processes are adapted with respect to the usual augmentation of the filtration generated by this Brownian motion.

Agents trade in two securities: a money market account in zero net supply and a stock in positive supply of one unit. The price of the riskless asset evolves according to

$$S_{0t} = 1 + \int_0^t S_{0u} r_u du$$

for $t > 0$ and an interest rate process r_t that is determined in equilibrium. The stock is a claim to a dividend process e_t that evolves according to a geometric Brownian motion with constant drift μ_e and volatility $\sigma_e > 0$. The stock price is denoted by S_t and evolves according to

$$S_t + \int_0^t e_u du = S_0 + \int_0^t S_u(\mu_u du + \sigma_u dZ_u)$$

for $t > 0$ for some initial value $S_0 > 0$, drift μ_t, and volatility σ_t that are determined in equilibrium.

A price system (S_{0t}, S_t) may contain risky arbitrages but no riskless arbitrages, for otherwise the market could not be in equilibrium. This implies that $\mu_t = r_t + \sigma_t \theta_t$ for some process θ_t. This process is referred to as the market price of risk and is uniquely defined on the set where volatility is non zero. Now consider the state price density defined by

$$\xi_t = \frac{1}{S_{0t}} \exp\left(-\int_0^t \theta_u dZ_u - \frac{1}{2} \int_0^t |\theta_u|^2 du \right). \tag{1}$$

The ratio $\xi_{t,u} = \xi_u / \xi_t$ is the pricing kernel used to characterize the feasible sets of agents facing complete markets, as well as to identify fundamental values and rational bubbles on any contingent claim.[6]

Bubbles. We refer to $F_t \equiv \mathbb{E}_t \left[\int_t^\infty \xi_{t,u} e_u du \right]$ as the *fundamental value* of the stock and to

$$B_t \equiv S_t - F_t = S_t - \mathbb{E}_t \left[\int_t^\infty \xi_{t,u} e_u du \right] \geq 0$$

as the *bubble* on its price. Bubbles are consistent with equilibrium because they constitute only *limited* arbitrage opportunities due to wealth constraints.[7] To see this, assume that the stock has a bubble and consider the textbook strategy that sells short $x > 0$ units of the stock, buys the portfolio that replicates the corresponding dividends and invests the proceeds in the riskless asset until some fixed date T. The wealth process of this trading strategy is given by

$$A_t(x;T) = x\left(F_t(T) - S_t\right) + x S_{0t}\left(S_0 - F_0(T)\right) = x\left(S_{0t} B_0(T) - B_t(T)\right) \tag{2}$$

where $F_t(T) \equiv \mathbb{E}_t \left[\int_t^T \xi_{t,u} e_u du + \xi_{t,T} S_T \right]$ denotes the fundamental value of the stock over the interval $[t, T]$, and

$$B_t(T) \equiv S_t - F_t(T) = S_t - \mathbb{E}_t \left[\int_t^T \xi_{t,u} e_u du + \xi_{t,T} S_T \right] \geq 0 \tag{3}$$

denotes the corresponding finite horizon bubble. This dynamic trading strategy requires no initial investment and has positive terminal value $A_T(x;T) = x B_0(T) S_{0T} > 0$. Note

[6]See e.g. Santos and Woodford (1997), Loewenstein and Willard (2000a), Heston et al. (2007).

[7]In the model we develop in Section 4, regular stockholders will keep nonnegative wealth whereas arbitrageurs will have access to a credit line that allows them to withstand periods of negative wealth. See Dybvig and Huang (1988) for a discussion of the use of wealth constraints to prevent doubling strategies versus other constraints, such as integrability constraints on portfolio strategies under a risk-neutral probability measure. As we explain below, within our equilibrium model such a measure may not exist.

however that this arbitrage opportunity is *risky* because it entails interim losses with strictly positive probability and, therefore, cannot be implemented to an arbitrary scale by the agents in the economy as they face wealth constraints.

Bubbles are similarly defined for the riskless asset. Indeed, over a time interval $[0, T]$ the money market account can be viewed as a derivative that pays a single lump dividend equal to S_{0T} at date T. The fundamental value of such a security is $F_{0t}(T) = \mathbb{E}_t [\xi_{t,T} S_{0T}]$ whereas its market value is simply S_{0t}, and this naturally leads to defining the finite horizon bubble on the riskless asset as

$$B_{0t}(T) \equiv S_{0t} - F_{0t}(T) = S_{0t} \left(1 - \mathbb{E}_t \left[\xi_{t,T} \frac{S_{0T}}{S_{0t}}\right]\right). \tag{4}$$

As was the case for the stock, a bubble on the riskless asset is consistent with both optimal choice and the existence of an equilibrium in our economy.[8]

2.2 Trading strategies

A trading strategy is a pair $(\pi_t; \phi_t)$, where π_t represents the amount invested in the stock and ϕ_t represents the amount invested in the riskless asset. A trading strategy is said to be self-financing given initial wealth w and consumption rate $c_t \geq 0$ if the corresponding wealth process satisfies

$$W_t = \pi_t + \phi_t = w + \int_0^t (\phi_u r_u + \pi_u \mu_u - c_u)\, du + \int_0^t \pi_u \sigma_u dZ_u. \tag{5}$$

Implicit in the definition is the requirement that the trading strategy and consumption plan be such that the above stochastic integrals are well defined.

2.3 Agents and endowments

The economy is populated by agents indexed by $k \in \{1, 2, 3\}$ whose preferences are represented by

$$U_k^i(w_k) \equiv \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log(c_t) dt\right]$$

[8]Equation (4) shows that the riskless asset has a bubble over $[0, T]$ if and only if the process $M_t \equiv S_{0t}\xi_t$ satisfies $\mathbb{E}[M_T] < M_0 = 1$. Since the economy is driven by a single source of risk this process is the unique candidate for the density of the risk-neutral probability measure and it follows that the existence of a bubble on the riskless asset is equivalent to the non existence of the risk-neutral probability measure.

for some subjective discount rate $\rho > 0$, where $i \in \{C, I\}$ stands for complete and incomplete markets, respectively. Agents have homogeneous preferences and beliefs but may differ in their trading opportunities and endowments. Agents are free to choose any strategy whose wealth remains non-negative, so that $W_{kt} \geq 0$, $\forall k$. Agent 1 is endowed with $w_1 = S_0 - K > 0$, i.e., the stock and a short position in cash, Agent 2 is endowed only with cash $w_2 = K > 0$, and Agent 3 has null initial endowment. Since $S_{00} = 1$, the cash amount K is the number of bonds held at the initial price. The endowment structure for the first two agents is the same as in the two-agent model of Basak and Cuoco (1998), but the economy has the additional Agent 3.

Assumption 1. *Let* $K = ne_0/\rho$ *with* $n \in (0, 1)$.

2.4 Definition of equilibrium

The concept of equilibrium that we use is similar to that of equilibrium of plans, prices and expectations introduced by Radner (1972).

Definition 1. *An equilibrium is a pair of security price processes* (S_{0t}, S_t) *and an array* $\{c_{kt}, (\pi_{kt}; \phi_{kt})\}_{k=1}^{3}$ *of consumption plans and trading strategies such that (1) given* (S_{0t}, S_t) *the plan* c_{kt} *maximizes* U_k *over the feasible set of Agent* k *and is financed by the trading strategy* (π_{kt}, ϕ_{kt})*; (2) markets clear:* $\sum_{k=1}^{3} \phi_{kt} = 0$, $\sum_{k=1}^{3} \pi_{kt} = S_t$ *and* $\sum_{k=1}^{3} c_{kt} = e_t$.

2.5 Complete markets

When agents trade on both securities it is well known that there exists a Pareto optimal no-trade equilibrium. The stock and riskless asset prices are given by $S_t = P_t \equiv e_t/\rho$, $S_{0t} = e^{rt}$, where the interest rate and market price of risk processes are constants, respectively given by $r = \rho + \mu_e - \sigma_e^2$ and $\theta = \sigma_e$. The optimal consumption plans are described by

$$c_{1t} = (1 - n)e_t, \quad c_{2t} = ne_t, \quad c_{3t} = 0, \tag{6}$$

so that the welfare indices are obtained from simple calculations.

Proposition 1. *Welfare functions of Agents 1 and 2 are given by*

$$U_1^C(n) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log\left((1-n)e_t\right)dt\right] = U_0 + \frac{1}{\rho}\log(1-n), \tag{7}$$

$$U_2^C(n) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log\left(ne_t\right)dt\right] = U_0 + \frac{1}{\rho}\log(n), \tag{8}$$

where the constant U_0 is given by

$$U_0 \equiv \frac{1}{\rho}\log e_0 + \frac{\mu_e - \sigma_e^2/2}{\rho^2}. \tag{9}$$

The welfare of Agent 3 is $U_3^C(0) = -\infty$.

The utility indices in (7)–(8) depend on initial wealth and exhibit similar dependence on the moments of the aggregate dividend growth rate because $U^C(w_k) = (1/\rho)\log(\rho w_k) + (\mu_e - \sigma_e^2/2)/\rho$ for $k \in \{1, 2\}$.[9]

2.6 Incomplete markets with exogenous constraint

Consider now the case of incomplete markets. As in Basak and Cuoco (1998), Agent 1 has access to a complete financial market, while Agent 2 is *exogenously* limited to trade only in the money market account.[10] Agent 3's null initial endowment naturally results in no trading. Prices can be readily characterized as follows.

Proposition 2. *There exists a unique equilibrium. The stock price and money market account are given by*

$$S_t = P_t, \quad S_{0t} = e^{\rho t}\frac{P_t}{P_0}\frac{s_t}{s_0}, \tag{10}$$

where $s_t = c_{2t}/e_t$ corresponds to the consumption share of Agent 2 and $P_t = e_t/\rho$. The interest rate and the market price of risk are time-varying and given by

$$r_t = \rho + \mu_e - \sigma_e^2\left(1 + \frac{s_t}{1-s_t}\right),$$

$$\theta_t = \sigma_e\left(1 + \frac{s_t}{1-s_t}\right),$$

[9] Setting $U_k^C(n) = U^C(w_k)$ for $k \in \{1, 2\}$ and substituting the values of endowments w_k evaluated at the equilibrium price $S_0 = e_0/\rho$ gives the formulas stated.

[10] The dynamic model in Basak and Cuoco (1998) builds on a number of models incorporating agents with restricted access to the stock market that were limited to one- or two-period settings, see e.g., Merton (1987), Hirshleifer (1988), Allen and Gale (1994) and Balasko et al. (1995).

where the consumption share process s_t evolves according to

$$ds_t = -s_t\sigma_e\left(dZ_t + \frac{s_t}{1-s_t}\sigma_e dt\right), \tag{11}$$

with initial condition $s_0 = n \in (0,1)$.

Given logarithmic utility function, the optimal consumption policy of Agent $k \in \{1,2,3\}$ is proportional to wealth: $c_{kt} = \rho W_{kt}$. Agent 3, whose wealth remains null at all times, neither consumes nor invests. By definition of the consumption shares $((1-s_t), s_t)$, we can also write $W_{1t} = (1-s_t)e_t/\rho$ and $W_{2t} = s_t e_t/\rho$. The stock price, which by market clearing equals aggregate wealth, is then given by $S_t = W_{1t} + W_{2t} = (1-s_t)e_t/\rho + s_t e_t/\rho$, leading to the formula in (10). At a given time t, it can be interpreted as the value of a perpetuity with constant future cash flow e_t in a market with discount factor $b_t^{(\rho)} = e^{-\rho t}$. The bond price is related to the optimal wealth of Agent 2. Since optimal consumption of Agent 2 is proportional to wealth $dW_{2t}/W_{2t} = dS_{0t}/S_{0t} - \rho dt = d(b_t^{(\rho)}S_{0t})/(b_t^{(\rho)}S_{0t})$, which gives $W_{2t} = b_t^{(\rho)}S_{0t}$ up to a multiplicative constant. Using the normalization $S_{00} = 1$ and the expression for W_2 gives the formula stated (see Appendix for details).

It is no surprise that this exogenously imposed constraint hinders non-participants' welfare, as we see next. We use the superscripts $\{p, np\}$ to denote if the agent *p*articipates or does *n*ot *p*articipate in the stock market and provide welfare functions, which are available in closed-form.

Proposition 3. *The welfare functions of Agents 1 and 2 are given by*

$$U_1^{I,p}(n) \equiv \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log\left((1-s_t)e_t\right)dt\right] = U_0 - \frac{1}{1-n}\sum_{j=1}^{\infty}\frac{1}{j}u(n,j), \tag{12}$$

where

$$u(n,j) \equiv \frac{n^j - n^\kappa}{\rho - j(j-1)\frac{\sigma_e^2}{2}} - \frac{n^{j+1} - n^\kappa}{\rho - j(j+1)\frac{\sigma_e^2}{2}},$$

and

$$U_2^{I,np}(n) \equiv \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log\left(s_t e_t\right)dt\right] = U_0 + \frac{1}{\rho}\log n - \frac{\sigma_e^2}{2\rho^2}\frac{1+n-2n^\kappa}{1-n}, \tag{13}$$

where the value of U_0 is given in (9). Agent 3's welfare function is $U_3^I(0) = -\infty$.

The next result provides limits and comparative statics with respect to initial wealth and formalizes an intuitive result. As shown in (14) and (15), for the same level of initial wealth, summarized by n, Agent 1 (Agent 2) is better (worse) off in the economy with incomplete markets, and the effects are decreasing (increasing) in the cash endowment n.[11]

Lemma 1. *The following inequalities hold for* $n \in (0, 1)$,

$$U_1^{I,p}(n) \geq U_1^C(n), \tag{14}$$

$$U_2^{I,np}(n) \leq U_2^C(n), \tag{15}$$

$$sign\left(\partial U_1^{I,p}/\partial n\right) = sign\left(\partial U_1^C/\partial n\right) \leq 0,$$

$$sign\left(\partial U_2^{I,np}/\partial n\right) = sign\left(\partial U_2^C/\partial n\right) \geq 0.$$

Furthermore, the limits when $n \downarrow 0$ *or* $n \uparrow 1$ *are given by*

$$\lim_{n\to 0} U_1^{I,p}(n) = \lim_{n\to 0} U_1^C(n) = U_0,$$

$$\lim_{n\to 0} U_2^{I,np}(n) = \lim_{n\downarrow 0} U_2^C(n) = -\infty,$$

$$\lim_{n\to 1} U_1^{I,p}(n) = U_0 - \frac{1}{\rho}\left(H_{\kappa-1} - \frac{\kappa-1}{\kappa}\right),$$

$$\lim_{n\to 1} U_2^{I,np}(n) = U_0 - \frac{\sigma_e^2}{2\rho^2}(2\kappa - 1),$$

where H_x *denotes the Harmonic Number.*

The equilibrium with incomplete markets is very compelling as both primitive securities in this equilibrium contain bubbles. The next proposition, adapted from (Hugonnier and Prieto, 2015, Prop. 5), shows the relative size of the bubbles on both the risky and the riskless asset over any investment horizon.

Proposition 4. *Over the time interval* $[t, T]$ *the stock and the riskless asset include bubble components that satisfy*

$$\frac{B_t(T)}{S_t} = H\left(T - t, s_t; 2\kappa - 1\right) \frac{B_t}{S_t} \leq H\left(T - t, s_t; 1\right) = \frac{B_{0t}(T)}{S_{0t}}, \tag{16}$$

[11]Since the stock price is invariant to Agent 2's portfolio position, welfare indices would be the same if Agent 2 were endowed with an amount n of shares of the stock and no cash ($K = 0$) and were allowed to divest at time zero.

where $B_t = s_t^{\kappa} P_t$ *and*

$$H(\tau, s; a) = N(d_+(\tau, s; a)) + s^{-a} N(d_-(\tau, s; a)), \tag{17}$$

$$\kappa = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\rho}{\sigma_e^2}}, \tag{18}$$

with $d_{\pm}(\tau, s; a) = \frac{1}{\sigma_e \sqrt{\tau}} \left(\log(s) \pm \frac{a}{2} \sigma_e^2 \tau \right)$ *and* $N(x)$ *is the standard normal cumulative distribution function. The fundamental value of the stock is* $F_t = S_t - B_t = (1 - s_t^{\kappa}) P_t$.

There are a number of observations that one can draw from this result. First, as seen in (16) for each dollar of investment, the bubble on the riskless asset is larger than that on the stock over any investment horizon. This relative ordering will have a direct interpretation on the stockholder's holdings. Second, this price system supports an equilibrium with no trade in the stock, but continuous trading in the money market account. Since Agent 1 is subject to a non-negative wealth constraint and cannot short both securities at the same time, she chooses a strategy that exploits the bubble on the riskless asset because it is more profitable. Her equilibrium portfolio consists thus of holding the stock and borrowing, at the risk free rate, an amount equal to Agents 2's wealth,

$$W_{1t} = c_{1t}/\rho = (1 - s_t) P_t = P_t - W_{2t}. \tag{19}$$

One can alternatively view Agent 1's portfolio holdings as one that shorts the riskless asset bubble and uses the stock as collateral, since she must maintain non-negative wealth. The wealth of Agent 1 expressed as a self-financing strategy in the stock and the riskless asset bubble over the interval $(t, T]$ can be written as

$$W_{1t} = \phi_{1t}^{S}(T) + \phi_{1t}^{B_0}(T) \tag{20}$$

where the positions have the signs

$$\phi_{1t}^{S}(T) = \frac{1 - \Sigma_{0t}(T)(1 - s_t)}{1 - \Sigma_{0t}(T)} P_t \geq 0$$

$$\phi_{1t}^{B_0}(T) = W_{1t} - \phi_{1t}^{S}(T) = -\frac{s_t}{1 - \Sigma_{0t}(T)} P_t \leq 0$$

and the process $\Sigma_{0t}(T)$ is the diffusion coefficient of $(1/\sigma_e) \log B_{0t}(T)$, with $\lim_{T \to \infty} \Sigma_{0t}(T) = 0$. Note that this limit recovers the primitive strategy in (19).

Remark 1. *Over an infinite investment horizon, the bubble on the riskless asset converges to the market value of the riskless asset*

$$\lim_{T\to\infty} B_{0t}(T) = \lim_{T\to\infty} H\left(T-t; s_t; 1\right) S_{0t} = S_{0t}. \tag{21}$$

As a result, the fundamental value of the riskless asset over an infinite investment horizon is zero *and it follows that the money market account is akin to fiat money, micro-founding the emergence of assets with no intrinsic value, such as cryptocurrencies, as store of value.*

3 Investor heterogeneity and bubble formation

We solve an extension to the benchmark model which explains the emergence of asset bubbles in equilibrium models based on liquidity flows and portfolio imbalances.

3.1 Marginal utility process

The behavior of the relative marginal utility process

$$\lambda_t = \frac{u_1'\left(c_{1t}\right)}{u_2'\left(c_{2t}\right)} = \frac{c_{2t}}{c_{1t}} = \frac{s_t}{1-s_t} = \frac{c_{20}}{c_{10}} + \int_0^t \sigma_e \lambda(s_u)\left(1+\frac{s_u}{1-s_u}\right) dZ_u, \tag{22}$$

determines the existence of bubbles on the stock and the riskless asset. Indeed, since

$$\begin{aligned} S_t &= W_{1t} + W_{2t} = \mathbb{E}_t\left[\int_t^\infty \xi_{t,u}(e_u - c_{2u})du\right] + \frac{c_{2t}}{\rho} \\ &= F_t + s_t P_t - \mathbb{E}_t\left[\int_t^\infty \xi_{t,u} s_u e_u du\right], \text{ and} \end{aligned}$$

$$\xi_{t,u} = b_{t,u}^{(\rho)}\frac{c_{1t}}{c_{1u}} = b_{t,u}^{(\rho)}\frac{(1-s_t)e_t}{(1-s_u)e_u}; \qquad \xi_{t,u}\frac{s_u e_u}{s_t e_t} = b_{t,u}^{(\rho)}\frac{\lambda_u}{\lambda_t}; \qquad \xi_{t,T}\frac{S_{0T}}{S_{0t}} = \frac{\lambda_T}{\lambda_t}$$

where $b_{t,u}^{(\rho)} = b_u^{(\rho)}/b_t^{(\rho)}$ and $F_t = \mathbb{E}_t\left[\int_t^\infty \xi_{t,u} e_u du\right]$, the stock and bond price bubbles are

$$B_t = S_t - F_t = \rho s_t P_t \int_t^\infty e^{-\rho(u-t)}\left(1 - \mathbb{E}_t\left[\frac{\lambda_u}{\lambda_t}\right]\right) du, \tag{23}$$

$$B_{0t}(T) = S_{0t}\left(1 - \mathbb{E}_t\left[\xi_{t,T}\frac{S_{0T}}{S_{0t}}\right]\right) = S_{0t}\left(1 - \mathbb{E}_t\left[\frac{\lambda_T}{\lambda_t}\right]\right). \tag{24}$$

In our baseline economy, prices contain bubble components as λ_t is a positive strict local martingale, hence a strict supermartingale, implying $\lambda_t > \mathbb{E}_t[\lambda_u]$.[12] Inspecting (22), it is

[12] For a detailed derivation, see (A.9) in the Appendix

clear that the non-participation constraint exogenously imposed on Agent 2 determines the emergence of bubbles, since absent the constraint, the relative consumption process is fixed at level $\lambda_0 = c_{20}/c_{10}$ and bubbles would be identically zero in (23)–(24).

3.2 Heterogeneous risk aversion

Portfolio frictions are necessary but not sufficient for bubbles to emerge. We explain next how investor heterogeneity sheds light on bubble formation with an extension. We let Agent 1 have CRRA preferences, $u_1(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ with $\gamma > 0$, as in He and Krishnamurthy (2013) and relax Agent 2's participation constraint. We introduce *partial* participation in the stock market by limiting Agent 2's portfolio volatility, that is,

$$\mathcal{C}_t = \left\{ |\bar{\pi}_t \sigma_t|^2 \leq (\varepsilon \sigma_e)^2 \right\},$$

with $\bar{\pi}_t = \pi_t/W_t$. The parameter $\varepsilon \in [0, 1)$ is a measure of Agent 2's risk bearing capacity, and hence, of her stock market participation.[13] The optimal policy is given by $\bar{\pi}_{2t} = k_t \theta_t / \sigma_t$ where the process $k_t = \left[1 + (|\theta_t| / (\varepsilon \sigma_e) - 1)^+\right]^{-1} \in [0, 1]$ is the reduction in risk taking induced by the constraint. The fact that k_t depends only on the absolute value of the market price of risk, and not on the price level or stock volatility, is the key to a tractable characterization of equilibrium. There are two regions which are determined by the primitives of the economy.

Proposition 5. *The market price of risk is given by*

$$\theta(s_t) = \begin{cases} \gamma \sigma_e \frac{1-\varepsilon s_t}{1-s_t}, & R(s_t) > \varepsilon, \\ \frac{\gamma \sigma_e}{1+(\gamma-1)s_t}, & R(s_t) \leq \varepsilon, \end{cases} \tag{25}$$

where $R(s) = \gamma[1 + (\gamma - 1)\, s]^{-1}$ *is aggregate relative risk aversion and* $s_t = c_{2t}/e_t$.

The constraint is active in states where the aggregate relative risk aversion is relatively high, $R(s_t) > \varepsilon$. The pair $(\varepsilon, \gamma) = (0, 1)$ recovers the baseline model. To understand the interaction between risk bearing capacity and risk aversion, let $\gamma > 1$. The constraint is active in all states since $R(s_t) \geq 1 > \varepsilon$. In these states, Agent 2 is less risk averse and the constraint will limit her position in the stock. Now let $\gamma < 1$, which implies that $\gamma \leq R(s_t) \leq 1$. The constraint binds in all states when $\varepsilon \in [0, \gamma)$. As the risk bearing capacity of Agent 2 increases, such that $\varepsilon \in [\gamma, 1)$, the constraint is inactive since $k_t = 1$.

[13] Partial equilibrium implications of risk constraints in dynamic settings have been studied by Cuoco et al. (2008), Gârleanu and Pedersen (2007) among others.

The relative marginal utility process under this extended model is given by

$$\lambda_t = \frac{u_1'(c_{1t})}{u_2'(c_{2t})} = \frac{c_{2t}}{c_{1t}^{\gamma}} = s_t(1-s_t)^{-\gamma} e_t^{1-\gamma} = \lambda_0 + \int_0^t \lambda_u \Phi(s_u) du$$

where $\Phi(s_t) = (k(s_t) - 1)\theta(s_t)$. As in the baseline model, its behavior determines the emergence of bubbles in equilibrium.

Proposition 6. *The stock contains a bubble if* $\gamma \geq 1$*. The stock is free of bubbles if* $\gamma < 1$.

The emergence of bubbles in the price system reflects an intuitive equilibrium mechanism: since Agent 1 clears markets with Agent 2, bubbles arise to encourage Agent 1 to do so, in other words, to mitigate an implicit liquidity provision constraint faced by Agent 1. The latter is determined by the severity of the portfolio constraint and the risk aversion disparity across agents. Bubbles thus are linked to how costly the constraint for Agent 2 is. The importance of both dimensions follows from the fact that the region where the constraint binds is determined by both parameters. This result connects the literature on continuous time bubbles with theories in macro-finance where risk absorption and asset shortages are the channels behind securities mispricing (see e.g., Caballero (2006), Caballero and Krishnamurthy (2009), and Caballero and Simsek (2016)).[14]

Revisiting the baseline model with $\varepsilon = 0$, so that Agent 2 cannot hold the stock as in Basak and Cuoco (1998), provides the intuition. Bubbles arise in the stock and in the riskless asset if Agent 1 is equally or more risk averse than Agent 2 (i.e. , $\gamma \geq 1$). That is, they emerge when the stockholder must be levered on the stock in order to hold a position that is compatible with market clearing, even though it is, a priori, a position opposite her preferences absent the participation friction. Prices are free of bubbles if the stockholder (Agent 1) is less risk averse than the liquidity provider (Agent 2). These positions with $\varepsilon = 0$, in contrast, coincide with the unconstrained flows whereby Agent 1, because she is less risk averse, would hold a levered on the stock and absorb all of the market risk, while borrowing from Agent 2 at the risk free rate.

4 Endogenous stock market participation

In this section, we examine how stock market participation frictions arise *endogenously* by extending the model in Hugonnier and Prieto (2015) where some agents investors tap

[14]Unlike these contributions, our equilibrium is constructed in a model with aggregate risk and risk averse investors.

an exogenous credit line to better exploit asset bubbles. An important contrast with their model is that in our model credit lines are priced under incomplete markets using explicit individual welfare functions, and non participating agents are fully compensated. This result confirms the intuition that bubbles emerge as an equilibrium outcome to incentivize *unconstrained* agents.

We provide existence results that highlight the different forms of entry and change of roles in equilibrium. For example, a stock market participant may become non-participant, and this shift may be important to explain the levels and dynamics of asset prices. We also provide a welfare result that formalizes a pecuniary externality at play that impacts stockholders.

4.1 Investor types

We construct equilibria with three types of agents, stockholders, liquidity providers and arbitrageurs, described as follows:

1. Stockholders (h) are agents subject to a standard wealth constraint $W_{ht} \geq 0$. They are akin to Agent 1 in our baseline model.

2. Liquidity providers (ℓ) are non-participating agents who are compensated upfront and trade only through the money market account. They are subject to a standard wealth constraint $W_{\ell t} \geq 0$. They are akin to Agent 2 of our baseline model with the fundamental difference that they are compensated.

3. Arbitrageurs (a) compensate non-participants and have access to a credit line that allows them to withstand short-term deficits provided their wealth satisfies the lower bound[15]

$$W_{at} + \psi_a S_t \geq 0, \qquad t \geq 0 \tag{26}$$

where $\psi_a > 0$ is a positive constant.

To grasp how an arbitrageur can exploit the stock bubble, consider again the strategy in (2). This arbitrage opportunity is *risky* because it entails the possibility of temporary losses, $\mathbb{P}(A_t(x;T) \leq 0) \geq 0$, however, arbitrageurs can implement this trade up to size ψ_a because of the credit line embedded in (26), and thus generate arbitrage rents.

[15] See e.g., Loewenstein and Willard (2000b, 2013) for additional examples for how wealth constraints allow investors to exploit bubbles.

This credit facility may represent other types of short-term loan arrangements, including non-traditional channels of liquidity creation that have boomed over the past years with crypto lending and various DeFi platforms. These lending channels, based on financial assets with zero fundamental value such as our riskless asset, see (21), bypass the traditional financial firms that act as gatekeepers for loans or other products.[16]

As we will see below, arbitrageurs will pay liquidity providers an upfront fee to access it.[17] To capture the fact that the availability of arbitrage capital tends to be procyclical (Ang et al., 2011, Ben-David et al., 2012), the facility is proportional to the market portfolio.

4.2 Optimal policies

The optimal consumption–portfolio policy of an arbitrageur in the presence of stock bubbles and the credit facility is given by

$$c_{at} = \rho \left(W_{at} + \psi_a B_t \right), \tag{27}$$

$$\pi_{at} = (\theta_t/\sigma_t)(W_{at} + \psi_a B_t) - \psi_a(\Sigma_t^B/\sigma_t), \tag{28}$$

where the process Σ_t^B denotes the diffusion coefficient of the bubble process B_t and we assume for now (and later verify) $\sigma \neq 0$. For regular stockholders and liquidity providers, the optimal policies are summarized by $(c_{ht}, \pi_{ht}) = W_{ht}(\rho, \theta_t/\sigma_t)$ and $(c_{\ell t}, \pi_{\ell t}) = W_{\ell t}(\rho, 0)$, respectively.

The optimal consumption policy of an arbitrageur, in (27), is proportional to total wealth, i.e., liquid wealth augmented by the value of the credit line. The optimal portfolio, in (28), has two components. The first is a standard mean-variance component, but now proportional to total wealth. The second is a hedging term against fluctuations of the bubble. Even though the standard logarithmic investor is myopic, this is not the case here as her total wealth is exposed to fluctuations in the bubble, which prompts her to hedge that exposure. In contrast, the consumption-portfolio policies of the regular stockholders and the liquidity providers have the standard structures. The result in (27)

[16] One of DeFi's most noteworthy products is an unsecured *flash loan*, which is mainly used to exploit arbitrage opportunities and yield farming strategies (Schär, 2021).

[17] This is, of course, a simplified structure. A typical credit line contract specifies a maximum amount that can be drawn over a given period (i.e., the commitment), an interest rate that applies to the amount borrowed, and various fees, such as an *upfront* commitment fee, an annual fee levied on the total amount committed and a usage fee levied annually on the undrawn portion of the commitment. Few credit lines, however, carry all three types of fees, as most of them usually have a usage fee combined with an upfront fee (Loukoianova et al., 2007).

and (28) is notable because it indicates that non-zero consumption plans are feasible even for investors like Agent 3 endowed with zero initial wealth. The credit facility effectively allows for an additional income stream due to arbitrage profits.

Optimal consumption policies can alternatively be written in terms of aggregate consumption shares

$$c_{ht} = \frac{1}{1+\nu}(1-s_t)e_t; \qquad c_{lt} = s_t e_t; \qquad c_{at} = \frac{\nu}{1+\nu}(1-s_t)e_t$$

where ν is a positive constant and s_t a stochastic sharing rule. Optimal (total) wealth of each agent is proportional to consumption as described in (27) for an arbitrageur.

4.3 Prices and bubbles

Optimal policies displayed above have important equilibrium implications. From market clearing, the stock price satisfies

$$S_t = W_{1t} + W_{2t} + W_{3t} = \frac{1}{1+\nu}(1-s_t)P_t + s_t P_t + \frac{\nu}{1+\nu}(1-s_t)P_t - \psi_a B_t$$

so that

$$S_t = P_t - \psi_a B_t = F_t + B_t = (1-\alpha)P_t + \alpha F_t \tag{29}$$

with $\alpha \equiv \psi_a/(1+\psi_a)$. The stock price is thus a linear combination between the benchmark bubbly price P_t and its fundamental value F_t. From equation (29) and noting that from Proposition 4, the fundamental value of the stock is given by $F_t = (1-s_t^\kappa)P_t$ where κ is given in (18), $s_t = c_{\ell t}/e_t$ is the consumption share of the liquidity provider and the stock price and its bubble are given by

$$S_t = \left(1-\alpha s_t^\kappa\right) P_t, \tag{30}$$

$$B_t = S_t - F_t = (1-\alpha)s_t^\kappa P_t. \tag{31}$$

Absent the credit line, that is when $\alpha = 0$, the equilibrium price of the stock is given by the usual valuation formula with logarithmic utility $S_t = P_t = e_t/\rho$. Applying Ito's lemma to (30) shows that the stock price volatility is strictly positive and excess volatility is given by $\sigma_t - \sigma_e = v(s_t;\alpha)\sigma_e$ where $v(s;\alpha) \equiv \kappa\alpha s^\kappa/(1-\alpha s^\kappa)$. Excess volatility in this model thus increases with both the liquidity provider's consumption share s_t and the size of the credit line, summarized by α, and goes in the right direction of matching the empirical counterparts, as illustrated in Figure 1.

[Insert Figure 1 about here]

Notably, the equilibrium price system with bubbles (30)–(31) will exist as long as there are well-defined consumption sharing rules $s_{kt} = c_{kt}/e_t$ for $k \in \{1, 2, 3\}$.

4.4 Aggregation and entry

Before we explain how each investor will be classified under a given type $\{a, h, \ell\}$ in our equilibrium, we make two remarks about the credit facility that will help provide intuition for our results. First, one can think of the arbitrageur as a *representative* arbitrageur composed by I investors with access to credit lines of size ψ_i. Indeed, the representative arbitrageur's consumption in (27) under this setting would correspond to

$$c_{at} = \rho \underbrace{\sum_{i \in I} W_{it}}_{W_{at}} + \rho \underbrace{\sum_{i \in I} \psi_i}_{\psi_a} B_t,$$

so that $\alpha = \sum_i \psi_i / (1 + \sum_i \psi_i) = \psi_a / (1 + \psi_a)$ as before. Second, its size ψ_a can be determined by entry. Take the following example: Assume that the economy is populated by a unit mass continuum of price taking arbitrageurs who do not hold any initial capital prior to entry and let entry be subject to a one-time fixed cost $X > 0$, which amounts to the compensation received by liquidity providers as we explain in the next section. Conditional on entry, a given arbitrageur benefits from a credit facility of size $\omega \geq 0$ and the cross-sectional distribution is represented by an exogenously given cumulative distribution function $\Psi(\omega)$ with support $[0, \infty)$.

An arbitrageur enters the market if and only if her credit facility is such that

$$\omega b - X > 0$$

where $b \geq 0$ gives the initial bubble on the stock price. The aggregate consumption by arbitrageurs is

$$c_{at} = \rho \psi(b) (e^{-\rho t} / \xi_t) b \quad \text{with} \quad \psi(b) = \int_0^\infty (\omega - X/b)^+ \, d\Psi(\omega).$$

This in turn implies that the equilibrium bubble is explicitly given by

$$B_t(b) = (1 - \alpha(b))\, s_t^{\kappa} P_t$$

where s_t denotes the liquidity provider's consumption share in an economy with $\alpha(b) = \frac{\psi(b)}{1+\psi(b)}$. The initial value of the bubble on the stock solves the fixed point equation

$$f(b) = -b + B_0(b) = 0.$$

In what follows, we abstract from aggregation and focus on the participation decision and the cost X. However, the take-away is that our economy with credit lines can be easily mapped into more sophisticated versions of types. We get back to this interpretation when we compute for the credit line's optimal size in Section 5.

4.5 Participation decision

The participation decision is at date 0. Non-participation is incentive compatible, meaning that if an existing market participant $\ell = \{1, 2\}$ becomes the liquidity provider, the compensation amount X_ℓ she receives must be such that she is at least as well off by non-participating,

$$U^{\mathrm{C}}(w_\ell) \leq U^{\mathrm{I,np}}(w_\ell + X_\ell). \tag{32}$$

This inequality compares the utility of the agent under complete markets and endowment w_ℓ with her non-participating utility but compensated upfront so that she holds initial wealth $w_\ell + X_\ell$.[18] If Agent 3 becomes the liquidity provider, agents $a \in \{1, 2\}$ may be arbitrageurs as long as

$$U^{\mathrm{C}}(w_a) \leq U^{\mathrm{I,p}}(w_a + \psi_a B_0 - X_{a\ell}). \tag{33}$$

This inequality compares the utility of the agent under complete markets and endowment w_a with her utility under incomplete markets and access to the credit line of size ψ_a after compensating the liquidity provider by $X_{a\ell}$. Thus, agents a and ℓ optimally decide

[18] Comparison across different equilibria shares similarities with Khorrami (2022), and also Grossman and Stiglitz (1980) where agents buy information ex-ante and this gives rise to an equilibrium with endogenous information structure, or Alvarez and Jermann (2000) where agents weigh between participation and autarky.

whether to assume the roles prescribed. Agent 3, since she is endowed with zero wealth, will take on the role assigned to her, as any outcome that allows her to enter is incentive compatible from her viewpoint.[19]

The value of the compensation to the liquidity provider corresponds to the utility valuation commonly used in models with incomplete markets.[20] Throughout, we assume that a reverse compensation (clawback) provision in the contract ensures that the liquidity provider will never find it optimal to renege on non-participation at a later date.[21]

The definition of equilibrium now takes into account these date 0 decisions and transfers:

Definition 2. *An equilibrium is a pair of security price processes* (S_{0t}, S_t) *and an array* $\{c_{kt}, (\pi_{kt}; \phi_{kt})\}$ *with* $k \in \{a, h, \ell\}$ *of consumption plans and trading strategies such that (1) given* (S_{0t}, S_t) *the plan* c_{kt} *maximizes* U_k *over the feasible set of Agent* k *and is financed by the trading strategy* (π_{kt}, ϕ_{kt}) *with initial endowments and transfers that satisfy* (32) *and* (33)*; (2) markets clear:* $\phi_{at} + \phi_{ht} + \phi_{\ell t} = 0$, $\pi_{at} + \pi_{ht} = S_t$ *and* $c_{at} + c_{ht} + c_{\ell t} = e_t$.

We first examine equilibria with only two types of agents. Equilibria with types $\{a, h\}$ or $\{h, \ell\}$ are easily ruled out: An equilibrium with types $\{a, h\}$ cannot occur without a liquidity provider, whereas an outcome $\{h, \ell\}$ is suboptimal as no agent would voluntarily relinquish access to markets without adequate compensation, see Lemma 1. This means that models with exogenous segmentation like Basak and Cuoco (1998), Hugonnier (2012) are not nested.

An equilibrium with types $\{a, \ell\}$,[22] on the other hand, deserves closer scrutiny. To simplify the intuition, suppose there is only one stockholder at the outset, so that $K = 0$, i.e., $n = 0$. By inspecting the limits in Lemma 1 we see that a single agent who owns the stock has utility U_0 under complete markets. If she changes type to arbitrageur, and a liquidity provider with zero-initial wealth enters, she cannot improve upon U_0 because she is sharing the dividend with the liquidity provider and must pay to tap the credit

[19] We assume that roles are prescribed by the regulator. The only issue from an agent's point of view is whether they are willing to participate in those roles or not. If not, the credit line shuts down and the complete market equilibrium prevails. An alternative rationale is that roles correspond to natural skills of agents and that mismatches between roles and skills are costly and un-implementable.

[20] Applications of utility-based valuation under incomplete markets can be found Karatzas and Kou (1996), Hugonnier et al. (2005), among others.

[21] An extreme form of clawback provision confiscates the wealth of the liquidity provider upon a breach. In this instance, consumption becomes null and ex-ante utility goes to $-\infty$.

[22] For example, stockholders become arbitrageurs and are financed by a liquidity provider with zero-initial wealth ($\{a, a, \ell\}$); or in a different configuration, a stockholder becomes a liquidity provider and an arbitrageur with zero-initial wealth enters, joining an existing stockholder who changes type ($\{\ell, a, a\}, \{a, \ell, a\}$).

line. Specifically, $W_{1t} = W_{at} = S_t - W_{\ell t}$, $W_{1t} + \psi_a B_t = (1-s_t)e_t/\rho$, and $W_{\ell t} = s_t e_t/\rho$ with $s_0 = p$. What if roles are reversed, i.e., the stockholder becomes the liquidity provider and the zero-initial wealth agent enters as an arbitrageur. In this case, Agent 1 sells her shares to the arbitrageur at price $S_0 = (1 - \alpha s_0^\kappa)P_0$ and collects $pP_0 = s_0 P_0$ for the credit facility, leaving her with $W_{10} = (1 - \alpha s_0^\kappa + p)P_0$. This is strictly less than her position $W_{10} = P_0$ if she chooses to remain an investor. Here welfare would decline as her maximal utility would be $U_0 - \frac{\sigma_e^2}{2\rho^2}(2\kappa - 1) < U_0$. This rules out existence of an equilibrium with types $\{a, \ell\}$ and a single initial stockholder. We note that it does not make any difference if we split the stockholder in two.

This analysis suggests that for an equilibrium to exist, there must be agents with initial positive endowment who do not change their type. As we will see in Section 4.8, the mechanism behind non-participation creates a pecuniary externality for those stockholders who do not change types. In what follows, we construct equilibria where either Agent 2 or 3 may become a liquidity provider whereas Agent 1 remains a regular stockholder.[23] We assume Agents 2 and 3 are sophisticated agents who are able to compare welfare across equilibria and decide whether to assume the roles assigned to them in equilibrium. In contrast, Agent 1 is a standard price taker acting on the basis of a given price system.

Our next proposition describes changes to the consumption rules that sustain an equilibrium with three types of agents.

Proposition 7. *Let $X_\ell = pP_0$ and assume that an equilibrium exists where Agent 1 participates, another agent acts as a liquidity provider and the last agent as an arbitrageur. In such an equilibrium, the stock price and its bubble are explicitly given by (30) and (31) and the money market account is described in (10). The consumption shares of agents are given by*

$$s_{1t} = s_{ht} = \frac{1}{1+\nu}(1-s_t); \qquad s_{\ell t} = s_t; \qquad s_{at} = \frac{\nu}{1+\nu}(1-s_t)$$

where the consumption share of the liquidity provider s_t satisfies (11) and

$$(s_0, \nu) = \begin{cases} \left(n + p, \frac{\alpha(n+p)^\kappa - p}{1-(\alpha(n+p)^\kappa + n)}\right); & \text{if } s_2 = s_\ell \\ \left(p, \frac{n + \alpha p^\kappa - p}{1-(\alpha p^\kappa + n)}\right); & \text{if } s_3 = s_\ell. \end{cases}$$

[23] Equilibrium configurations $\{a, h, \ell\}$ and $\{\ell, h, a\}$ lead to qualitatively similar results which we omit for brevity.

If Agent 2 is the liquidity provider $(s_{2t}, s_{3t}) = (s_{\ell t}, s_{at})$*, otherwise* $(s_{2t}, s_{3t}) = (s_{at}, s_{\ell t})$.

We highlight a few common elements across equilibria. First, the functional forms for the interest rate and the market price of risk are identical to the baseline model in Proposition 2. Second, the bubble ordering described in (16) carries over and the pair (S_t, B_t) is now defined in (30) and (31). Third, the bubble on the stock vanishes as $\alpha \uparrow 1$ (unlimited borrowing capacity). This confirms the intuition that says if arbitrageurs are allowed to increase the scale of the arbitrage that exploits the mispricing in the stock, the bubble decreases in size, to the extent that, in the limit, it disappears and the stock price converges to its fundamental value. This interpretation is starker if one thinks in terms of more arbitrageurs entering the market. The bubble in the money market account, on the other hand, persists as an equilibrium outcome that incentivizes stockholders to clear markets.

Finally, the notion of fundamental value is inherently linked to the trading conditions under which agents operate and, as a corollary, both fundamental values and bubbles will be agent-specific. Take for example an investor i with access to credit facility ψ_i and a security with payoff V_T, at time $T > 0$. The minimal cost portfolio strategy that replicates the payoff under agent i's investment opportunity set is given by

$$F_t(V_T, \psi_i) = F_t(V_T) - \psi_i B_t(T) \tag{34}$$

where $F_t(V_T) = \mathbb{E}_t [\xi_{t,T} V_T]$ is the non-negative replication value of the claim over an interval $[0, T]$ and the term $-\psi_i B_t(T)$ arises from exploiting the limited arbitrage in the stock by tapping a credit facility of size ψ_i. This has implications for pricing beyond primitive assets and hence for financial innovation: As valuation depends on credit conditions, contingent claim pricing out of this economy is not obvious. Note too that characterizing fundamental value using the minimal cost strategy in (34) leads to a redefinition of the security's bubble component. Taking the stock as an example, its fundamental value for Agent i is thus

$$F_t(T, \psi_i) = F_t(T) - \psi_i B_t(T), \tag{35}$$

and it is easy to see that the bubble on the stock, now defined as the difference between the stock price and (35)

$$B_t(T, \psi_i) = S_t - F_t(T, \psi_i) = S_t - (F_t(T) - \psi_i B_t(T)) = (1 + \psi_i) B_t(T),$$

would amount to more than 100% of the market price, since the process (35) could be negative in some states.[24]

4.6 Existence of equilibrium

We now detail the parametric restrictions that allow the equilibria described in Proposition 7 to exist.

Proposition 8. *Assume* $n+\alpha > 1$ *and define the following constants* $\bar{n} \equiv \min(\exp(-\sigma_e^2(2\kappa-1)/2\rho), 1)$, $\underline{p} \equiv ((1-\bar{n})/\alpha)^{1/\kappa}$ *and* $\bar{p} \equiv ((1-n)/\alpha)^{1/\kappa}$ *in the unit interval.*

(i) $\{h, \ell, a\}$*: Fix* $n \in (0, \bar{n})$*. Let* $s_0 \in (\underline{p}, \bar{p})$ *be a solution of* $\frac{1}{\rho}\log(s_0/n) - \frac{\sigma_e^2}{2\rho^2}\frac{1+s_0-2s_0^\kappa}{1-s_0} = 0$*, then there is a unique equilibrium where Agent 2 is the liquidity provider with* $p = s_0 - n$ *and Agent 3 is the arbitrageur.*

(ii) $\{h, a, \ell\}$*: Let* $s_0 \in (0, \bar{p})$ *be a solution of* $\frac{1}{\rho}\log\left(\frac{n+\alpha s_0^\kappa - s_0}{n(1-s_0)}\right) - \frac{1}{1-s_0}\sum_{j=1}^{\infty}\frac{1}{j}u(s_0, j) = 0$ *and assume* $U_1^{I,p}(\bar{p}) - U_0 > \log(n)/\rho$*, then there is a unique equilibrium where Agent 3 is the liquidity provider with* $p = s_0$ *and Agent 2 is the arbitrageur.*

Necessary parametric constraints for $\nu > 0$ for both types of equilibria can be summarized by $s_0 - \alpha s_0^\kappa < n < 1 - \alpha s_0^\kappa$. These inequalities ensure that initial wealth positions can support positive consumption for all agents and the required transfers to support equilibrium. The right hand side $n < 1 - \alpha s_0^\kappa$ implies that the wealth of Agent 1 must be positive, $w_1 = S_0 - nP_0 = (1 - \alpha s_0^\kappa - n)P_0 > 0$. The left hand side $s_0 - \alpha s_0^\kappa < n$ ensures that arbitrage profits extracted at time 0, $\psi_a B_0 = \alpha s_0^\kappa P_0$, are sizable enough to compensate liquidity providers, $w_{0\ell} + pP_0 > 0$, and allow for the arbitrageur's initial wealth to be positive, $w_{0a} + \psi_a B_0 - pP_0 > 0$. The sharing rule $\nu(n, \alpha)$ for both types of equilibria is depicted in Figure 2 Panels (a)–(c).

[Insert Figure 2 about here]

However, there are differences across types of equilibria since the identity of the arbitrageur varies. If Agent 2 (Agent 3) is the liquidity provider (arbitrageur), arbitrage profits must cover the price of the credit line, $\psi_a B_0 - pP_0 = (\alpha s_0^\kappa - p)P_0 > 0$, which is

[24] For the bubble in the money market account, note that the redefined bubble includes a long position on the stock price bubble

$$B_{0t}(T, \psi_i) = S_{0t} - F_{0t}(T, \psi_i) = S_{0t} - (F_{0t}(T) - \psi_i B_t(T)) = B_{ot}(T) + \psi_i B_t(T)$$

which, for the same reason, could exceed the price of the money market account.

enough to ensure Agent 3 chooses to act as an arbitrageur. As long as this condition is verified, the utility indifference condition fixes the compensation as a function of Agent 2's wealth level only, $p(n)$, since her endowment only depends on cash, nP_0. This is shown in Figure 2 Panel (b). This result will be useful when we discuss optimal levels of liquidity in the next section, as quantities will be available in closed-form.

If Agent 3 (Agent 2) is the liquidity provider (arbitrageur), Agent 2's initial wealth must be strictly positive after payment, $w_2 + \psi_a B_0 - pP_0 = (n + \alpha s_0^{\kappa} - p)P_0 > 0$. The utility indifference condition fixes the price $p(n, \alpha)$ as a function of Agent 2's wealth level and the size of the credit line since her endowment depends on cash nP_0 and the arbitrage profits $\psi_a B_0$, as illustrated in Figure 2 Panel (d). Agent 3 is automatically better off by choosing to provide liquidity, provided the compensation is positive.

4.7 Entry costs

Khorrami (2022) questions the risk concentration channel by arguing the mechanism relies on implausibly costly financial friction. Traditional models, e.g., (Basak and Cuoco, 1998, Vissing-Jørgensen, 2002) rely on exogenously fixed types and no entry. Khorrami (2022) studies an OLG extension of (Basak and Cuoco, 1998) based in Blanchard (1985), Gârleanu and Panageas (2015) that renders the equilibrium bubble free. In his baseline model, non-participant new cohorts may pay a one-time cost to begin trading in risky asset markets forever after. The intuition is that if entry is not too costly, the risk concentration channel is completely severed as markets becomefully integrated and agents share aggregate risk equally.[25]

We offer a rehabilitation of the concentration channel that highlights the importance of the various forms of segmentation that arise with asset bubbles. These imply different *entry* costs, as shown in Proposition 7, and are substantially different across equilibria.

Take the equilibrium $\{h, \ell, a\}$. Here Agent 2 shifts type to liquidity provider, a role change that is suboptimal in Khorrami (2022), and Agent 3 enters as arbitrageur to compensate Agent 2 so that their initial consumption shares are given by $(s_{20}, s_{30}) = (n+p, \alpha s^{\kappa} - p)$. As shown in Figure 2 Panel (b), the price (p) could amount to as little as

[25]Since participation provides an extra average return on wealth, which translates into a large present discounted utility gain that outweighs small entry costs, in other words, small participation costs cannot dissuade investors from taking these benefits. The corollary is that if limited participation is the mechanism generating large and volatile risk premia, implied participation costs must be as large as 90% of aggregate wealth to induce enough segmentation and hence empirically realistic asset prices in his calibration.

12% aggregate wealth (10% with Khorrami (2022) calibration), using the pre-entry price P_0 as the basis, and then it increases with the initial cash holdings n up to 30%.

In contrast, in the equilibrium labeled $\{h, a, \ell\}$ Agent 2 shifts the type to arbitrageur and Agent 3 enters as liquidity provider so that their initial consumption shares are given by $(s_{20}, s_{30}) = (n + \alpha p^{\kappa} - p, p)$, echoes Khorrami (2022). Figure 2 Panel (d), the price (p) amounts to large fractions of aggregate wealth, yet it decreases with the amount of arbitrage profits extracted by Agent 2, dictated by α, and her cash endowment n.

4.8 Welfare impact of the credit line

The credit line has two *negative* effects on Agent 1. The first is a wealth reduction for Agent 1 for any level of α, since her initial wealth $w_1 = P_0(1 - n - \alpha s_0^{\kappa})$ is decreasing in α. The second is a dynamic effect, through the consumption share s_0. By expressing the wealth of Agent 1 as a self-financing strategy in the stock and the riskless asset bubble over the interval $(t, T]$ as in (20) and computing the ratio bubble-to-stock holdings sensitivity to the size of the credit line α

$$\frac{\partial}{\partial \alpha} \left| \frac{\phi_{1t}^{B_0}(T)}{\phi_{1t}^{S}(T)} \right| = \frac{-v(s_t)(1 - s_t) + s_t}{s_t + (1 - \Sigma_{0t}(T))\,(1 - s_t)} \leq 0 \tag{36}$$

shows how the stockholder decreases her relative position on the riskless asset bubble as the stock becomes more volatile. More formally, using the results from Propositions 7 and 8, it follows that:

Proposition 9. *For any given $\alpha \in (1 - n, 1]$, Agent 1, the regular stockholder, is worse off in the equilibria with incomplete markets of Proposition 6 relative to the complete market equilibrium.*

To understand the comparison result take the counterargument $U_1^{I,p}(\cdot) > U_1^{C}(\cdot)$. This inequality implies that there is a feasible consumption allocation that Pareto dominates the allocation in (6), since Agent 2 is indifferent and Agent 3's consumption is nonzero. This is a contradiction with respect to the first welfare theorem that states that if there is an equilibrium in which markets are complete, then the corresponding consumption allocation is Pareto optimal. At work, there is a pecuniary externality that arises from the interaction between the participation decision and the loosening of the solvency constraint via the credit line in (26).

The arbitrageur benefits, to the detriment of Agent 1, from a larger credit facility because the liquidity provider's consumption is invariant to α. As such it introduces a novel channel in the literature on the transmission of macroeconomic shocks through credit markets with redistributive effects, whereby the stockholder faces a crowding out effect due to increased volatility.[26]

[Insert Figure 3 about here]

Figure 3 illustrates portfolio positions for the arbitrageur and the regular stockholder for various levels of the credit line. Panel (a) shows the arbitrageur takes a much larger position on the stock than the regular stockholder and this position decreases with α as the stock becomes more volatile. Panel (b) shows the extent to which the arbitrageur exploits the credit line to finance her strategy relative to the regular stockholder.

5 Optimal liquidity

In this section, we develop a notion of optimal liquidity that determines the credit line size, using a social welfare function that represents the concerns of a planner/regulator that cares about zero-wealth investors. We then show this is a distinct channel by which financial innovation may increase fragility, in the sense that too much liquidity leaves the system open for larger losses for lenders upon unanticipated shocks.

There are well-known examples of optimal securities design based on social welfare functions, based on a sizable literature on financial innovation.[27] It is clear that the opening of a credit facility does not improve the risk sharing of *existing* market participants. On the other hand, there is evidence that new assets are endogenously introduced by agents with profit incentives (Simsek, 2013), who may possibly manipulate the social or political environment without creating, necessarily, new wealth (Krueger, 1974, Velasco, 2025).

5.1 Planner's problem

In our economy, since Agent 3's welfare is $-\infty$ when she does not consume, a planner maximizing a weighted average of (all) agents' utilities would choose an equilibrium with bubbles that allows for Agent 3's participation. This is true provided the planner puts

[26]See e.g. Brunnermeier and Sannikov (2015), Dávila and Korinek (2017) for additional examples of pecuniary externalities for competitive economies with financial constraints.

[27]See e.g., Allen and Gale (1988), Demange and Laroque (1995), Athanasoulis and Shiller (2000, 2001), Simsek (2013), among others. There is also a growing literature, see, e.g., Schilling and Uhlig (2019), Biais et al. (2024) that analyzes the welfare implications of introducing cryptocurrencies; however, their null intrinsic/fundamental value is part of the assumptions rather than an endogenous result.

positive weight (even if very small) on Agent 3's welfare, i.e., the planner cares about Agent 3's welfare.[28] In order to calculate an optimal ψ (equivalently α) for the economy, we proceed as follows. Consider a social welfare function, i.e., the sum of agents expected utilities calculated using a set of non-negative weights η such that $\sum_{k=1}^{3} \eta_k = 1$, gives $\mathcal{U} = \sum_{k=1}^{3} \eta_k U_k^i(\Theta_k)$, where Θ_k includes the initial endowment, credit line and transfers, and the superscript i denotes the market structure, complete or incomplete, under which each agent operates. Maximizing the welfare criterion $\mathcal{U}$ with respect to α provides the optimal design of the credit line with respect to the social welfare function.

Proposition 10. *(i) $\{h, \ell, a\}$: Assume Agent 2 (Agent 3) is the liquidity provider (arbitrageur). The optimal amount of liquidity α^* is given by*

$$\alpha^* = \min\left(1, \frac{1}{(n+p(n))^\kappa}\frac{\eta_1 p(n) + \eta_3(1-n)}{\eta_1+\eta_3}\right); \qquad p(n) = s_0 - n. \tag{37}$$

(ii) $\{h, a, \ell\}$: Assume Agent 3 (Agent 2) is the liquidity provider (arbitrageur). The optimal amount of liquidity α^* is given by the solution to

$$\alpha^* = \underset{\alpha\in(1-n,1]}{\operatorname{argmax}} \left\{ -\frac{\eta_1}{\rho}\log\left(\frac{n+\alpha p(n,\alpha)^\kappa - p(n,\alpha)}{1-(\alpha p(n,\alpha)^\kappa + n)}\right) + \eta_3\left(\frac{\log p(n,\alpha)}{\rho} - \frac{\sigma_e^2}{2\rho^2}\frac{1+p(n,\alpha)-2p(n,\alpha)^\kappa}{1-p(n,\alpha)}\right)\right\} \tag{38}$$

In case (i), the regulator weights the impact of liquidity α on the consumption shares of Agents 1 (regular stockholder) and 3 (arbitrageur). Agent 2 (liquidity provider) does not influence this choice because her compensation $p(n)$ does not depend on liquidity. The regulator's decision then boils down to the choice

$$\max_{\alpha\in[0,1]}\left[\eta_1 \log\frac{1}{1+\nu(\alpha)} + \eta_3 \log\frac{\nu(\alpha)}{1+\nu(\alpha)}\right]; \qquad \nu(\alpha) = \frac{c_{3t}}{c_{1t}} = \frac{\alpha(n+p(n))^\kappa - p(n)}{1-\alpha(n+p(n))^\kappa - n}$$

where the relative consumption weight $\nu(\alpha)$ is an increasing function of liquidity. The solution is driven by the fact that an increase in α favors the arbitrageur at the expense of the regular stockholder. Balancing these effects gives $\nu(\alpha^*) = \eta_3/\eta_1$, leading to (37). This expression shows that optimal liquidity is a weighted average of the compensation $p(n)$ paid by the arbitrageur and the endowment $1-n$ of the regular stockholder, normalized by the marginal benefit of liquidity for arbitrage profits $(n+p(n))^\kappa$. The average is over post-transfer endowments of Agents 1 and 2 and the weights applied reflect the

[28] The role of the planner could be expanded to also include the choice between equilibria.

regulator's preferences. Furthermore, inspection of the formula shows that the limits $(\eta_1, \eta_3) = (1, 0)$, $(\eta_1, \eta_3) = (0, 1)$ do not satisfy necessary conditions for existence of equilibria $p(n) < \alpha(n + p(n))^\kappa < 1 - n$. Finally, if $(\eta_1 p(n) + \eta_3(1 - n))/(\eta_1 + \eta_3) \geq (n + p(n))^\kappa$ then unlimited borrowing capacity ($\alpha^* = 1$) is optimal. The trigger n^* for this to happen equates the marginal benefit of liquidity for arbitrage to the average post-transfer endowments of Agents 1 and 2.

[**Insert Figure 4 about here**]

Figure 4 illustrates the behavior of optimal liquidity in (38) as cash endowment n increases. It shows liquidity is maximal at 1, i.e., the stock bubble is arbitraged away, if n is sufficiently low. In this instance, the marginal benefit of α for arbitrage profits is low, prompting the regulator to optimally select maximal liquidity. When cash n reaches the threshold n^*, the marginal benefit becomes large enough to reduce optimal liquidity at the margin. The regulator then balances the benefits and costs of liquidity for Agents 1 and 3 as described above.

The mechanisms underlying case (ii) are more intricate. Here, the regulator impacts a priori all agents, as the equilibrium compensation $p(n, \alpha)$ depends on liquidity. Incentive compatibility, however, ensures that Agent 2, who now serves as an arbitrageur, attains the same utility as in the complete market model. It follows that her utility, ultimately, does not depend on α and does not affect the regulator's decision. Optimal regulation, again, mandates a choice between the impact of liquidity on Agents 1 (regular stockholder) and 3 (liquidity provider), as described in the proposition. In this instance, the utility of Agent 1 taking account of incentive compatibility, leads to $-\eta_1 \log(\nu(\alpha))$ where

$$\nu(\alpha) = \frac{c_{2t}}{c_{1t}} = \frac{n + \alpha(n + p(n, \alpha))^\kappa - p(n, \alpha)}{1 - \alpha(n + p(n, \alpha))^\kappa - n}$$

whereas the utility of Agent 3 leads to the second term. There are now two effects of liquidity, direct and indirect. The direct effect balances the marginal impact on the relative consumption shares of Agents 3 and 1, through arbitrage profits, keeping compensation $p(n, \alpha)$ frozen. The indirect effect arises through the adjustment in equilibrium compensation $p(n, \alpha)$. It affects the relative consumption shares as well as the utility of Agent 3. The optimal liquidity policy in this case is not obtained in explicit form, but can be calculated numerically using standard optimization routines.

[**Insert Figure 5 about here**]

Figure 5 plots the social welfare function for different weights $\zeta = \eta_3/\eta_1$. The characterization is slightly more challenging but it shows liquidity is maximal at 1 for the example, i.e., the stock bubble is arbitraged away for all levels of the cash endowment n.

This showcases a distinct channel by which financial innovation increases portfolio risk and, as we will see in the next section, induces fragility, in the sense that unlimited borrowing capacity leaves the system open for larger losses for lenders upon unanticipated shocks.

5.2 Fragility

We explore the impact of an exogenous liquidity shock that decreases the value of α and thus might force investors to settle their positions. This type of shock can represent a credit crunch originating from an unanticipated change in regulation or from an unanticipated change in the economic environment prompting a reduction in the flexibility afforded to arbitrageurs. Its impact is similar to agents neglecting some of the risk (Gennaioli et al., 2012, Bolton et al., 2018) whereby credit is not rolled over if the arbitrageur violates the wealth constraint in (26).[29]

Let us assume that the tightening liquidity shock occurs at time $\tau > 0$ and decreases the size of the credit facility to $0 \leq \tilde{\alpha} < \alpha$. Depending on the distribution of wealth in the economy, the shock will either lead to an equilibrium in which arbitrageurs are still active but on a smaller scale, or to an equilibrium in which they are no longer present because they are unable to satisfy the wealth constraint given the new prices.

To set up the main example, consider first the non-default outcome, that is, that satisfies the inequality in (26). Denote by $(\pi_{k\tau}, \phi_{k\tau})$ the portfolio holdings of the agents prior to the occurrence of the shock:

$$
\begin{aligned}
(\pi_{1\tau}, \phi_{1\tau}) &= \left(\frac{1}{(1+\nu)(1+v(s_\tau))}; \frac{v(s_\tau) - (1+v(s_\tau))s_\tau}{(1+\nu)(1+v(s_\tau))} \right) P_\tau \\
(\pi_{\ell\tau}, \phi_{\ell\tau}) &= (0; 1)\, s_\tau P_\tau
\end{aligned}
$$

[29] There is extensive literature, see, e.g., Gromb and Vayanos (2008), Krishnamurthy (2010), Brunnermeier et al. (2020), arguing that large financial shocks are often preceded by periods of credit expansion during which market participants become increasingly vulnerable to a reversal in funding conditions, i.e., a liquidity dry-up. These episodes are bound to occur in the crypto space (Hermans et al., 2022, Financial Stability BIS, 2023, Lowrey, 2025).

and

$$(\pi_{a\tau}, \phi_{a\tau}) = \nu(\pi_{1\tau}, \phi_{1\tau}) + \left(\frac{\kappa - 1}{1 + v(s_\tau)}; -\frac{\kappa + v(s_\tau)}{1 + v(s_\tau)}\right) \alpha s_\tau^\kappa P_\tau. \tag{39}$$

Given these holdings, an equilibrium exists if and only if there exists $\tilde{\nu} > 0$ such that

$$\underbrace{\phi_{1\tau} + \pi_{1\tau}(\tilde{S}_\tau / S_\tau)}_{\text{after}} = \frac{1}{1 + \tilde{\nu}}(1 - s_\tau)P_\tau > \underbrace{\frac{1}{1 + \nu}(1 - s_\tau)P_\tau}_{\text{prior}} \tag{40}$$

where $S_\tau = (1 - \alpha s_\tau^\kappa)P_\tau$, $\tilde{S}_\tau = (1 - \tilde{\alpha} s_\tau^\kappa)P_\tau$ give the equilibrium price of the stock an instant prior and after the liquidity shock, respectively. As the regular stockholder is long in the stock, this price appreciation benefits her unambiguously, while the liquidity provider keeps her consumption path unaltered. This is why the same consumption share s_τ appears on both sides of the inequality.

Agent 1 has an incentive to reduce the size of the credit line in place for the arbitrageur to zero, as this price adjustment leads to a decrease in the consumption share of the arbitrageur

$$\underbrace{\phi_{a\tau} + \pi_{a\tau}(\tilde{S}_\tau / S_\tau) + \tilde{\alpha} s_\tau^\kappa P_\tau}_{\text{after}} = \frac{\tilde{\nu}}{1 + \tilde{\nu}}(1 - s_\tau)P_\tau < \underbrace{\frac{\nu}{1 + \nu}(1 - s_\tau)P_\tau}_{\text{prior}}.$$

Risky liquidity. Next, consider a liquidity shock at time τ that shuts down the credit line entirely. The economy would move to an equilibrium where the stock price is equal to P_τ. If the arbitrageur's wealth is nonnegative, then the economy looks like the baseline economy of Section 2.6 where the arbitrageur is indistinguishable from a regular stockholder, and her sharing rule $\tilde{\nu} < \nu$ is determined by a special case of (40)

$$\underbrace{\phi_{a\tau} + \pi_{a\tau}(P_\tau / S_\tau)}_{\text{after}} = \frac{\tilde{\nu}}{1 + \tilde{\nu}}(1 - s_\tau)P_\tau < \underbrace{\frac{\nu}{1 + \nu}(1 - s_\tau)P_\tau}_{\text{prior}}.$$

On the other hand, if the arbitrageur is in violation of the constraint in (26),

$$W_{a\tau} = \phi_{a\tau} + \frac{\pi_{a\tau}}{S_\tau}P_\tau = \frac{1}{1 + \nu}\left(\nu(1 - s_\tau) - \frac{\alpha s_\tau^\kappa}{1 + (\kappa - 1)\alpha s_\tau^\kappa}\right)P_\tau \leq 0 \tag{41}$$

the portfolio holdings of the remaining agents reflect the losses generated by her default/exit. Since arbitrageurs hold a long position in the stock prior to the shock, upon her exit the liquidity provider would simply take over her stock position in partial repayment of her debt.

Let s_{τ^+} denote the consumption share of the liquidity provider post shock. Assuming there are not direct costs of default or other inefficiencies (e.g., costly repossession), it follows from (41) that the change in the liquidity provider's wealth is always negative

$$\frac{W_{\ell\tau^+}}{W_{\ell\tau}} - 1 = \frac{s_{\tau^+}}{s_\tau} - 1 = \frac{-\phi_{1\tau} + \frac{\pi_{a\tau}}{S_\tau} P_\tau}{-\phi_{1\tau} - \phi_{a\tau}} - 1 = \frac{W_{a\tau}}{W_{\ell\tau}} < 0 \tag{42}$$

The arbitrageur's exit impacts the liquidity provider through two channels: losses from default and price changes. Keeping prices fixed, we have that the default imposes direct losses on her. On the other hand, the price shift from S_t to P_t mitigates the direct losses inflicted on the liquidity provider by increasing the value of her remaining stock holdings. This type of event is akin to an LTCM or a sudden stop episode where the arbitrageur is wiped out and the price recovers, all of it happening in an instant of time. To grasp the magnitudes, the first term in (39) shows that the arbitrageur's strategy mimics the *positive* net position of the stockholder $(\pi_{1\tau}, \phi_{1\tau})$, in proportion to her weight ν. The second term is a *negative* net position proportional to the stock bubble and the credit line, since $\alpha s_\tau^\kappa P_\tau$ is equivalent to $\psi_a B_t$. The process $W_{a\tau}$ is monotonically decreasing in s_τ with positive and negative extremes, $\lim_{s_\tau \to 0} W_{a\tau}/P_\tau = \frac{\nu}{1+\nu}$ and $\lim_{s_\tau \to 1} W_{a\tau}/P_\tau = \frac{-\alpha}{(1+\nu)(1+(\kappa-1)\alpha)}$.

Losses occur in states where s_τ is high, that is, $W_{a\tau}$ is negative in bad times, and are expected to be greater for an equilibrium with small ν and high α, that is, when the net negative position is relatively larger.

[Insert Figure 6 about here]

Figure 6 illustrates that losses for the liquidity provider could be substantial and they are increasing in α,

$$\frac{\partial}{\partial \alpha} |W_{a\tau}/W_{\ell\tau}| = \frac{s_\tau^\kappa}{(1+\nu)\left(1+(\kappa-1)\alpha s_\tau^\kappa\right)^2} > 0.$$

The rise of non-traditional financial channels, such as crypto-lending and decentralized finance (DeFi) (OECD, 2022), exacerbates the risk to financial stability posed by markets awash with liquidity. Regulators face a significant challenge (Azar et al., 2022) in this environment, particularly as major players (e.g., BlackRock, Fidelity) enter the crypto market, further amplifying liquidity and investor participation.

6 Conclusion

This study demonstrates how the interplay of limited stock market participation and credit lines generates asset price bubbles within a simplified three-agent exchange economy. Increased stock price volatility, driven by arbitrageurs leveraging these credit facilities, reduces the profitability of bubbly prices for regular stockholders, negatively impacting their welfare

Credit lines play a dual role in the dynamics of asset bubbles. They ease arbitrage limitations imposed by wealth constraints, enabling arbitrageurs to actively participate in market correction. Conversely, credit lines are provided by liquidity providers who, by remaining non-participants in the stock market, contribute to asset bubble formation. The interplay between credit lines and liquidity provider behavior underscores the complex and intertwined relationship between liquidity, funding constraints, and asset bubbles.

Our model yields significant quantitative implications, demonstrating that credit lines facilitate low-cost entry for arbitrageurs, thereby rehabilitating the risk concentration channel of limited participation models.

Finally, we show how optimal liquidity levels balance the benefits and costs of arbitrage; however excessive liquidity can increase market fragility. This is particularly relevant in the current financial landscape, where the rise of cryptocurrencies and decentralized finance platforms presents novel challenges for regulators.

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A Proofs

Proof of Proposition 1. The welfare of agents 1 and 2 follows from computing the utility functions using the optimal consumption rules in (6). ■

Proof of Proposition 2. We slightly generalize the setup in Section 2.6 to account for heterogeneity in risk aversion where γ defines Agent 1's relative risk aversion and a weaker limited participation constraint $\mathcal{C}_t = \{|\bar{\pi}_t\sigma_t|^2 \leq (\varepsilon\sigma_e)^2\}$, with $\bar{\pi}_t = \pi_t/W_t$. We then employ this result in Section 3. Agent 1 faces a complete financial market and thus uses process ξ_t in (1) as a state price density. Using standard probabilistic methods, it is well known that the solution of the unconstrained agent's problem is given by $c_{1t} = (e^{\rho t}y_1\xi_t)^{-\frac{1}{\gamma}}$, $\sigma_t\bar{\pi}_{1t} = \theta_t + h_{1t}(\xi_t W_{1t})^{-1}$, with $\bar{\pi}_{1t} = \pi_{1t}/W_{1t}$ and $W_{1t} = \mathbb{E}_t\left[\int_t^\infty \xi_{t,u}c_{1u}du\right] = \frac{1}{\xi_t}\left[H_{1t} - \int_0^t \xi_u c_{1u}du\right]$ represents the agent's wealth along the optimal path, h_1 is the integrand in the stochastic integral representation of the martingale $H_1 = \mathbb{E}_t\left[\int_0^\infty \xi_u c_{1u}du\right]$ and the strictly positive constant y_1 is chosen in such a way that $W_{10} = w_1$. Agent 2 solves the program $\sup_{c,\bar{\pi}\in\mathcal{A}(w_2)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(c_{2t})\,dt\right]$, subject to

$$\log(W_{2t}) = \log(W_{20}) + \int_0^t \left(r_u + \bar{\pi}_u\sigma_u\theta_u - \frac{1}{2}\left|\sigma_u^\top\bar{\pi}_u\right|^2 - \bar{c}_{2u}\right)du + \int_0^t \bar{\pi}_u\sigma_u dZ_u$$

where $\mathcal{A}(w_2) = \left\{(\bar{\pi}, c) : \bar{\pi} \in \mathcal{C}_t \text{ and } W_{2t}^{w_2,\bar{\pi},c} \geq 0,\ t \in [0,\infty)\right\}$ and $\bar{c}_2 = c_2/W_2$. Using the objective function and the budget constraint, the problem can be expressed as the maximization of

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log(a_t) + \log(W_{20}) + \int_0^t \left(r_u + \bar{\pi}_u\sigma_u\theta_u - \frac{1}{2}|\sigma_u\bar{\pi}_u|^2 - a_u\right)du\right)dt\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left[\log(a_t) - \rho^{-1}a_t + \log(W_{20}) + \rho^{-1}r_t + \rho^{-1}\left(\bar{\pi}_t\sigma_t\theta_t - \frac{1}{2}|\sigma_t\bar{\pi}_t|^2\right)\right]dt\right]
\end{aligned}$$

where we conjectured a consumption policy of the form $\bar{c}_{2t} = a_t$. The problem is solved by a pointwise optimization of $\sup_{a>0}\{\log(a) - \rho^{-1}a\}$, which admits a unique solution given by $a = \rho$, and the mean variance program

$$\sup_{\bar{\pi}=\pi/W\in\mathcal{C}_t}\left\{\bar{\pi}\sigma_t\theta_t - \frac{1}{2}|\sigma_t\bar{\pi}|^2\right\}. \tag{A.1}$$

Since $\mathcal{C}_t$ is a closed convex subset of $\mathbb{R}$, the mean variance problem in (A.1) admits a unique solution given by $\sigma_t\bar{\pi}_t = k_t\theta_t$ with $k_t = \left[1 + (|\theta_t|/(\varepsilon\sigma_e) - 1)^+\right]^{-1}$. We construct an equilibrium using the consumption sharing rule $s_t = c_{2t}/e_t$ and the optimality conditions of both agents. We conjecture and verify that the consumption share process follows an Itô process given by

$$ds_t = s_t\mu_s(\cdot)dt + s_t\sigma_s(\cdot)dZ_t, \tag{A.2}$$

where the coefficients (μ_s, σ_s) are determined jointly with the interest rate and the market price of risk. We briefly outline the steps to construct an equilibrium. (i) The state price density is obtained from the first order condition of the unconstrained agent $\xi(t, s_t, e_t) = e^{-\rho t} y_1^{-1}(1 - s_t)^{-\gamma} e_t^{-\gamma}$, and thus, (ii) an application of Itô's lemma to this function identifies the market price of risk and the interest rate as functions of the drift and diffusion terms of the consumption share and the dividend dynamics,

$$\theta_t = \gamma \left(\sigma_e - \frac{s_t \sigma_s(\cdot)}{1 - s_t} \right), \tag{A.3}$$

$$r_t = \rho + \gamma \mu_e - \frac{1}{2}(1+\gamma)\gamma\sigma_e^2 - \gamma \frac{s_t \mu_s(\cdot)}{1 - s_t} + \frac{2\gamma^2 \sigma_e (1 - s_t) s_t \sigma_s(\cdot) - (1+\gamma)\gamma s_t^2 \sigma_s(\cdot)^2}{2(1 - s_t)^2}.$$

(iii) An application of Itô's lemma to the process $s_t = c_{2t}/e_t = \rho W_{2t}/e_t$, where W_{2t} is the wealth process of the constrained agent along the optimal path, with dynamics $dW_{2t}/W_{2t} = (r_t + k_t \theta_t^2 - \rho)dt + k_t \theta_t dZ_t$, pins down the drift and volatility in (A.2),

$$\mu_s(\cdot) = r_t + \sigma_e \theta_t - \mu_e - \rho + (k_t \theta_t - \sigma_e)(\theta_t - \sigma_e),$$

$$\sigma_s(\cdot) = k_t \theta_t - \sigma_e. \tag{A.4}$$

Using equations (A.3) and (A.4) and the fact that the process k_t depends only on θ_t, we obtain a nonlinear equation for θ,

$$\theta_t = \gamma \left(\sigma_e - \frac{s_t (k_t \theta_t - \sigma_e)}{1 - s_t} \right) \tag{A.5}$$

The solution of this problem is then used to express $(r(\cdot), \mu_s(\cdot), \sigma_s(\cdot))$ as functions of the consumption share only. The pair $(\mu_s(\cdot), \sigma_s(\cdot))$ is given in (B.2) and (B.3) in the Online Appendix. The starting point, $s_0 \in (0,1)$, is a solution to the equation $W_{20} = P_0 s_0 = nP_0$, where $P_0 = e_0/\rho$ by definition and $w_2 = K = nP_0$ by Assumption 1, when $\gamma = 1$. When $\gamma \neq 1$, s_0 is solved numerically. The lagrange multiplier of the unconstrained agent is set to $y_1 = (1 - s_0)^{-\gamma} e_0^{-\gamma} > 0$. The equilibrium requires two conditions: (i) $s_0 \in (0,1)$ and (ii) the process s_t never reaches either zero or one in $t \in [0, \infty)$. Condition (i) implicitly restricts the size of the initial portfolios such that the initial endowments w_k are strictly positive. Condition (ii) indicates that boundaries cannot be reached when the process starts from $s_0 \in (0,1)$, otherwise, the consumption policies would not be optimal and equilibrium would fail to exist. In the Online Appendix B.1 we offer a procedure to check for (ii). To obtain (10)–(11) for the case $(\gamma, \varepsilon) = (1, 0)$, we first note that $k = 0$ when $\varepsilon = 0$, then substitute the relevant parameter values in $(\theta, r, \mu_s(\cdot), \sigma_s(\cdot))$ above. ■

Proof of Proposition 3. Let $\beta \equiv 1 + \nu$, we have that $d \log s_t = -\frac{\sigma_e^2}{2\beta}(2\lambda_t + \beta)dt - \sigma_e dZ_t$. Next, using integration by parts, we get that the utility of Agent 2 with incomplete markets is given by

$$U_2^{I,np}(n) - U_0 = \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log s_t dt\right] \tag{A.6}$$

$$= \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log s dt\right] - \frac{\sigma_e^2}{2\beta}\mathbb{E}\left[\int_0^\infty e^{-\rho t} \int_0^t (2\lambda_u + \beta) du \, dt\right]$$

$$= \frac{\log s}{\rho} - \frac{\sigma_e^2}{2\beta\rho}\left(\frac{\beta}{\rho} + 2\mathbb{E}\left[\int_0^\infty e^{-\rho t} \lambda_t dt\right]\right).$$

To compute $\mathbb{E}\left[\int_0^\infty e^{-\rho t}\lambda_t dt\right]$, we recall the definition of the fundamental value and use straightforward manipulations

$$\begin{aligned}F_0 &= e_0(1-s)\mathbb{E}\left[\int_0^\infty e^{-\rho t}/(1-s_t)dt\right] = e_0\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\frac{1-s}{1-s_t}-1+1\right)dt\right]\\ &= e_0\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\frac{\lambda_t-\lambda}{\beta+\lambda}+1\right)dt\right] = \frac{e_0}{\beta+\lambda}\left(\mathbb{E}\left[\int_0^\infty e^{-\rho t}\lambda_t dt\right]+\frac{\beta}{\rho}\right)\end{aligned}$$

and get $\mathbb{E}\left[\int_0^\infty e^{-\rho t}\lambda_t dt\right] = \frac{F_0(\beta+\lambda)}{P_0\rho} - \frac{\beta}{\rho} = \frac{(1-s^\kappa)(\beta+\lambda)-\beta}{\rho}$ where we use $P_t = e_t/\rho$ and $F_t = P_t(1-s_t^\kappa)$. Using the above and continuing calculations in (A.6)

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log s_t dt\right] &= \frac{\log s}{\rho} - \frac{\sigma_e^2}{2\beta\rho}\left(\frac{2(1-s^\kappa)(\beta+\lambda)-\beta}{\rho}\right)\\ &= \frac{\log s}{\rho} - \frac{\sigma_e^2}{2\rho^2}\left(2(1-s^\kappa)(1+\lambda/\beta)-1\right),\end{aligned}$$

the expression in (13) follows by using $s = \lambda/(\beta+\lambda)$ and $\lambda = \beta s/(1-s)$. Next, we use the following series representation for the logarithm $\log(1-x) = -\sum_{j=1}^\infty \frac{1}{j}x^j$ for $x \in (0,1)$. Since $s_t \in (0,1)$ we then obtain that

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(1-s_t)dt\right] = -\sum_{j=1}^\infty \frac{1}{j}\mathbb{E}\left[\int_0^\infty e^{-\rho t}s_t^j dt\right] \tag{A.7}$$

and define $g(\lambda, j) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}s_t^j dt\right]$ as the Laplace transform of n-th moment of s_t. Using Itô's lemma and the SDE for λ, the function g is the unique bounded solution to the Sturm-Liouville problem

$$\frac{1}{2}\frac{\sigma_e^2}{\beta^2}\lambda^2(\beta+\lambda)^2 g''(\lambda,j) + \left(\frac{\lambda}{\beta+\lambda}\right)^j = \rho g(\lambda,j) \tag{A.8}$$

with $g(0,j) = 0$. First solving the homogeneous equation and then by finding the particular solution to (A.8), we get the following general solution

$$g(\lambda,j) = g^h(\lambda) + g^p(\lambda,j)$$

where $g^h(\lambda) = C_1\lambda\left(\frac{\lambda}{\beta+\lambda}\right)^{\kappa-1} + C_2\lambda\left(\frac{\lambda}{\beta+\lambda}\right)^{-\kappa}$ and $g^p(\lambda,j) = A\left(\frac{\lambda}{\beta+\lambda}\right)^j + B\lambda\left(\frac{\lambda}{\beta+\lambda}\right)^j$ with

$$A(j) = \frac{1}{\rho - j(j-1)\frac{\sigma_e^2}{2}}, \qquad B(j) = \frac{1}{\beta}\left(\frac{1}{\rho - j(j-1)\frac{\sigma_e^2}{2}} + \frac{1}{j(j+1)\frac{\sigma_e^2}{2} - \rho}\right).$$

Then using that $g(0,j) = 0$ and g is bounded at infinity we deduce that $C_2 = 0$ and $C_1 = -B$, and therefore g can be written by straightforward simplifications as

$$g(s,j) = A(j)s^j + B(j)\lambda(s)(s^j - s^{\kappa-1})$$

where we used $\lambda/\beta = s/(1-s)$. We note that g does not depend on β. Now by plugging the expression for g into equation (A.7) the formula in (12) obtains. ■

Proof of Lemma 1. We show that $U_1^{I,p}(n) \geq U_1^C(n)$ for all $n \in (0,1)$. For this we apply Itô's lemma and obtain that

$$d\log(1-s_t) = \frac{\sigma_e}{\beta}\lambda_t dZ_t + \frac{\sigma_e^2}{2\beta^2}\lambda_t^2 dt$$

so that $\log(1-s_t)$ has a local martingale term and an increasing bounded variation part. We also note that $\log(1-s_t)$ is a negative process and hence standard arguments based on Fatou's lemma show that $\log(1-s_t)$ is a submartingale. By Fubini's theorem,

$$U_1^{I,p}(n) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log((1-s_t)e_t)dt\right] \geq U_0 + \frac{\log(1-n)}{\rho} = U_1^C(n)$$

which gives the desired result. Next, it is clear that $\frac{\sigma_e^2}{2\rho^2}\frac{1+n-2n^\kappa}{1-n} \geq 0$ and thus $U_2^C(n) \geq U_2^{I,np}(n)$ for $n \in (0,1)$. The inequalities for partial derivatives with respect to n follow from the comparison principle for SDEs. Finally, the limits at the boundary points can be computed either directly by inserting $n=0$ and $n=1$ into the welfare expressions or by L'Hôpital rule. In particular, we have that

$$\lim_{n\to 1} U_1^{\mathrm{I,p}}(n) = U_0 + \sum_{j=1}^\infty \frac{1}{j}\left(\frac{j-\kappa}{\rho - j(j-1)\frac{\sigma_e^2}{2}} - \frac{j+1-\kappa}{\rho - j(j+1)\frac{\sigma_e^2}{2}}\right) = U_0 + \frac{1}{\rho}\left(\frac{\kappa-1}{\kappa} - H_{\kappa-1}\right),$$

$$\lim_{n\to 1} U_2^{\mathrm{I,np}}(n) = U_0 - \frac{\sigma_e^2}{2\rho^2}(2\kappa-1)$$

which completes the proof. ■

Proof of Proposition 4. We recall two lemmas (B.2, B.3) from Hugonnier and Prieto (2015). First, the stochastic differential equation

$$Y_t(a) = 1 - \int_0^t Y_u(a)\left(a + \frac{s_u}{1-s_u}\right)\sigma_e dZ_u \tag{A.9}$$

where $a \in \mathbb{R}$ is a constant and s_t follows (11), admits a unique strictly positive solution which satisfies

$$\mathbb{E}_t\left[Y_{t+T}(a)\right] = Y_t(a)\left(1 - H(T, s_t; 2a-1)\right), \tag{A.10}$$

where $H(\tau, s, a)$ is defined in (17). In particular, the process $Y_t(a)$ is a strictly positive local martingale but not a martingale (Lemma B.3). Second, let $T > 0$ and the function defined by

$$\begin{aligned} q_t(T) &= \rho s_t \int_t^T e^{-\rho(u-t)}(1 - \mathbb{E}_t[\lambda_u/\lambda_t])du \\ &= s_t^\kappa H(T-t, s_t; 2\kappa-1) - e^{-\rho(T-t)} s_t H(T-t, s_t; 1) \end{aligned} \tag{A.11}$$

where $H(\tau, s, a)$ is defined in (17) (Lemma B.2). The bubble on the stock in (23) follows from taking the limit in (A.11), $\frac{B_t}{P_t} = \lim_{T\to\infty} q_t(T) = s_t^\kappa$. Using (3) and the law of iterated expectations, we obtain that the finite horizon bubble on the stock is given by

$$\begin{aligned} B_t(T) &= S_t - \mathbb{E}_t\left[\int_t^T \xi_{t,u}e_u du + \xi_{t,T}S_T\right] = S_t - \mathbb{E}_t\left[\int_t^\infty \xi_{t,u}e_u du - \int_T^\infty \xi_{t,u}e_u du + \xi_{t,T}S_T\right] \\ &= B_t - \mathbb{E}_t\left[\xi_{t,T}\left(S_T - \mathbb{E}_T\int_T^\infty \xi_{T,u}e_u du\right)\right] = B_t - \mathbb{E}_t\left[\xi_{t,T}B_T\right]. \end{aligned}$$

We next compute the right hand side. We include a constant ν to accommodate the model in Section 4. In the baseline model $\nu = 0$. We have,

$$\frac{B_t - \mathbb{E}_t[\xi_{t,T} B_T]}{P_t} = s_t^\kappa - e^{-\rho(T-t)}\mathbb{E}_t\left[\frac{1+\nu+\lambda_T}{1+\nu+\lambda_t} s_T^\kappa\right] = s_t^\kappa - \rho\mathbb{E}_t\left[\int_T^\infty e^{-\rho(u-t)}\frac{\lambda_T - \lambda_u}{1+\nu+\lambda_t}du\right]$$

$$= s_t^\kappa + q_t(T) - \lim_{\Theta\to\infty} q_t(\Theta) - e^{-\rho(T-t)}\mathbb{E}_t\left[\frac{\lambda_T - \lambda_t}{1+\nu+\lambda_t}\right] = s_t^\kappa H(T-t, s_t, 2\kappa - 1).$$

Note that the bubble on the riskless asset in (24) follows from a direct application of (A.10)

$$B_{0t}(T)/S_{0t} = 1 - \mathbb{E}_t\left[\xi_{t,T}\frac{S_{0T}}{S_{0t}}\right] = 1 - \mathbb{E}_t\left[\lambda_T/\lambda_t\right] = H(T-1, s_t; 1).$$

To complete the proof it remains to show that the relative bubble on the stock is dominated by the relative bubble on the riskless asset over any horizon. Consider the function defined by $G(\tau; s; a) = s^{\frac{1+a}{2}} H(\tau, s; a)$. A direct calculation using (17) shows that

$$\frac{\partial}{\partial a}(G(\tau; s; a)) = s^{\frac{1-a}{2}} \log(s)\mathcal{G}(s),$$

$$\frac{\partial}{\partial a}(s^{-a}G(\tau; s; 2a-1)) = 2s^{1-2a}\log(1/s)N(d_-(\tau; s; 2a-1)) \geq 0,$$

with the function defined by $\mathcal{G}(s) = [s^a N(d_+(\tau; s; a)) - N(d_-(\tau; s; a))]/2$, and since $\mathcal{G}(0) = 0 < N(d_+(\tau; 1, b)) - 1/2 = \mathcal{G}(1)$, and, $\mathcal{G}'(s) = (b/2)x^{b-1}N(d_+(\tau; s, b)) \geq 0$, we conclude that the functions $G(\tau; s, a)$ and $s^{-a}G(\tau; s, 2a-1)$ are respectively decreasing and increasing in a. Using these properties and the facts that $s_t \in (0, 1)$, $\kappa > 1$, we then deduce $B_t(t+T)/S_t = G(T, s_t, 2\kappa - 1)) \leq G(T, s_t, 1) \leq s_t^{-1}G(T, s_t, 1) = B_{0t}(t+T)/S_{0t}$. ■

Asymptotic behaviour of the consumption share process in (21). . Since the consumption share process is a nonnegative supermartingale we have that it converges to a well-defined limit. On the other hand, an application of Itô's lemma to (11) shows that

$$0 \leq s_t = s_0 e^{-\int_0^t \frac{\sigma_e^2}{1-s_u}du - \frac{1}{2}\sigma_e^2 t - \sigma_e Z_t} \leq \bar{s}_t = s_0 e^{-\frac{1}{2}\sigma_e^2 t - \sigma_e Z_t}$$

and the desired result then follows from the fact that, by well-known results on geometric Brownian motion, the process $\bar{s}_t$ converges to zero. ■

Proof of Proposition 5. Using the portfolio choice of the constrained agent in the market price of risk in equation (A.5) gives, $\theta = \left[1 - \left(1 - \frac{1}{1+(|\theta|/\varepsilon\sigma_e - 1)^+}\right) R(s)s\right]^{-1} R(s)\sigma_e$ which is uniquely solved by the positive, continuous and piecewise differentiable function in (25). ■

Proof of Proposition 6. Define the strictly positive process λ_t by

$$\lambda_t = c_{2t}/c_{1t}^\gamma = s_t(1-s_t)^{-\gamma}e_t^{1-\gamma} = \lambda_0 + \int_0^t \lambda_u \Phi(s_u)dZ_u$$

with $\Phi(s_t) = (k(s_t) - 1)\theta(s_t)$, where $\theta(\cdot)$ is given in (25). The bubble on the stock can be characterized by the difference between Agent 2's wealth and the fundamental value of her consumption plan,

$$B_t = \rho^{-1}s_t e_t - \mathbb{E}_t\left[\int_t^\infty \xi_{t,u}s_u e_u du\right] = s_t e_t \mathbb{E}_t\left[\int_t^\infty e^{-\rho(u-t)}\left(1-\frac{s_u(1-s_u)^{-\gamma}e_u^{1-\gamma}}{s_t(1-s_t)^{-\gamma}e_t^{1-\gamma}}\right)du\right]$$
$$= e_t^\gamma(1-s_t)^\gamma \int_t^\infty e^{-\rho(u-t)}\left(\lambda_t - \mathbb{E}_t\left[\lambda_u\right]\right)du$$
$$= e_t s_t \int_t^\infty e^{-\rho(u-t)}\left(1-\mathbb{E}_t\left[\lambda_u/\lambda_t\right]\right)du.$$

which is identical to (23). Just as in the baseline case with $\gamma = 1$, the bubble on the stock depends on λ. To assess the the martingality of λ, we use the exponential local martingale

$$M_t^\lambda = \frac{\lambda_t}{\lambda_0} = e^{-\int_0^t \frac{1}{2}\Phi(s_u)^2 du + \int_0^t \Phi(s_u)dZ_u}, \tag{A.12}$$

where $\Phi(x) = \varepsilon\sigma_e - \gamma\sigma_e\frac{1-\varepsilon x}{1-x}$ as the density of a candidate equivalent change of measure $\mathbb{P}^\lambda$. We verify the properties of the consumption share process, whose dynamics under $\mathbb{P}^\lambda$ follows

$$ds_t = \mu_s^\lambda(s_t)s_t dt - (1-\varepsilon)\sigma_e s_t dZ_t^\lambda, \tag{A.13}$$
$$\mu_s^\lambda(x) = (\gamma-1)\frac{1-x}{1+(\gamma-1)x}\mu_e + (\gamma-1)\gamma\frac{[2-\varepsilon(2-\varepsilon)x]x-1}{2(1-x)[1+(\gamma-1)x]}\sigma_e^2 + (1-\varepsilon)^2\sigma_e^2,$$

where $dZ_t^\lambda = dZ_t - \Phi(s_t)dt$ is a $\mathbb{P}^\lambda-$ Brownian motion. In equilibrium, the consumption share lives in $(0,1)$ (see Appendix B) therefore, if under $\mathbb{P}^\lambda$ the process hits one of the boundaries with positive probability, $\mathbb{P}^\lambda$ could not be equivalent to $\mathbb{P}$ because this behavior is not possible under the objective measure. As shown in Heston et al. (2007, Theorem A.1), this is a test for the martingality of λ, i.e., the failure of equivalence of measures implies the process is a strict local martingale.

Define the stopping times $T_\Delta = \inf\{t \geq 0 : s_t \geq \Delta\}$, with $\Delta \in I \equiv (0,1)$, and let $T_1 = \lim_{\Delta\to 1} T_\Delta$, $T_0 = \lim_{\Delta\to 0} T_\Delta$.

When $\gamma = 1$, the process in (A.13) under $\mathbb{P}^\lambda$ corresponds to a geometric Brownian motion, and hence, it reaches 1 in finite time with positive probability, $\mathbb{P}^\lambda[T_1 < T] > 0$. This contradicts the equivalence between $\mathbb{P}$ and $\mathbb{P}^\lambda$, and thus, λ is a strictly positive local martingale but fails to be martingale.

When $\gamma > 1$, the drift diverges to plus infinity when $s = 1$ (note that the numerator in the second term is positive as $s \uparrow 1$), we get $\mathbb{P}^\lambda[T_1 < T] > 0$ by using a standard comparison argument (the drift can be bounded from below using a linear function). This contradicts the equivalence between $\mathbb{P}$ and $\mathbb{P}^\lambda$, and thus, λ is a strictly positive local martingale but fails to be martingale.

When $\gamma < 1$, unlike the previous cases, the behavior of the process under $\mathbb{P}^\lambda$ resembles its behavior under $\mathbb{P}$, that is, its drift diverges to negative infinity when $s \uparrow 1$. In order to obtain information about the behavior of the consumption share as it approaches 1, we construct its scale function, Sc(x),

$$\begin{aligned}\mathrm{Sc}(x) &= \int_c^x \exp\left[-2\int_c^y \frac{z\mu_s^\lambda(z)}{z^2\sigma_s(z)^2}dz\right]dy\\ &= \frac{(1-x)x}{b_3-1}\left(\frac{1-x}{1-c}\right)^{-(1+\gamma)}\left(\frac{x}{c}\right)^{b_3-1}\left(\frac{1+(\gamma-1)x}{1+(\gamma-1)c}\right)^{-b_2}F_1\left[1,b_1,b_2,b_3,x,\frac{\gamma x}{1+(\gamma-1)x}\right]\\ &- \frac{(1-c)c}{b_3-1}F_1\left[1,b_1,b_2,b_3,c,\frac{\gamma c}{1+(\gamma-1)c}\right]\end{aligned}$$

where $c \in I$ is an arbitrary constant and $F_1(\cdot)$ denotes the Appell Hypergeometric function of two variables,[30] with $b_1 = \frac{2\mu_e-\gamma[2-\varepsilon(2-\varepsilon)]\sigma_e^2}{(1-\varepsilon)^2\sigma_e^2}$, $b_2 = -2\gamma\frac{\mu_e-\frac{1}{2}\gamma\sigma_e^2}{(1-\varepsilon)^2\sigma_e^2}-1$, $b_3 = 2(1-\gamma)\frac{\mu_e-\frac{1}{2}\gamma\sigma_e^2}{(1-\varepsilon)^2\sigma_e^2}$. When $b_3-1>0$ or equivalently, $(1-\gamma)\left(\mu_e-\frac{1}{2}\gamma\sigma_e^2\right)-\frac{1}{2}(1-\varepsilon)^2\sigma_e^2>0$, we obtain $\lim_{\Delta\to 1}\mathrm{Sc}(\Delta)=\infty$, which is a sufficient condition that guarantees that starting from any point in I, the right boundary cannot be reached in finite time, i.e., $\mathbb{P}^\lambda[T_1<T]=0$ (Karatzas and Shreve, 1991, p. 348). To ensure that the share process does not reach 0, we use a comparison argument, similar to the procedure in Appendix B, we omit the details. ■

Proof of Propositions 7. Let N_t be defined by

$$\begin{aligned}N_t &= \xi_t W_{at} + \psi_a\xi_t S_t + \int_0^t \xi_u(c_{au}+\psi_a e_u)du\\ &= w_{a0}+\psi_a S_0 + \int_0^t \xi_u\left((\pi_{au}+\psi_a S_u)\sigma_u - (\psi_a S_u + W_{au})\theta_u\right)dZ_u \end{aligned} \tag{A.14}$$

with $\psi_a \geq 0$. N_t is a nonnegative local martingale for positive consumption plans, and hence, is a supermartingale. This implies that $\mathbb{E}\left[\int_0^T \xi_t(c_{au}+\psi_a e_u)du + \xi_T(\psi_a S_T + W_{aT})\right] \leq w_{a0}+\psi_a S_0$. The arbitrageur optimization problem can be formulated as

$$\sup_{c\geq 0}\mathbb{E}\left[\int_0^\infty e^{-\rho u}\log c_u du\right] \text{ subject to } \mathbb{E}\left[\int_0^\infty \xi_u c_u du\right] \leq w_{a0}+\psi_a B_0$$

where the static budget constraint follows by letting $T\to\infty$, fixing $w_{a0}=w_0-pP_0$ and applying the definition of the bubble component in (A.14). We will verify that the transversality conditions $\lim_{T\to\infty}\mathbb{E}[\xi_T W_T] = \lim_{T\to\infty}\mathbb{E}[\xi_T S_T]=0$ are satisfied in equilibrium. The solution to this problem and the corresponding wealth process are explicitly given by $c_{at}=(y_a e^{\rho t}\xi_t)^{-1}$ and $W_{at}+\psi_a B_t = \frac{1}{\xi_t}\mathbb{E}_t\left[\int_t^\infty \xi_u c_{au}du\right] = \frac{c_{at}}{\rho} = e^{-\rho t}\frac{w_0+\psi_a B_0 - pP_0}{\xi_t}$. for the Lagrange multiplier $y_a>0$ given $\frac{1}{y_a\rho}=w_0+\psi_a B_0 - pP_0$. On the other hand, applying Itô's lemma to the left hand side of the above expression and using the fact that $B_t = B_0+\int_0^t\left(r_s B_s ds + \Sigma_s^B(dZ_s+\theta_s ds)\right)$, we deduce that the optimal wealth process evolves according to

$$W_{at} = \int_0^t \left((r_s W_{au}-c_{au})du + \left((W_{au}+\psi_a B_u)\theta_u - \psi_a\Sigma_u^B\right)(dZ_u+\theta_u du)\right)$$

and the result in (28) now follows by comparing this expression to (5). This process satisfies by construction the constraint in (26).

As in the baseline model, we construct a sharing rule so that the consumption good market clears. The process $s_t \equiv \frac{c_{\ell t}}{e_t} = \frac{\lambda_t}{1+\nu+\lambda_t} \in (0,1)$ represents the consumption share of the liquidity

[30]See e.g., Whittaker and Watson (1990), Ex.22, p.300., or http://functions.wolfram.com/HypergeometricFunctions/AppellF1/

provider and $c_{\ell t} = s_t e_t$, $c_{1t} = \frac{1}{1+\nu}(1-s_t)e_t$, and $c_{at} = \frac{\nu}{1+\nu}(1-s_t)e_t$. The equilibrium state price density is given by $\xi_t = e^{-\rho t}\frac{e_0(1-s_0)}{e_t(1-s_t)}$. To compute the equilibrium price of the stock, we rely on the financial market clearing condition which requires $S_t = \sum_{k=1}^3 W_{kt}$. Combining this identity with the optimal consumption rules and the clearing condition of the consumption good market gives $S_t = P_t - \psi_a B_t = P_t - \psi_a(S_t - F_t)$ where the process F_t is the fundamental value of the stock. Setting $\alpha = \psi_a/(1+\psi_a)$ and solving for the stock price we get (30). The stock price includes the bubble, $B_t = \frac{(1-\alpha)e_t}{1+\nu+\lambda_t}\mathbb{E}_t\left[\int_t^\infty e^{-\rho(u-t)}(\lambda_t - \lambda_u)\,du\right]$ since an application of Itô's lemma gives $d\lambda_t = (1+\nu)d\left(\frac{s_t}{1-s_t}\right) = -\lambda_t(1+\nu+\lambda_t)\frac{\sigma_e}{1+\nu}dZ_t, = -\lambda_t\left(1+\frac{s_t}{1-s_t}\right)\sigma_e dZ_t$ and the uniqueness of the solution to (A.9) implies that $\lambda_t = Y_t(1)$, so that λ is a strict local martingale. The characterization of relative bubbles follows from an identical procedure to the one described in the proof of Proposition 4. We omit the details. To conclude, note that $\alpha s_T^\kappa \leq 1$ and the supermartingale property of the process λ implies that

$$\mathbb{E}[\xi_T S_T] = e^{-\rho T}\frac{P_0}{1+\nu+\lambda_0}\mathbb{E}[(1+\nu+\lambda_T)(1-\alpha s_T^\kappa)] \leq e^{-\rho T}P_0$$

and since $\rho > 0$ it follows that the transversality condition holds. ∎

Proof of Proposition 8. $\{h, \ell, a\}$: *Agent 2 as liquidity provider.* Let $\mathcal{A}k$ denote Agent $k \in \{1,2,3\}$. The equilibrium is determined by the following system of equations

$$(\mathcal{A}2) \qquad \frac{1}{\rho}\log(s) - \frac{\sigma_e^2}{2\rho^2}\frac{1+s-2s^\kappa}{1-s} = \frac{1}{\rho}\log(n), \tag{A.15}$$

$$(\mathcal{A}2) \qquad s = n + p, \tag{A.16}$$

$$(\mathcal{A}1) \qquad \frac{1}{1+\nu}(1-s) = 1 - \alpha s^\kappa - n, \tag{A.17}$$

$$(\mathcal{A}3) \qquad \frac{\nu}{1+\nu}(1-s) = \psi(1-\alpha)s^\kappa - p = \alpha s^\kappa - p. \tag{A.18}$$

Conditions for existence are characterized next. The left hand side in (A.15) is monotone increasing in s by the comparison theorem for solutions of SDEs with limits $\{-\infty, -(2\kappa-1)\sigma_e^2/(2\rho^2)\}$, so the single root $s \equiv s(n) \in (0,1)$ to (A.15) exists as long as $n \in (0,\bar{n})$, where $\bar{n} = \min(1,u)$ with $\{u : \frac{1}{\rho}\log(u) = \lim_{s\to 1}\frac{1}{\rho}\log(s) - \frac{\sigma_e^2}{2\rho^2}\frac{1+s-2s^\kappa}{1-s}\}$ is the positive constant defined in the Proposition. Furthermore, it follows from an application of the implicit function theorem in (A.15) that $p'(n) = s'(n) - 1 \geq 0$.[31] Next, we use (A.17) and (A.18) to get ν

$$\nu = \frac{\psi(1-\alpha)s^\kappa - p}{1 - \alpha s^\kappa - n}. \tag{A.19}$$

Agent 1 and 3's initial (effective) wealth must be strictly positive in (A.17) and (A.18). This implies that $\nu > 0$ if and only if

$$\psi(1-\alpha)s^\kappa - p = \alpha s^\kappa - p > 0, \tag{A.20}$$

$$1 - \alpha s^\kappa - n > 0. \tag{A.21}$$

From (A.16), we have that $p = s - n$, and substituting in the inequality (A.20), $n > s - \alpha s^\kappa$. From (A.21), we have $1 - \alpha s^\kappa - n > 0$. Taken together, $s - \alpha s^\kappa < n < 1 - \alpha s^\kappa$. We have two cases:

[31] This condition amounts to $(1-s)^2(s-n)\rho + n\sigma_e^2(s + s^\kappa((\kappa-1)s - \kappa)) \geq 0$ which is verified as $s \in (n,1)$.

Case 1: $n+\alpha > 1$, i.e., $(1-n)/\alpha < 1$. From the upper bound $n < 1-\alpha s^\kappa \Leftrightarrow s < \left(\frac{1-n}{\alpha}\right)^{1/\kappa}$. From the lower bound $p < \alpha s^\kappa \Leftrightarrow s > \left(\frac{p}{\alpha}\right)^{1/\kappa}$. Note that since $p'(n) > 0$ and $p(\bar{n}) = s(\bar{n}) - \bar{n} = 1-\bar{n}$, a sufficient condition is $s > \left(\frac{1-\bar{n}}{\alpha}\right)^{1/\kappa}$. Existence follows if the starting point s_0 belongs to the open set $\left(\left(\frac{1-\bar{n}}{\alpha}\right)^{1/\kappa}, \left(\frac{1-n}{\alpha}\right)^{1/\kappa}\right)$.

Case 2: $n+\alpha < 1$, i.e., $(1-n)/\alpha > 1$. The same steps apply, but the right hand side of the interval becomes $\min(1, ((1-n)/\alpha)^{1/\kappa}) = 1$. Existence follows if $s_0 \in \left(\left(\frac{1-\bar{n}}{\alpha}\right)^{1/\kappa}, 1\right)$ and $\bar{n}+\alpha > 1$.

$\{h, a, \ell\}$: *Agent 3 as liquidity provider.* $\mathcal{A}3$ receives compensation $pP_0 \geq 0$, and this quantity fixes the starting point for the consumption share of the liquidity provider. For this equilibrium to be incentive compatible,

$$(\mathcal{A}2) \qquad \frac{1}{\rho}\log(\nu/(1+\nu)) + U_1^{I,p}(p) - U_0 = \frac{1}{\rho}\log(n), \tag{A.22}$$

$$(\mathcal{A}3) \qquad s = p,$$

$$(\mathcal{A}1) \qquad \frac{1}{1+\nu}(1-p) = 1-\alpha p^\kappa - n, \tag{A.23}$$

$$(\mathcal{A}2) \qquad \frac{\nu}{1+\nu}(1-p) = n + \psi(1-\alpha)p^\kappa - p = n + \alpha p^\kappa - p. \tag{A.24}$$

where we have from (12), $U_1^{I,p}(p) - U_0 = -\frac{1}{1-p}\sum_{j=1}^{\infty}\frac{1}{j}u(p,j)$. Solving for ν using (A.23) and (A.24),

$$\nu = \frac{n+\alpha p^\kappa - p}{1-(\alpha p^\kappa + n)} \tag{A.25}$$

so that $\nu > 0$ when $p < \alpha p^\kappa + n < 1$, or equivalently, $p - \alpha p^\kappa < n < 1-\alpha p^\kappa$. From the upper bound $n < 1-\alpha p^\kappa \Leftrightarrow p < \left(\frac{1-n}{\alpha}\right)^{1/\kappa}$. Assume for now the lower bound $p - \alpha p^\kappa < n$ holds.We will verify later that it does. There is one root in (A.22) at $p = s = 0$, we look for an additional one. Substituting (A.24) into (A.22) and rearranging gives

$$\begin{aligned} f(p) &= \frac{n+\alpha p^\kappa - p}{1-p} = ne^{-\rho(U_1^{I,p}(p)-U_0)} = g(p), \\ f'(p) &= \frac{q(p)}{(1-p)^2}; \quad g'(p) = -\rho\partial U_1^{I,p}(p)g(p) \geq 0, \\ q(p) &= \alpha p^{\kappa-1}(\kappa + p(1-\kappa)) + n - 1, \\ q'(p) &= \alpha p^{\kappa-2}(\kappa-1)\kappa(1-p) > 0 \text{ for } p \in (0,1). \end{aligned}$$

The function $g(\cdot)$ is non-decreasing with respect to p, because $\partial U_1^{I,p}(p) \leq 0$ by Lemma 1, with limit values $g(0) = n$ and $g(1) < \infty$. Under the condition $\alpha + n - 1 > 0$, the function $f(\cdot)$ is decreasing-increasing with minimum at $p^o \in (0,1)$ such that $q(p^o) = 0$ and limit values $f(0) = n$ and $f(1) = +\infty$. In this case, there exists a positive solution in the open interval $(p^o, 1)$, denoted by p^*. If $\alpha + n - 1 < 0$, there is no $p^o \in (0,1)$ such that $q(p^o) = 0$, hence no solution in $(0,1)$. To complete the proof for the case $\alpha + n - 1 > 0$, we need to verify $\nu(p^*) > 0$, i.e., $p^* < \bar{p}$, where $\nu(p)$ is given by the right hand side of (A.25) and $\bar{p} = ((1-n)/\alpha)^{1/\kappa}$. This condition holds if $f(\bar{p}) > g(\bar{p})$. Since the left hand side equals $f(\bar{p}) = 1$, a sufficient condition is $U_1^{I,p}(\bar{p}) - U_0 > \log(n)/\rho$. ■

Proof of Proposition 9. The result follows from a direct application of the equilibrium allocations of Proposition 7 into the welfare functions described in Proposition 3. Assume $U_1^{I,p}(\cdot) > U_1^C(\cdot)$. Then there is a feasible consumption allocation rule that Pareto dominates the allocation in (6), since the liquidity provider is indifferent, and the arbitrageur consumption is nonzero. This is a contradiction with respect to the first welfare theorem that states that if there is an equilibrium in which markets are complete, then the corresponding consumption allocation is Pareto optimal. ■

Computation of Equations (20)–(36). The expression for the equilibrium trading strategies now follows by noting that we have $\pi_{kt} = \frac{1}{\sigma_t}\left[\frac{\partial W_{kt}}{\partial P_t} P_t \sigma_e - \frac{\partial W_{kt}}{\partial s_t} s_t \sigma_e\right]$ and $\phi_{kt} = W_{kt} - \pi_{kt}$ as a result of (5), (11) and Itô's lemma. To establish the sign of ϕ_{1t} we argue as follows. Using the fact that $v(s) \geq 0$ we deduce that

$$\operatorname{sign}[\phi_{1t}] = \operatorname{sign}[v(s) - s(1+v(s))] = \operatorname{sign}\left[-1 + \alpha s^{\kappa-1}(s + \kappa(1-s))\right].$$

Denote by $h(s)$ the continuous function inside the bracket. Since $\kappa > 1$ we have that this function is increasing with $h(0) = -1$ and $h(1) = \alpha - 1$ and the result now follows by continuity. Finally, since $-\phi_{at} = \nu\phi_{1t} - \phi_{3t} \geq 0$ we have that $\phi_{3t} \leq 0$. We find a pair of adapted processes such that $W_{1t} = \frac{1}{1+\nu}(1-s_t)P_t = \phi_{1t}^S(T) + \phi_{1t}^{B_0}(T)$ and $dW_{1t} = \phi_{1t}^S(T)\left(\frac{dS_t}{S_t} + \frac{e_t}{S_t}dt\right) + \phi_{1t}^{B_0}(T)\frac{dB_{0t}(T)}{B_{0t}(T)} - \rho W_{1t}dt$ where S_t denotes the equilibrium stock price and $B_{0t}(T)$ denotes the bubble on the riskless asset at horizon T. Expanding the dynamics of these two processes and using (5) shows that these equations are equivalent to

$$\begin{aligned}
&\phi_{1t}^{B_0}(T) + \phi_{1t}^S(T) = W_{1t} \\
&\phi_{1t}^{B_0}(T)\Sigma_{0t}(T) + \phi_{1t}^S(T)(1+v(s_t)) = \pi_{1t}(1+v(s_t))
\end{aligned}$$

and solving this system gives the desired decomposition. The sign of the riskless asset bubble position follows by noting that $\operatorname{sign}\phi_{1t}^{B_0}(T) = \operatorname{sign}\phi_{1t}$. To establish the sign of the position in the stock and complete the proof it suffices to show that $\Sigma_{0t}(T) \leq 0$ or, equivalently, that the function $H(\tau, s; a)$ is increasing in s. Differentiating in (17) shows that

$$s^{1+a}\frac{\partial H}{\partial s}(\tau, s; a) = aN(d_-(\tau, s; a)) + \frac{2}{\sigma_e\sqrt{\tau}}N'(d_-(\tau, s; a))$$

and the required result now follows by noting that the function on the right hand side is increasing on $[0, 1]$ and equal to zero at zero. The ratio in (36) depends on α only through $v(\cdot)$, and the latter is increasing in α. Thus it decreases when the size of credit facility goes up. ■

Proof of Proposition 10. The utility functions for each type are given by $a : \frac{1}{\rho}\log\frac{\nu}{1+\nu} + U_1^{I,p}(s)$, $h : \frac{1}{\rho}\log\frac{1}{1+\nu} + U_1^{I,p}(s)$, and $\ell : U_0 + \frac{1}{\rho}\log s - \frac{\sigma_e}{2\rho^2}\frac{1+s-2s^\kappa}{1-s}$.

$\{h, \ell, a\}$: since Agent 2 is the liquidity provider, then

$$\begin{aligned}
\mathcal{U}(\Theta) =& \eta_1\left(\frac{1}{\rho}\log\frac{1}{1+\nu(\alpha)} + U_1^{I,p}(n+p(n))\right) + \eta_2\left(U_0 + \frac{1}{\rho}\log(n)\right) \\
&+ \eta_3\left(\frac{1}{\rho}\log\frac{\nu(\alpha)}{1+\nu(\alpha)} + U_1^{I,p}(n+p(n))\right) \\
=& \eta_1\frac{1}{\rho}\log\frac{1}{1+\nu(\alpha)} + \eta_3\frac{1}{\rho}\log\frac{\nu(\alpha)}{1+\nu(\alpha)} + Q(n,\eta)
\end{aligned}$$

where $\nu(\alpha)$ is given in (A.19) and α is not an argument of $Q(\cdot)$. The first order condition gives $\nu(\alpha) = c_{3t}/c_{1t} = \eta_3/\eta_1$, and solving for α using (A.19) and $s(n) = n + p(n)$ gives (37).

$\{h, a, \ell\}$: since Agent 3 is the liquidity provider, then

$$
\begin{aligned}
\mathcal{U}(\Theta) =& \eta_1 \left(\frac{1}{\rho} \log \frac{1}{1+\nu(\alpha)} + U_1^{I,p}(p(n,\alpha)) \right) + \eta_2 \left(\frac{1}{\rho} \log \frac{\nu(\alpha)}{1+\nu(\alpha)} + U_1^{I,p}(p(n,\alpha)) \right) \\
&+ \eta_3 \left(U_0 + \frac{\log p(n,\alpha)}{\rho} - \frac{\sigma_e^2}{2\rho^2} \frac{1+p(n,\alpha)-2p(n,\alpha)^\kappa}{1-p(n,\alpha)} \right) \\
=& -\frac{\eta_1}{\rho} \log \nu(\alpha) + \eta_3 \left(\frac{\log p(n,\alpha)}{\rho} - \frac{\sigma_e^2}{2\rho^2} \frac{1+p(n,\alpha)-2p(n,\alpha)^\kappa}{1-p(n,\alpha)} \right) + T(n,\eta)
\end{aligned}
$$

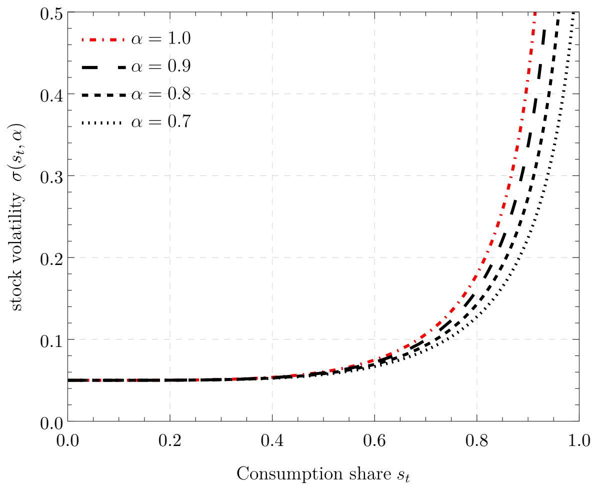
where we used the compatibility condition in (A.22), rewritten as

$$
U_1^{I,p}(p(n,\alpha)) = U_0 + \frac{1}{\rho} \log(n) - \frac{1}{\rho} \log \frac{\nu(\alpha)}{1+\nu(\alpha)}
$$

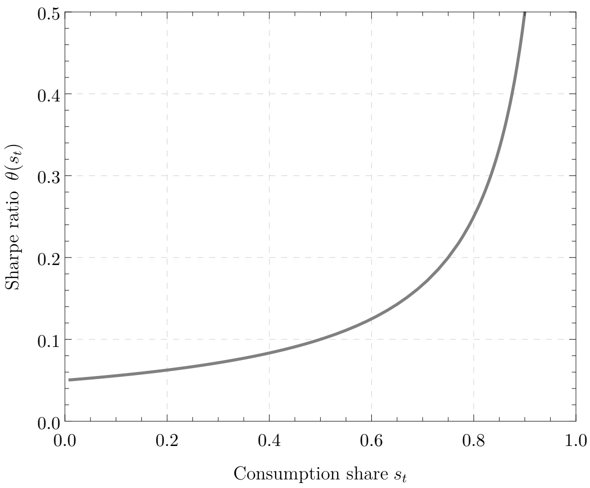
to deduce

$$
\begin{aligned}
\frac{1}{\rho} \log \frac{1}{1+\nu(\alpha)} + U_1^{I,p}(p(n,\alpha)) &= U_0 + \frac{1}{\rho} \log \frac{1}{1+\nu(\alpha)} + \frac{1}{\rho} \log(n) - \frac{1}{\rho} \log \frac{\nu(\alpha)}{1+\nu(\alpha)} \\
&= U_0 + \frac{1}{\rho} \log(n) - \frac{1}{\rho} \log \nu(\alpha)
\end{aligned}
$$

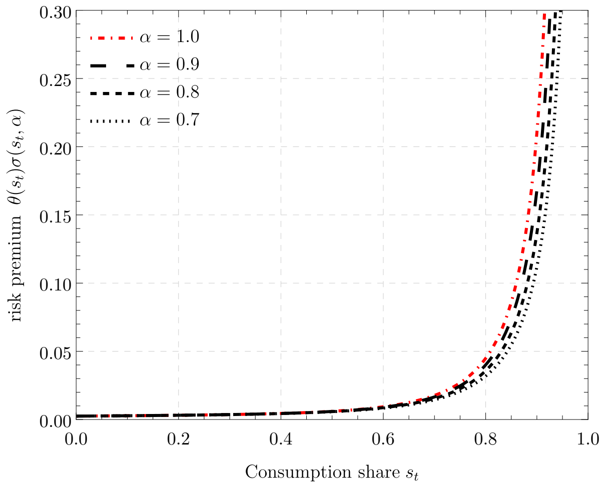
and simplify terms so as to arrive at the second equality. Finally, we note that the consumption sharing ratio $\nu(\alpha)$ is given in (A.25) and that α is not an argument of $T(\cdot)$. ■



(a) Stock volatility

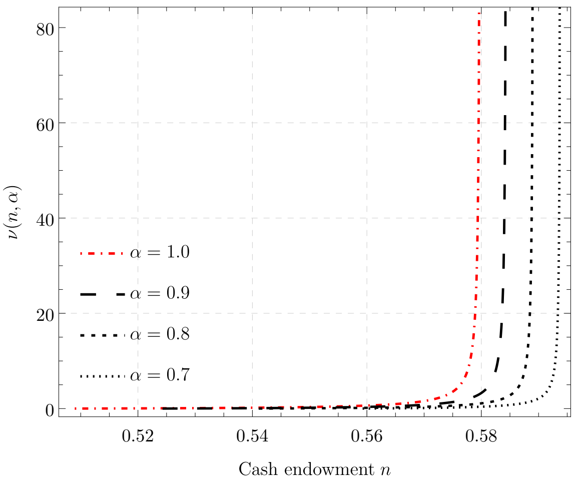


(b) Sharpe ratio

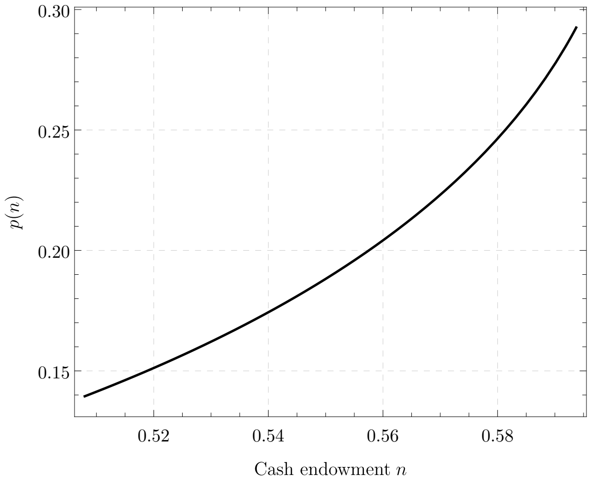


(c) Equity risk premium

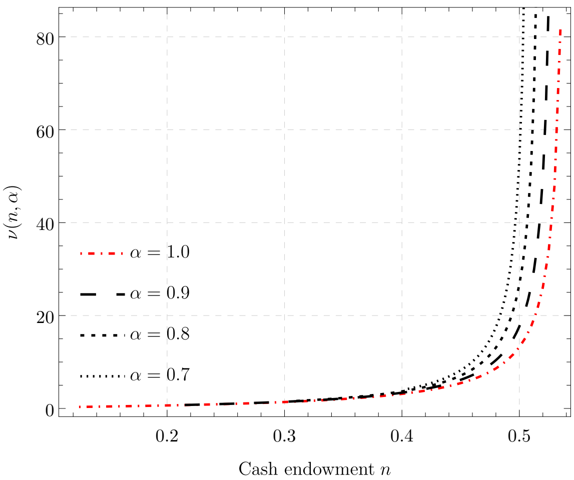
Figure 1: **Equilibrium quantities**. Parameters values are given by $\{\rho, \sigma_e\} = \{0.02, 0.05\}$. As shown by the figure the impact in equilibrium quantities is sizable and increases with both the consumption share of liquidity providers and the size of the credit line. For example, with 80% of liquidity providers, the stock volatility is about 3 times the dividend volatility, the equity premium near 5% and the Sharpe ratio around 0.25.



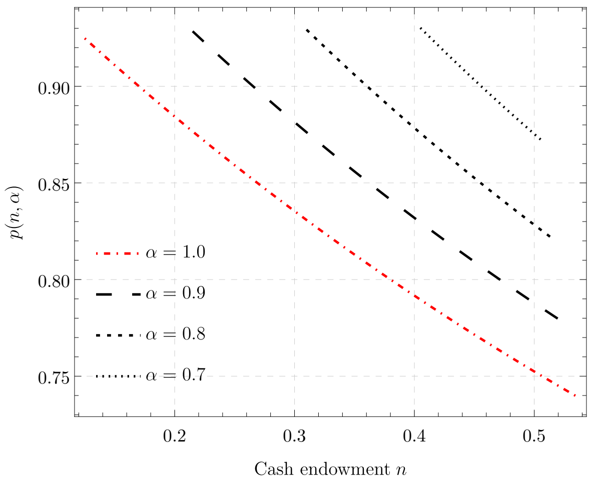
(a) $\{h, \ell, a\}$: Sharing rule $\nu(n, \alpha)$



(b) $\{h, \ell, a\}$: Credit line price $p(n)$

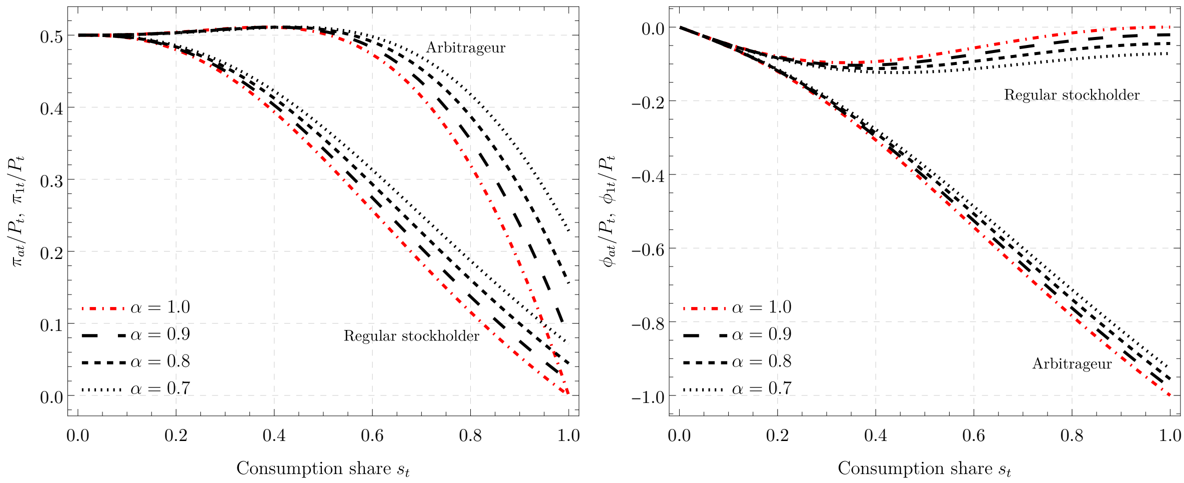


(c) $\{h, a, \ell\}$: Sharing rule $\nu(n, \alpha)$



(d) $\{h, a, \ell\}$: Credit line price $p(n, \alpha)$

Figure 2: **Equilibrium price and sharing rule**. Upper panels $\{h, \ell, a\}$: Agent 2 is the arbitrageur and Agent 3 is the liquidity provider. Lower panels $\{h, a, \ell\}$: Agent 2 is the liquidity provider and Agent 3 is the arbitrageur. Parameters values are given by $\{\rho, \sigma_e\} = \{0.02, 0.05\}$.



(a) Stock *(b)* Money market

Figure 3: **Portfolio positions**. Panel (a): Scaled portfolio positions on the stock, Panel (b): Scaled portfolio positions on the riskless asset. Parameters values are given by $\{\rho, \sigma_e, \nu\} = \{0.02, 0.1, 1\}$. Quantities are scaled by P_t instead of the agent's own wealth to facilitate comparison.

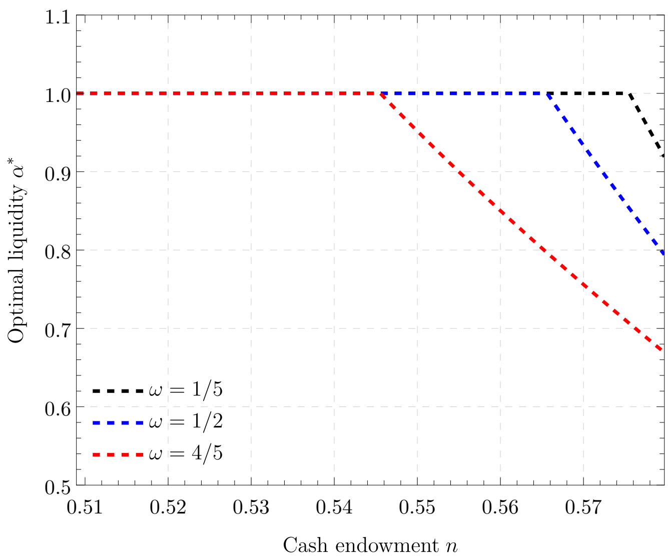


Figure 4: **Optimal α when Agent 2 is the liquidity provider, Agent 3 is the arbitrageur** $\{h, \ell, a\}$. Figure depicts the optimal liquidity level from the closed-form in (37). Parameters values are given by $\{\rho, \sigma_e\} = \{0.02, 0.05\}$.

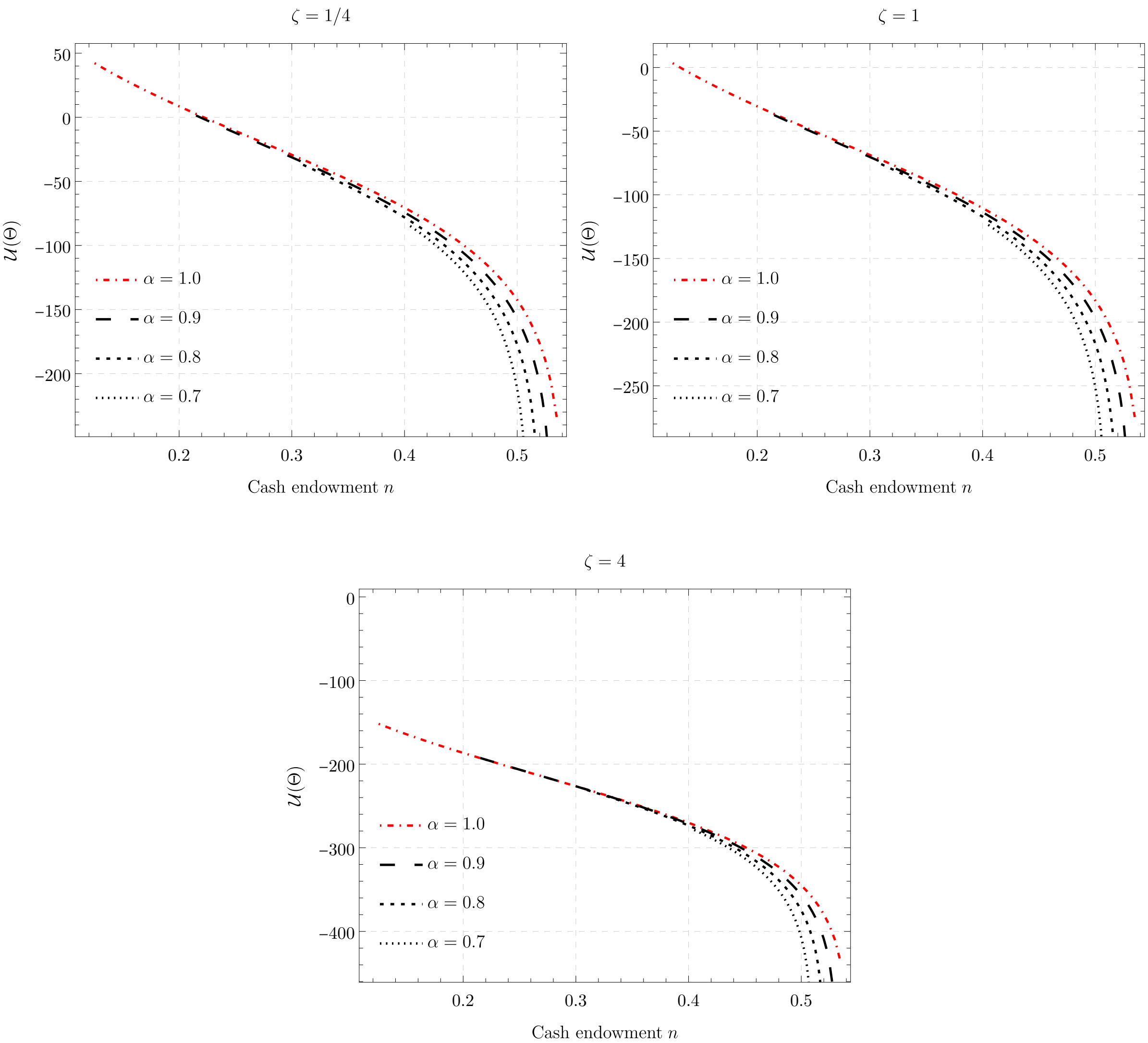


Figure 5: **Social welfare functions when Agent 2 is the arbitrageur, Agent 3 is the liquidity provider** $\{h, a, \ell\}$. The parameter $\zeta = \eta_3/\eta_1$ is the relative weight across investors in the social welfare function $-\frac{1}{\rho}\log\nu(\alpha) + \zeta\left(\frac{\log p(n,\alpha)}{\rho} - \frac{\sigma_e^2}{2\rho^2}\frac{1+p(n,\alpha)-2p(n,\alpha)^\kappa}{1-p(n,\alpha)}\right)$. Parameters values are given by $\{\rho, \sigma_e\} = \{0.02, 0.1\}$.

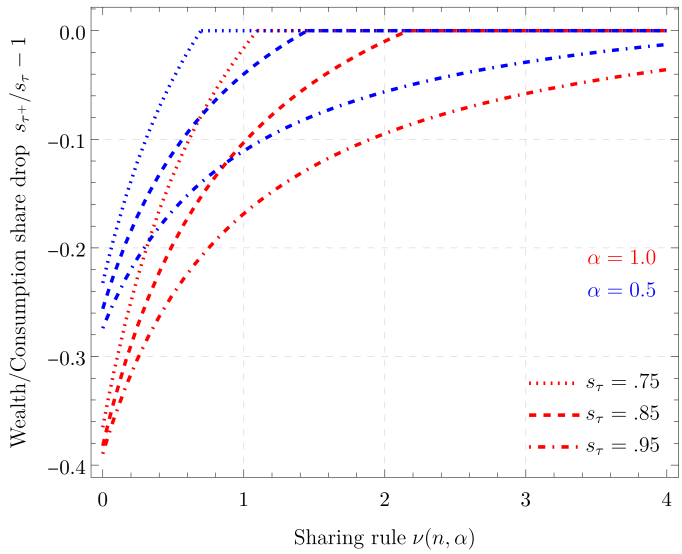


Figure 6: **Impact of the liquidity shock**. Figure depicts the impact of the liquidity shock in the wealth/consumption share of the liquidity provider, eq. (42). Parameters values are given by $\{\rho, \sigma_e\} = \{0.02, 0.1\}$.

B Online Appendix

B.1 Existence of equilibrium with heterogeneous agents

The dynamics of the consumption share process are given by

$$\frac{ds_t}{s_t} = \mu_s(s_t)dt + \sigma_s(s_t)dZ_t \tag{B.1}$$

with

$$\mu_s(x) = \begin{cases} (\gamma-1)\frac{1-x}{1+(\gamma-1)x}\mu_e + (\gamma-1)\gamma\frac{[2-\varepsilon(2-\varepsilon)x]x-1}{2(1-x)[1+(\gamma-1)x]}\sigma_e^2 \\ +(1-\varepsilon)\sigma_e^2\left(1-\gamma\frac{1-\varepsilon x}{1-x}\right), & R(x) > \varepsilon, \\ \\ (\gamma-1)\frac{1-x}{1+(\gamma-1)x}\mu_e + \frac{\gamma}{1+(\gamma-1)x}\sigma_e^2 - \frac{\gamma[1+\gamma+(\gamma-1)(1+2\gamma)x]}{2[1+(\gamma-1)x]^3}\sigma_e^2 \\ -(\gamma-1)\frac{(1-x)[1-\gamma+(\gamma-1)x]}{[1+x(\gamma-1)]^2}\sigma_e^2, & R(x) \leq \varepsilon, \end{cases} \tag{B.2}$$

and

$$\sigma_s(x) = \begin{cases} -(1-\varepsilon)\sigma_e, & R(x) > \varepsilon, \\ \\ \sigma_e\left(\frac{\gamma}{1+(\gamma-1)x} - 1\right), & R(x) \leq \varepsilon. \end{cases} \tag{B.3}$$

The drift diverges to minus infinity as $s \uparrow 1$ if the constraint binds. This behavior counterbalances the effect of the linear diffusion in (B.3), such that the process never reaches 1 from the interior. This observation, in conjunction with comparison arguments, gives us the following existence result.

Proposition B.1. *Suppose that the process in* (B.1) *has a starting point in* $(0, 1)$*, the equilibrium exists as the boundary points* $\{0, 1\}$ *cannot be reached in finite time.*

Proof. Set $I = (0, 1)$. (i) the drift and diffusion functions have continuous derivatives in I and (ii) $(s\sigma_s)^2 > 0$ in I. We also verify a (iii) local integrability condition, that for all $x \in I$, there exists $\epsilon > 0$ such that $\int_{x-\epsilon}^{x+\epsilon} \frac{1+|\mu_s(y)|}{\sigma_s(y)^2} dy < \infty$. It is well known that (i) implies that the coefficients are locally Lipschitz, a sufficient condition for pathwise uniqueness of the solution (Karatzas and Shreve, 1991, Th. 5.2.5). Also, conditions (i), (ii) and (iii) guarantee the existence of a weak solution (Karatzas and Shreve, 1991, Th. 5.5.15) possibly up to an explosion time. A weak solution combined with pathwise uniqueness imply that equation (A.2) admits a strong solution possibly up to an explosion time, i.e., when s_t hits one of the endpoints of I. Define the stopping times $T_\Delta = \inf\{t \geq 0 : s_t \geq \Delta\}$, with $\Delta \in I$, and let $T_1 = \lim_{\Delta\to 1} T_\Delta$, $T_0 = \lim_{\Delta \to 0} T_\Delta$. To rule out explosions, we proceed as follows. From equation (A.12), λ is a nonnegative supermartingale under $\mathbb{P}$,

$$\mathbb{E}[\lambda_T] \leq \lambda_0 > 0, \quad \text{for all} \quad T \in [0, \infty),$$

consequently, it is a.s. finite under the objective probability measure, which implies

$$\mathbb{P}[T_1 < T] = 0, \quad \text{for all} \quad T \in [0, \infty).$$

Let $\bar{s}_t = s_{t \wedge T_\Delta}$. Using a comparison argument (Karatzas and Shreve, 1991, Prop. 5.2.18), we bound the stopped process $\bar{s}_t$ from below by a process s_ℓ, with dynamics

$$ds_{\ell t} = \mu_{s_\ell}(s_{\ell t}) s_{\ell t} dt + \sigma_s(s_{\ell t}) s_{\ell t} dZ_t.$$

We construct this process so that it never reaches the left boundary, and such that its diffusion is the same as s_t, $x\sigma_s(\cdot)$. We fix $s_{\ell 0} = s_0$ and set the drift of s_ℓ such that $\mu_{s_\ell}(x) \leq \mu_s(x)$, which implies that for all constants $\Delta \in I$, with $\Delta > s_0$,

$$\mathbb{P}[T_0 < T_\Delta] = \mathbb{E}[\mathbf{1}_{\{T_0 < T_\Delta\}}] = 0, \tag{B.4}$$

that is, the probability of the consumption share process of hitting 0 before it reaches Δ is zero. In order to show that $\mathbb{P}[T_0 < T] = 0$, for all $T \in [0, \infty)$, it suffices to note that

$$\mathbb{E}[\lim_{\Delta \to 1} \mathbf{1}_{\{T_0 < T_\Delta\}}] \leq \lim_{\Delta \to 1} \mathbb{E}[\mathbf{1}_{\{T_0 < T_\Delta\}}] = 0,$$

which follows from Fatou's lemma, and (B.4), implying $\mathbb{P}[T_0 < T] = 0$, for all $T \in [0, \infty)$, since the probability of reaching 1 in finite time is zero a.s.. To close the proof, we find a candidate process s_ℓ, given by a geometric Brownian motion with dynamics

$$ds_{\ell t} = \mu_{s_\ell} s_{\ell t} dt - (1 - \varepsilon)\sigma_e s_{\ell t} dZ_t,$$

such that $\mu_s(x) \geq \mu_{s_\ell}$. ∎