Constrained Borrowing and Living Standard: Optimal Consumption/Savings and Investment Policies*

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Abstract

We study the optimal dynamic consumption and portfolio decisions of utility-maximizing agents who wish to maintain a living standard. These agents exhibit decreasing relative risk aversion as their living standard increases. Our findings reveal that the requirement to uphold a minimum living standard allows borrowing-constrained agents to endogenously determine a wealth threshold, which we refer to as subsistence wealth. Below this threshold, agents optimally choose to consume nothing. However, once their wealth surpasses the subsistence level, they significantly increase their consumption. This behavior aligns more closely with empirical estimates of marginal propensities to consume. Moreover, the presence of subsistence wealth lowers agents' effective risk aversion, leading them to favor riskier portfolios in the stock market.

Keywords : Living Standard, Subsistence Wealth, Constrained Borrowing, Decreasing Risk Aversion, Consumption

JEL classification : E21, G11

1 Introduction

> Britain's cost of living crisis is showing no sign of easing after new data showed UK inflation stuck above 10 percent in March, making it more likely that the Bank of England will increase interest rates next month. (Financial Times; April 19, 2023)

The dramatic rise of inflation since post-COVID-19 has significantly driven an increase in concern about the cost of living. The above quote from the Financial Times demonstrates little hope for an end to the cost of living crisis with a more tightening liquidity environment. Indeed, a number of people in the world face a significant shortage of a minimum living standard[1]. The paper responds to the urgent need to develop theories that address some interactions of consumption and investment policies with constrained borrowing to maintain a living standard in today's inflation crisis.

The living standard has been supported by a wide range of government subsidies and social security programs[2]. In particular, a universal basic income (UBI) comprised of periodic cash payments to an individual already offered in many countries (e.g., Canada, the U.S., Finland, and India) can be in part for the minimum level of subsistence for households. In the U.K., Income Support (as extra money) has been provided to those on a low income or none at all, and it is being replaced by Universal Credit, which is a payment to help with low-income people's living costs.

We study optimal dynamic consumption and portfolio policies of utility-maximizing agents who must maintain a living standard. The agents receive a constant labor income stream and choose (i) how much to consume and (ii) how to allocate their wealth between a riskless bond and a risky stock, facing a constant investment opportunity set. They are borrowing constrained, but they could partially borrow against the net present value

[1]44% of U.S. households cannot pay for just $400 emergency expense (Federal Reserve report, 2017), and approximately 218 million people in the European Union struggle how to meet future consumption needs provided earnings insecurity and volatility (European Commission statistics, 2017).

[2]For instance, a consumption voucher has been provided to households by many governments in the world for the purpose of restoring consumption during the recent COVID-19 pandemic period. Also, retirement benefits provided by a social security program can be thought of as the minimum subsistence provision for public welfare. Other direct money provisions for both liquidity and stimulus purposes (e.g., loans at a lower interest rate, massive tax cuts, providing perishable goods and services) are all for households to maintain a reasonable quality of life.

of their future income[3]. Their utility exhibits decreasing relative risk aversion with the living standard the agents wish to maintain.

We offer three insights that contribute to the consumption/savings and investment literature. We show that maintaining the minimum standard of living allows the borrowing-constrained agents to endogenize a certain wealth threshold, which we call subsistence wealth, below which they optimally choose to consume nothing[4]. The agents then find it optimal to increase consumption significantly once they accumulate assets above subsistence wealth.

The living standard tends to shift subsistence wealth upwards, whereas the greater extent to which agents are allowed to borrow against their human capital (implying more borrowing) brings subsistence wealth down. There are intuitive predictions. If an agent wishes to maintain a high living standard, far enough financial resources are required for the agent to sustain such a living standard, and therefore, the agent tends to target high subsistence wealth. In the limiting case for which agents are fully allowed to borrow, such subsistence wealth does not exist even in the presence of living standards. An agent who is endowed with lots of human capital from borrowing can easily finance living standards without characterizing subsistence wealth because the agent's total available financial resources are expanded by the opportunity of enough borrowing. This borrowing effect tends to somehow counteract the effects of living standards and produces a much greater decrease in subsistence wealth. The living standard levels matter for the agent when formulating subsistence wealth, but only when an agent is borrowing constrained.

Second, the model better matches empirical marginal propensities to consume (MPC) estimates. The model's capability of matching the MPCs is greatly improved with living standard, easily generating as high as 10% for high-wealth agents[5]. The MPCs produced

[3]We think of constrained borrowing as caused by market frictions (e.g., informational asymmetry, agency conflicts, and limited enforcement). In the most extreme case, the agents are disallowed from borrowing against their income.

[4]Here, subsistence wealth may be loosely related to a so-called subsistence wealth because income and wealth serve as strong social determinants so that the perception of psychological wealth is now increasingly being considered in UBI to maintain the minimum living standard. The anecdotal and empirical evidence have supported the positive psychological impacts of UBI on health conditions (Royal Society and Art report, 2022).

[5]Theoretically suggested MPCs by most macroeconomic models are about 4% (Wang (2003)), whereas the MPCs empirically observed from the data range from 20% to 60% (Carroll et al. (2017)). According to Fisher

by our model are even greater than 10% and tend to sharply increase as agents' wealth approaches subsistence wealth.

Contrary to classical consumption and portfolio models without living standards that support the low concavity of consumption function generates the MPCs that are quite low even for low wealth levels, the effects of consumption concavity in our model with the living standard exhibit the most striking differences for those with low levels of wealth (close to subsistence wealth). The agents' greater consumption responses (in MPCs) to wealth changes, especially when they have little wealth, are closely associated with an important discontinuity and significant change in optimal policies at subsistence wealth below which they optimally find it to not consume at all and above which they substantially increase the amount of consumption. The departure of our model from benchmark models without living standards is highest in *transitional* states around subsistence wealth, where there is the highest uncertainty concerning whether wealth levels are enough to support positive consumption. The agents' aggressive bidding for consumption occurs as an optimal response to subsistence wealth above *transitional* states because of the agents' affordability of consumption having maintained the minimum standard of living well.

Third, subsistence wealth endogenously determined by living standards tends to reduce agents' effective risk aversion, thus increasing their risk-taking in the stock market. We show the agents' dynamic portfolio decision to deviate significantly from that of benchmark models without living standards, especially when they have low levels of wealth. Agents with high wealth act similarly to benchmark agents, investing a similar fraction of their wealth in the stock market with one more unit of wealth. However, as wealth decumulates, the agents begin to increase their equity exposure with one more unit of wealth, then approaching subsistence wealth closely so that in *transitional* states, they invest a much higher fraction of their wealth in stocks with an increase in wealth compared to benchmark cases without living standard.

Intuitively, in the presence of living standards, the decision to buy more or less stocks is influenced, to a large extent, by the way in which agents' desire to maintain the standard of living is satisfied. When wealth is large, agents are already in a liquid position so that they are actually not constrained to maintain the living standard, thus leading their portfolio decisions to be similar to those derived by the benchmark models without living standards. When wealth is small but still above subsistence wealth, agents are in low-

et al. (2020), the lower bound of the empirical MPC range is around 10%.

liquid states, so it is very likely that agents will end up in unfavorable economic conditions, maintaining the living standard only with the help of income support and having zero consumption. However, as long as their living standard is kept, not all hope is lost, and agents attempt to finance a high level of wealth by taking on large equity positions, should the fundamentals be favorable over the investment horizon.

The paper sits squarely within the continuous-time optimal consumption and portfolio choice framework with a focus on constrained borrowing developed by Merton (1969) and Merton (1971). Extending the Merton framework, Detemple and Serrat (2003), Farhi and Panageas (2007), Dybvig and Liu (2010), Jang et al. (2013), Kim and Shin (2018), Jang et al. (2019), Jang et al. (2020), Park and Jang (2014), Holm (2018), Ahn et al. (2019), Kim et al. (2020), Park (2022), and others have considered constrained borrowing on financial wealth. For instance, these studies reflect the fact that, in reality, borrowing against the net present value of future income (human capital) is not fully allowed because of market frictions (e.g., informational asymmetry, agency conflicts, and limited enforcement), thus requiring that financial wealth be nonnegative or negative up to the maximum debt-to-income ratio (Park and Jang (2014); Ahn et al. (2019); Park (2022)). Apart from such borrowing limits depending upon current income, Grossman and Vila (1989), Basak (1995), and Grossman and Zhou (1996) have considered the portfolio insurer who keeps her horizon wealth above some floor in all states. Nesting this portfolio insurer case, Basak and Shapiro (2001) have incorporated a Value-at-Risk (VaR) constraint into the agent's portfolio optimization in which the agent is constrained to maintain the probability of his horizon wealth falling below some floor. None of these papers, however, explore the effects of constrained borrowing associated with a living standard agent wish to maintain on their optimal policies.

The paper also contributes to the literature on some interactions between a living standard and optimal consumption. In association with a wide range of government subsidies, social security programs, and insurance supporting households to sustain the minimum living standard, Sethi et al. (1992), Gong and Li (2006), Elmendorf and Kimball (2000), Gormley et al. (2010), Bae et al. (2020), Kosar et al. (2023), and others have investigated the effects of the government subsidy on households' optimal consumption/savings and investment choices. Thinking about the government subsidy as financial support for subsistence consumption, Sethi et al. (1992) and Gong and Li (2006) have studied households' optimal strategies. Elmendorf and Kimball (2000) and Gormley et al. (2010) have demonstrated that private insurance and various types of government safety nets play a pivotal role in household investment and savings choices. Bae et al. (2020) have explored the effects of retirement benefits provided by social insurance programs on optimal strategies. More recently, Kosar et al. (2023) have documented that people with low net wealth-to-income ratios have an incentive to pay down debt first when there is a stimulus from the government. We extend the literature in sufficiently novel ways by offering new insight into the relationship between living standards and household portfolio decisions. We first show that maintaining the living standard, in turn, allows the borrowing-constrained agents to endogenously determine subsistence wealth for meaningful consumption, and agents exhibit distinct consumption/savings and investment behavior around subsistence wealth.

The paper is organized as follows. In Section 2, we describe the consumption and savings model with living standards and subsistence wealth. In Section 3, we provide analytically tractable results for optimal consumption/savings and investment strategies. In Section 4, we conduct numerical analyses to discuss various implications of living standards and subsistence wealth on optimal strategies. In Section 5, we conclude the paper.

2 The Model

2.1 Basic Setting

The Investment Opportunity Set. The model considers a financial market in which there are two tradable assets: a riskless bond and a risky stock. The bond price grows at the risk-free interest rate $r > 0$. The stock price, S_t, is assumed to follow a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where $\mu > r$ is the expected stock return, $\sigma > 0$ is the stock volatility, and B_t is the standard one-dimensional Brownian motion. We assume that r, μ, σ are constant, i.e., we consider the constant investment opportunity.

Constrained Borrowing. As is fairly standard in consumption and savings models with constrained borrowing caused by market frictions (e.g., informational asymmetry, agency conflicts, and limited enforcement), an agent is allowed to borrow against the net present value (NPV) of human capital, but her borrowing is constrained up to some proportion of the NPV. In the presence of this constrained borrowing situation, financial wealth W_t can be negative up to the NPV proportion as follows:

$$W_t \geq -\nu \frac{y}{r}, \quad \text{for all } t \geq 0, \tag{2.1}$$

where $\nu \in [0, 1)$ is an exogenously given parameter that determines the tightness of borrowing (or credit) limit, and y is the agent's constant labor income stream. Note that in our constant investment opportunity consideration, y/r is the NPV of the agent's human capital as in Friedman (1957) and Hall (1978).

2.2 Living Standard and Income Support

Income support is particularly crucial for those who are on a low income or unable to work due to forced unemployment due to various reasons (e.g., disability, firm closure, health issues). A scheme to help with the living costs of these people has been pledged, especially during economic downturns (e.g., the 2008 global financial crisis and the recent COVID-19 pandemic).

Benchmark without Income Support. We first establish a benchmark against which to determine the effectiveness of income support for those who have no income sources, which is of utmost importance to both policy design and the basis of sound financial advice. The people in the benchmark economy follow Merton (1969) so that they choose (i) how much to consume and (ii) how to allocate their wealth between riskless bonds and risky stocks in the financial market with constant investment opportunities. Their maximized discounted expected CRRA (constant relative risk aversion) consumption utilities are then formulated as the following value function:

$$U(w) = \max_{(c,\pi)} E\Big[\int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt\Big],$$

where w is the initial wealth of the people, $\rho > 0$ is their subjective discount rate, $\gamma > 0$ ($\gamma \neq 1$) is their constant coefficient of relative risk aversion, and c_t is their consumption amount. The value function $U(w)$ is subject to the following wealth dynamics:

$$dW_t = [rW_t + \pi_t(\mu - r) - c_t]\, dt + \sigma \pi_t dB_t, \; W_0 = w > 0, \tag{2.2}$$

where π_t is the dollar amount invested in the stock market.

Following Merton (1969), the value function $U(w)$ has the following closed-form solution:

$$U(w) = K^{-\gamma} \frac{w^{1-\gamma}}{1-\gamma},$$

where K is the so-called Merton constant, and it is given by[6]

$$K = \frac{\gamma - 1}{\gamma}\Big(r + \frac{\theta^2}{2\gamma}\Big) + \frac{\rho}{\gamma} > 0, \quad \theta = \frac{\mu - r}{\sigma}. \tag{2.3}$$

We also know the following relation between optimal consumption c_t and wealth W_t:

$$c_t = KW_t. \tag{2.4}$$

The Economy with Income Support. Having constructed the benchmark without income support to isolate and very closely investigate the new issues introduced by income support, we now assume that income support is put in place for low-income people when income disaster (causing forced unemployment or resulting in severe financial constraints) occurs at an uncertain time. In this paper, the timing at which income disasters occur is not deterministic, so its probability changes over time. The uncertain time when income disaster occurs is assumed to be the first jump time τ of an independent Poisson process with intensity $\delta > 0$.

The people in the economy with income support optimize their discounted expected CRRA consumption utility post-disaster over the following value function:

$$U(w) = \max_{(c,\pi)} E\Big[e^{-\rho\tau}\int_{\tau}^{\infty} e^{-\rho(t-\tau)}\frac{c_t^{1-\gamma}}{1-\gamma}dt\Big]. \tag{2.5}$$

Our income support considered in the paper continues making payments for life. We denote $L \geq 0$ by the level of living standard that the agent wishes to maintain.[7] The living standard considered in this paper is thought to be supported by public insurance and various income support schemes (e.g., government subsidies, social security programs, a UBI, Universal Credit). The people at time τ when income disaster occurs receive income support $\$rL$ per year continuously so that

$$\int_{\tau}^{\infty} e^{-r(t-\tau)}rLdt = L. \tag{2.6}$$

So, the agent's wealth dynamics are given by

$$dW_t = [rW_t + \pi_t(\mu - r) - c_t + rL]\,dt + \sigma\pi_t dB_t, \quad W_\tau > -L. \tag{2.7}$$

[6]To make the value function well-defined, we assume $K > 0$ throughout the paper. When $K < 0$, infinite utility can be obtained for the agent to delay consumption. If we assume risk aversion $\gamma > 1$ consistent with the data, K is always positive. Otherwise, K can be negative in some parameter conditions.

[7]The living standard L is exogenously given by the government. We could monetize social security benefits to impute L.

Following Merton (1971) with consideration of income sources, the people's value function post-disaster can be restated as follows:

$$U(W_\tau) = E\Big[e^{-\rho\tau}K^{-\gamma}\frac{\big(W_\tau + L\big)^{1-\gamma}}{1-\gamma}\Big]. \tag{2.8}$$

The value function can then be rewritten for people who are just before income disaster, i.e., those at time $\tau-$, as

$$\begin{aligned} U(w) &= E\Big[e^{-\rho\tau-}K^{-\gamma}\frac{(W_{\tau-} + L)^{1-\gamma}}{1-\gamma}\Big] \\ &= E\Big[e^{-\rho\tau-}K^{-\gamma}\frac{(K^{-1}c_{\tau-} + L)^{1-\gamma}}{1-\gamma}\Big] \\ &= E\Big[e^{-\rho\tau-}\frac{1}{K}\frac{(c_{\tau-} + KL)^{1-\gamma}}{1-\gamma}\Big] \\ &= E\Big[\int_0^\infty e^{-(\rho+\delta)t}\frac{\delta}{K}\frac{(c_t + KL)^{1-\gamma}}{1-\gamma}dt\Big], \end{aligned} \tag{2.9}$$

where the second equality results from the relation (2.4).

Remarkably, the newly derived value function $U(w)$, as stated above, demonstrates that random time τ of income disaster introduces new friction into the economy so that the people with income support, especially at times of income disaster, exhibit decreasing relative risk aversion with the level L of living standard[8]. The relative risk aversion (RRA) is given by $(\gamma c_t)/(c_t + KL)$ so that γ reduces to the constant coefficient of relative risk aversion (CRRA) if $L = 0$[9]. We demonstrate that the RRA then decreases as L increases so that the people show a decreasing relative risk aversion with the level of living standard.

2.3 Optimization and Subsistence Wealth

Income support is extra money to help people with little or no income. We now consider a representative agent who just receives a small but constant labor income stream $y > 0$ so that the wealth dynamics are given by

$$dW_t = [rW_t + \pi_t(\mu - r) - c_t + y]\,dt + \sigma\pi_t dB_t,\ W_0 = w > -\nu\frac{y}{r}, \tag{2.10}$$

where W_t represents the state of a stochastic dynamic system that is to be controlled by the pair $\{c_t, \pi_t\}$.

[8] Mechanically, the presence of L shifts the utility function in a parallel fashion, bringing some exciting economic behavior around wealth at zero.

[9] The higher γ is associated with people's more increased relative risk aversion, implying that they would be less willing to consume more. Given this immediate relation between γ and the concept of risk aversion, we can still treat γ as the risk aversion coefficient.

We then optimize the utilities (2.9) of the agent with income support who exhibits decreasing relative risk aversion over the following value function:

$$V(w) = \max_{(c,\pi)} E\Big[\int_0^\infty e^{-(\rho+\delta)t}\frac{\delta}{K}\frac{(c_t+KL)^{1-\gamma}}{1-\gamma}dt\Big], \tag{2.11}$$

which is subject to the constrained borrowing (2.1) and wealth dynamics (2.10).

We conjecture that maintaining the living standard allows the borrowing-constrained agent to endogenize a certain threshold of wealth, the so-called subsistence wealth, below which her optimal decision is to consume nothing. We denote $\widetilde{w}(L)$ the subsistence wealth that varies over the levels of living standard L. With the constrained borrowing given in (2.1), the agent's optimal consumption is equal to zero as long as her wealth is smaller than $\widetilde{w}(L)$:

$$c_t = 0, \quad -\nu\frac{y}{r} \le W_t < \widetilde{w}(L), \tag{2.12}$$

where $\widetilde{w}(L)$ is to be endogenously determined. In this zero consumption case, the agent still derives utilities from $(KL)^{1-\gamma}/(1-\gamma)$ by effectively consuming L provided by income support, maintaining the living standard well. Notice that all hope is not lost even in this zero consumption case because the agent may be able to finance a high level of wealth by collateralizing the NPV of her human capital according to the constrained borrowing (2.1) and attempting to bet on a favorable realization of positive equity positions.

3 Analytic Results

To provide an intuitive insight into how the presence of living standard affects the borrowing-constrained agent's optimal consumption/savings and investment decisions, especially with the endogenously determined subsistence wealth, we proceed pedagogically with our analysis and develop insights by solving three models, which are sorted by constrained borrowing and the living standard as follows:

Model 1. Consumption/savings and investment only (Merton (1971)).

Model 2. Consumption/savings and investment with constrained borrowing (Ahn et al. (2019)).

Model 3. Consumption/savings and investment with both constrained borrowing and the living standard.

The three models are formulated as follows.

Model 1. The agent aims to maximize her CRRA consumption utilities over the infinite horizon:

$$V(w) = \max_{(c,\pi)} E\Big[\int_0^\infty e^{-\rho t}\frac{c_t^{1-\gamma}}{1-\gamma}\Big],$$

subject to the wealth dynamics (2.10) with

$$W_t \geq -\frac{y}{r}, \quad \text{for all} \quad t \geq 0, \tag{3.1}$$

implying that the agent is fully allowed to borrow against the NPV of her human capital.

Model 2. The same as Model 1, except that the agent is partially allowed to borrow against the NPV of her human capital according to the constrained borrowing (2.1).

Model 3. The same as Model 2, except that the agent aims to maintain the living standard as well in her optimization as given in (2.11).

Model 1 is the case of Merton (1971). Moving to Model 2 isolates the effects of constrained borrowing on optimal strategies. Here, Model 2 is a close relative of Ahn et al. (2019) with time-varying liquidity constraints that reduce to constrained borrowing when the agent's income stream is constant. Subsequently, moving to Model 3 isolates the effects of the living standard for the agent to maintain optimal policies.

We provide analytic solutions that enable an intuitive understanding of how constrained borrowing and the living standard affect the agent's optimal consumption/savings and investment choices.

Theorem 3.1. *(Model 1). The optimal consumption and investment strategies are*

$$c_t = K\Big(w + \frac{y}{r}\Big)$$

and

$$\pi_t = \frac{\theta}{\gamma\sigma}\Big(w + \frac{y}{r}\Big),$$

respectively.

Proof. Refer to the Appendix. □

The solutions of the original problem (2.11) reduce to Model 1 solutions given in Theorem 3.1 with neither constrained borrowing nor the living standard ($L = 0$). That is, the agent's optimal decision for Model 1 is merely to consume and invest proportionally out of her total financial resources, which are the sum of financial wealth w and the human capital y/r.

We now consider two constants $m_+ > 0$ and $m_- < -1$ that solve the following quadratic equation:

$$\frac{1}{2}\theta^2 m^2 + \left(\rho - r + \frac{1}{2}\theta^2\right) m - r = 0. \tag{3.2}$$

Theorem 3.2. *(Model 2). The optimal consumption and investment strategies are*

$$c_t = K\Big(w + \frac{y}{r} - A_\nu \lambda^{m_+}\Big),$$

and

$$\pi_t = \frac{\theta}{\gamma\sigma}\Big(w + \frac{y}{r} - (\gamma m_+ + 1) A_\nu \lambda^{m_+}\Big),$$

respectively, where

$$A_\nu = \frac{1}{\gamma K m_+} \widehat{\lambda}^{-\frac{1}{\gamma} - m_+} > 0, \quad \widehat{\lambda} = \left[\frac{m_+(1-\nu)\frac{y}{r}}{(m_+ + 1/\gamma)\frac{1}{K}}\right]^{-\gamma} > 0,$$

and λ is the solution to the following algebraic equation:

$$w = A_\nu \lambda^{m_+} + \frac{1}{K}\lambda^{-\frac{1}{\gamma}} - \frac{y}{r}.$$

Proof. Refer to the Appendix. □

In the presence of constrained borrowing, Model 2 solutions given in Theorem 3.2 demonstrate that the borrowing-constrained agent's optimal decision is to consume less and invest less in the stock market than without constrained borrowing (i.e., as compared to Model 1 solutions), and the extent to which the agent reduces consumption and stock investment is determined by how tight the agent's borrowing is constrained with levels of the credit tightening parameter ν. Such choices to reduce both consumption and equity positions are for the agent to secure extra cash reserves to be in a liquid position, avoiding binding constrained borrowing.

We now introduce the concept of a minimum standard of living by demonstrating that there exists a certain threshold of living standard L^* such that $\Psi < 0 \Leftrightarrow L > L^*$, where

$$L^* := -(\gamma m_- + 1)(1-\nu)y > 0. \tag{3.3}$$

Throughout the paper, we call L^* the minimum standard of living and mainly focus on the levels of living standard greater than L^*, i.e., the case for which $L > L^*$[10].

[10]We could focus on developed countries so that $L > L^*$ always. Put another way, the U.S., for example, always provides enough of a safety net where this condition is satisfied.

Theorem 3.3. *(Model 3) For an agent who wishes to maintain at least the minimum standard of living, i.e., when $L > L^*$, the optimal consumption and investment strategies are*

$$c_t = \begin{cases} 0, & \text{if } -\nu\dfrac{y}{r} \le W_t < \widetilde{w}(L), \\ K\Big(w + \dfrac{y+L}{r} - D_1\lambda_2^{m_+}\Big) - L, & \text{if } W_t \ge \widetilde{w}(L), \end{cases}$$

and

$$\pi_t = \begin{cases} -\dfrac{\theta}{\sigma}\Big(m_+C_1\lambda_1^{m_+} + m_-C_2\lambda_1^{m_-}\Big), & \text{if } -\nu\dfrac{y}{r} \le W_t < \widetilde{w}(L), \\ \dfrac{\theta}{\gamma\sigma}\Big(w + \dfrac{y+L}{r} - (\gamma m_+ + 1)D_1\lambda_2^{m_+}\Big), & \text{if } W_t \ge \widetilde{w}(L), \end{cases}$$

where

$$C_1 = -\frac{m_-}{m_+ - m_-}\frac{(1-\nu)y}{r}\widehat{\lambda}^{-m_+} > 0, \tag{3.4}$$

$$C_2 = -\frac{(m_+ + 1)(m_- + 1)}{(1-\gamma)m_-(m_+ - m_-)}\left\{\frac{1}{\rho} - \frac{\gamma m_+ + 1}{\gamma(m_+ + 1)K}\right\}L^{1+\gamma m_-} > 0, \tag{3.5}$$

$$\widehat{\lambda} = \left[-\frac{(m_+ + 1)(m_- + 1)}{(1-\gamma)m_+m_-}\frac{r}{(1-\nu)y}\left\{\frac{K}{\rho} - \frac{\gamma m_+ + 1}{\gamma(m_+ + 1)}\right\}\frac{L}{K}\right]^{-\frac{1}{m_-}} L^{-\gamma} > 0, \tag{3.6}$$

$$D_1 = C_1 - \frac{(m_+ + 1)(m_- + 1)}{(1-\gamma)m_+(m_+ - m_-)}\left\{\frac{1}{\rho} - \frac{\gamma m_- + 1}{\gamma(m_- + 1)K}\right\}L^{1+\gamma m_+} > 0,$$

$$\widetilde{w}(L) = C_1L^{-\gamma m_+} + C_2L^{-\gamma m_-} - \frac{y}{r} > -\frac{y}{r}, \tag{3.7}$$

and λ_1 and λ_2 are the solutions to the following algebraic equations

$$w = C_1\lambda_1^{m_+} + C_2\lambda_1^{m_-} - \frac{y}{r} \tag{3.8}$$

and

$$w = D_1\lambda_2^{m_+} + \frac{1}{K}\lambda_2^{-\frac{1}{\gamma}} - \frac{y+L}{r}, \tag{3.9}$$

respectively.

Proof. Refer to the Appendix. □

In the presence of both constrained borrowing and the living standard that the agent wishes to maintain, Model 3 solutions given in Theorem 3.3 show that the agent endogenously characterizes a certain threshold of wealth (subsistence wealth) below which it is optimal to consume nothing and the threshold varies over the levels of living standard. The wealth threshold plays a crucial role in the characterization of the agent's optimal policies with the living standard.

An important discontinuity and significant change in optimal consumption strategies is observed at subsistence wealth below, where the agent does not consume at all, and above, where the agent dramatically increases the amount of optimal consumption. Intuitively, the agent would be in *transitional* states around subsistence wealth because there is the highest uncertainty regarding whether financial resources are enough to finance positive consumption whilst maintaining the required living standard. Having maintained the minimum standard of living well, the agent's optimal response above subsistence wealth is to aggressively bid for consumption, maximizing lifetime utilities derived from consumption. The agent's greater consumption responses to changes in wealth around subsistence wealth can also be invoked to rationalize the MPC puzzle by generating high MPCs, which is to be further investigated in our numerical analyses.

We arguably state that subsistence wealth endogenously determined by living standards tends to reduce the agent's effective risk aversion, thus increasing equity exposure. For the agent with far enough wealth, optimal investment policies are reduced to those obtained from Model 1. The agent with high wealth above subsistence wealth acts similarly to the benchmark agent without considering the living standard, as in Model 2 with constrained borrowing. Interestingly, agents with low levels of wealth below subsistence wealth optimally choose to invest in the stock market even with zero consumption.

The reason behind such differing investment decisions with respect to levels of wealth is that the agent attempts to finance a high level of wealth by taking on large equity positions around subsistence wealth, should the fundamentals be favorable over the investment horizon. Basically, the agent begins to increase investment in the stock market with one more unit of wealth as wealth decumulates, then approaching subsistence wealth closely so that in *transitional* states, the agent invests a much higher fraction of wealth to equity with an increase in wealth compared to benchmark models without living standard (Model 1 and Model 2).

The agent's decision to buy more or fewer stocks with living standards is influenced, to a large extent, by the way in which the agent's desire to maintain the standard of living is satisfied. When wealth is large, the agent is already in a liquid position so that she is actually not constrained to maintain the living standard. Therefore, the agent's optimal policies are similar to benchmark agents from Model 1 and Model 2 without consideration of living standards. When wealth is small but above subsistence wealth, the agent is in low liquid states, so it is highly likely that the agent will end up in unfavorable economic conditions, maintaining the living standard only with the help of income support and having zero consumption. However, as long as the agent's living standard is maintained, not all hope is lost, and the agent can bet on a favorable realization of large equity positions to finance a high level of wealth not only for positive consumption but eventually for effective wealth accumulation.

Proposition 3.1. *The endogenously determined wealth threshold (subsistence wealth)* $\widetilde{w}(L)$ *rises with the levels of living standard* L*.*

Proof. Refer to the Appendix. □

Proposition 3.1 demonstrates that the agent who wishes to maintain a high level of living standard tends to target high subsistence wealth, thus implying more financial resources are to be reserved for financing high consumption and large equity exposure.

Proposition 3.2. *The endogenously determined wealth threshold (subsistence wealth)* $\widetilde{w}(L)$ *decreases with the extent to which the agent is allowed to borrow against her human capital.*

Proof. Refer to the Appendix. □

Proposition 3.2 shows that the agent is inclined to target low subsistence wealth when she is less constrained to borrow against human capital. In the limiting case for which the agent is fully allowed to borrow, the agent does not endogenously determine subsistence wealth below which she does not consume, i.e., subsistence wealth no longer plays a role in the agent's optimal policies. The intuition is that the agent's total available financial resources for consumption are already strongly held up by her enough human capital borrowing so that she can maintain the living standard without characterizing subsistence wealth. The result can be reversed with the presence of subsistence wealth if the agent is strictly constrained from borrowing against human capital.

4 Quantitative Analysis

We now perform an extensive quantitative analysis to discuss various properties of agents' optimal consumption and investment decisions. The following values are used for the model's baseline parameters:

$$\gamma = 2, r = 0.01, \mu = 0.05, \sigma = 0.2, \rho = 0.015, y = 0.1, \nu = 0.5.$$

The relative risk aversion coefficient $\gamma = 2$ is moderately chosen so that it lies well in the empirically plausible range of risk aversion that is less than 10. The parameter values for asset returns r, μ, and σ are taken from Ahn et al. (2019). The subjective discount rate $\rho = 0.015$ that is higher than the risk-free rate $r = 0.01$ can be regarded as a mortality-risk-perceived subjective discount rate[11].

Under the above baseline parameter values, the Merton constant K given in (2.3) is obtained as 0.035 so that income disaster is assumed to occur once every 28.5 years by setting $\delta = K$. This parameter choice is consistent with the literature on rare disasters[12].

The agents' constant labor income y is set to 0.1 so that the NPV y/r of human capital becomes 10. The minimum standard of living L^* given in (3.3) is obtained 0.2658 so that the minimum income support \$$rL^* = 0.02658$ by (2.6) is provided per year continuously, replacing about 26.5% of the agents' labor income in case of income disaster.

The extent to which agents' borrowing is restricted is set to $\nu = 0.5$ so that they can borrow up to half of their human capital NPV 10. Our choice for the agent's initial income level y and borrowing capacity ν conveniently allows us to focus on agents' optimal policies with levels of wealth around zero or negative, which is consistent with reality[13].

Living standard L	10% above L^*	50% above L^*	100% above L^*	200% above L^*
Subsistence wealth $\widetilde{w}(L)$	-4.9988	-4.9765	-4.9239	-4.8557
Human capital borrowing ν	0%	25%	50%	99%
Subsistence wealth $\widetilde{w}(L)$	0.2885	-2.2836	-4.8557	-9.8971

Table 1: Levels of subsistence wealth

We show that maintaining the minimum standard of living $L > L^*$ allows the borrowing-constrained agents to endogenously determine a certain wealth threshold $\widetilde{w}(L)$, denoted

[11] In the case for which the subjective discount rate is higher than the risk-free rate, agents become relatively impatient in the bond market so that they tend to save less and consume more.

[12] Rare disaster possibility is estimated as 1.7% per year in Barro (2006), and our income disaster possibility per year is $1 - e^{-\delta \times t} = 1 - e^{-0.035 \times 1} = 3.4\%$. The random arrival rate of large negative income shocks is chosen as 5% in Wang et al. (2016), which is greater than our choice 3.5%.

[13] Most people have zero or negative wealth given their debt. For instance, my wealth is likely very low because I took out the mortgage to buy my home.

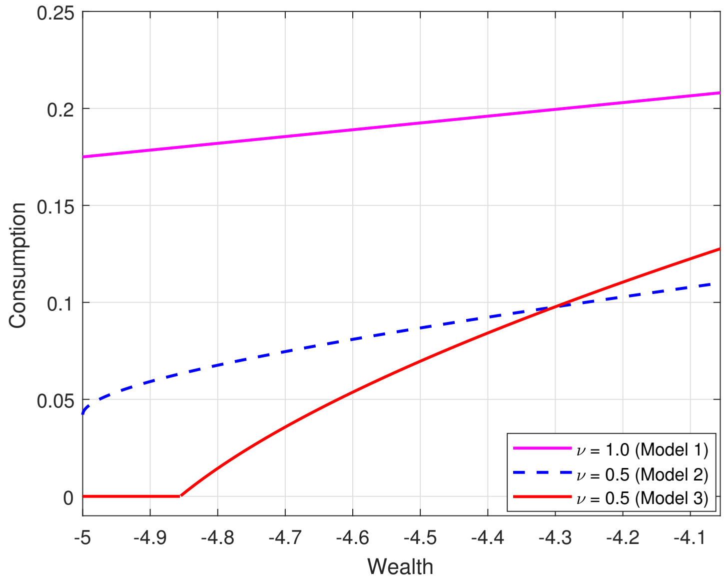

Figure 1: Optimal consumption

as subsistence wealth (Table 1) below which they find it optimal to consume nothing (Model 3 in Figure 1). In particular, compared to benchmark models (Model 1 and Model 2) without consideration of living standards, the borrowing-constrained agents sharply increase their consumption once they accumulate financial resources above subsistence wealth. Here, our consumption results may support the stimulus channel to pay down debt (Kosar et al. (2023)) in that agents with low wealth levels below subsistence wealth save rather than consume, thereby reducing borrowing (or repaying debt).

The living standard L tends to shift subsistence wealth $\widetilde{w}(L)$ upwards, whereas the greater extent ν to which agents are allowed to borrow against their human capital (implying more borrowing) brings subsistence wealth down (Table 1). Intuitively, far enough financial resources are required for agents to maintain a higher living standard, thus targeting greater subsistence wealth.

In the limiting case for which agents are fully allowed to borrow, subsistence wealth $\widetilde{w}(L)$ approaches the NPV of agents' human capital so that they do not endogenize a wealth threshold below which their optimal decision is to consume nothing. Therefore, the presence of living standards in this case does not matter for agents' optimal policies. Agents who are endowed with lots of human capital from borrowing would be in a relatively better position to finance living standards without characterizing subsistence

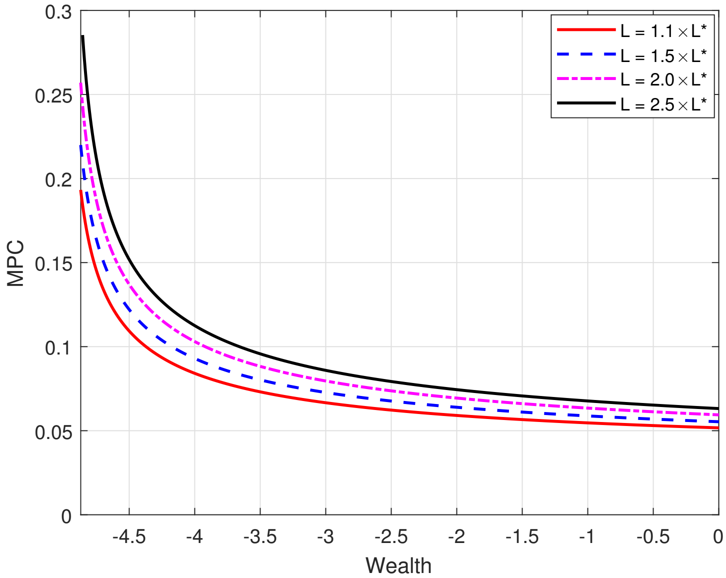

Figure 2: Marginal propensities to consume (MPC)

wealth because agents' total available financial resources are expanded by the opportunity of enough borrowing. This borrowing effect tends to somehow counteract the effects of living standards and produces a much greater decrease in subsistence wealth. The levels of living standard are crucial for agents to formulate subsistence wealth only when they are borrowing constrained.

We find that our model greatly improves the ability to match empirical MPC estimates (Figure 2). Most macroeconomic models theoretically suggest around 4% MPC (Wang (2003)), whereas empirically plausible MPC ranges are from 12% to 30% (Parker et al. (2013)), from 20% to 60% (Carroll et al. (2017)), or at least more than 10% (Fisher et al. (2020)), which gives rise to the MPC gap between theory and the empirics. The living standard channel can be invoked to rationalize the MPC puzzle if we focus on an important discontinuity and dramatic change in agents' optimal consumption at subsistence wealth below which they optimally do not consume at all and above which they sharply increase the amount of optimal consumption (Figure 1). As opposed to the low concavity of consumption function obtained from benchmark models without living standards (Model 1 and Model 2), the consumption concavity is high, especially for low levels of wealth around subsistence wealth. The departure of our model from the benchmark models is highest in *transitional* states around subsistence wealth, where there is the highest uncertainty

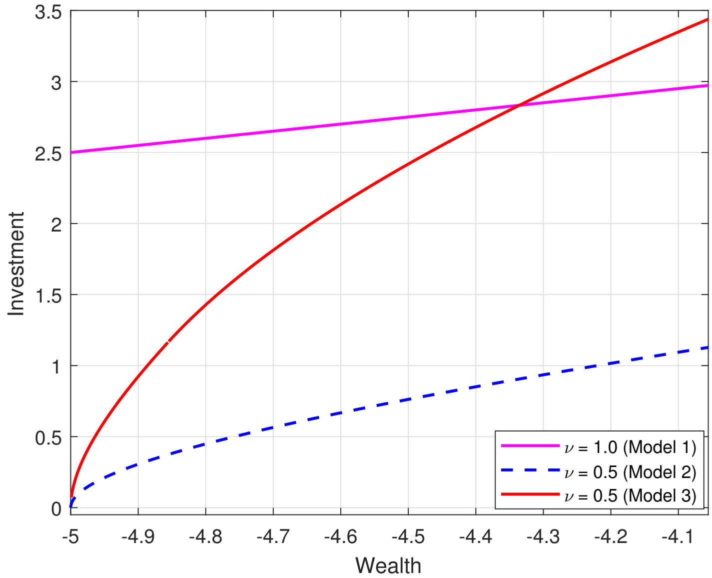

Figure 3: Optimal investment

regarding whether agents have enough wealth to support positive consumption. The agents' aggressive bidding for consumption occurs as an optimal response to subsistence wealth above *transitional* states because of the agents' affordability of consumption having maintained the minimum living standard well. The agents' greater consumption responses (measured by MPCs) to wealth changes are therefore obtained due to such interactions amongst consumption, subsistence, and living standards.

The decision to buy more or less equity is influenced, to a large extent, by the way in which the exogenously given constrained borrowing interacts with endogenously determined subsistence wealth with living standards. Compared to benchmark models (Model 1 and Model 2 in Figure 3) with no interaction between constrained borrowing and subsistence wealth due to the absence of living standard consideration, agents find it optimal to dramatically increase their equity exposure as wealth increases from subsistence wealth (Model 3).

We arguably state that subsistence wealth tends to reduce agents' effective risk aversion, especially in light of the growing risk-taking in the stock market by agents, even when their wealth is small (but above subsistence wealth). When agents have far enough wealth, they are already in a liquid position so that they act similarly to benchmark agents, where the effects of subsistence wealth with living standards can be safely ignored (not reported). When agents' wealth is small but above subsistence wealth, they are in low liquid states, so they may end up in unfavorable fundamentals, maintaining the living standard only with the help of income support and consuming nothing. At the same time, however, agents are in *transitional* states so that as long as their living standard is well maintained, not all hope is lost; they are encouraged to take on high risk with large equity positions to finance a high level of wealth, should economic conditions become favorable over the investment horizon. The amount of wealth is, therefore, a driving factor that underlines the agents' decision to invest in the stock market, which crucially determines whether they accumulate enough wealth so that they can be above *transitional* states.

5 Conclusion

We have developed a dynamic continuous-time model to understand the interactions among living standards, consumption/savings, and portfolio choice, especially in light of today's growing attention to government subsidies and social security programs. We have proposed that agents endogenize a certain wealth threshold, the so-called subsistence wealth, to maintain the minimum standard of living. We have demonstrated that agents consume nothing when their wealth levels are below subsistence wealth; instead, they save and invest to sustain consumption for longer. Indeed, once the agents accumulate assets above subsistence wealth, they find it optimal to sharply increase consumption. Such a subsistence wealth channel can also be invoked to rationalize the marginal propensities to consume (MPC) puzzle by generating high MPCs as observed in the data, contrary to very low MPCs obtained by most theoretical macroeconomic models. Finally, we generate an empirically testable hypothesis on portfolio choice: very low-wealth households could take a risky portfolio under an income support scheme for a slight chance of increasing their wealth dramatically.

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Appendix

Proof of Theorem 3.1

Referring to Merton (1971), the value function is given by

$$V(w) = K^{-\gamma}\frac{w^{1-\gamma}}{1-\gamma},$$

thus obtaining from the first-order conditions (FOCs) for consumption and investment the optimal strategies in the theorem.

Proof of Theorem 3.2

The HJB equation associated with Model 2 is given by

$$\max_{(c,\pi)}\left[\Big(rw+\pi(\mu-r)-c+y\Big)V'(w)+\frac{1}{2}\sigma^2\pi^2V''(w)-\rho V(w)+\frac{c^{1-\gamma}}{1-\gamma}\right]=0, \tag{5.1}$$

for any $w>-\nu y/r$, subject to the constrained borrowing (2.1). The FOCs for consumption c and investment π are given by

$$c = V'(w)^{-\frac{1}{\gamma}} \tag{5.2}$$

and

$$\pi = -\frac{\theta}{\sigma}\frac{V'(w)}{V''(w)}, \tag{5.3}$$

respectively. With the substitution of the FOCs above in the HJB equation (5.1), we obtain

$$(rw+y)V'(w)-\frac{1}{2}\theta^2\frac{(V'(w))^2}{V''(w)}-\rho V(w)+\frac{\gamma}{1-\gamma}V'(w)^{1-\frac{1}{\gamma}}=0, \tag{5.4}$$

for any $w>-\nu y/r$, subject to the constrained borrowing (2.1). By differentiating the both sides of (5.1) with respect to w, we get

$$(r-\rho-\theta^2)V'(w)+(rw+y)V''(w)+\frac{1}{2}\theta^2\frac{V'(w)^2V'''(w)}{V''(w)^2}-V'(w)^{-\frac{1}{\gamma}}V''(w)=0, \tag{5.5}$$

for any $w>-\nu y/r$, subject to the constrained borrowing (2.1).

To apply the convex-duality approach of Bensoussan *et al.* (2016) to the HJB equation (5.1), we introduce a dual variable defined as the first derivative of the value function:

$$\lambda := \lambda(w) = V'(w).$$

The equation (5.5) can then be restated as

$$(r-\rho-\theta^2)\lambda(w)+(rw+y)\lambda'(w)+\frac{1}{2}\theta^2\frac{\lambda(w)^2\lambda''(w)}{\lambda'(w)^2}-\lambda(w)^{-\frac{1}{\gamma}}\lambda'(w)=0, \tag{5.6}$$

for any $w > -\nu y/r$, subject to the constrained borrowing (2.1).

We next introduce a convex-dual function defined as the sum of financial wealth and the present value of future income:

$$G(\lambda) := G(\lambda(w)) = w + \frac{y}{r} \tag{5.7}$$

with

$$G'(\lambda)\lambda'(w) = 1 \quad \text{and} \quad G''(\lambda)\lambda'(w)^2 + G'(\lambda)\lambda''(w) = 0.$$

We now rewrite the equation (5.6) as

$$\frac{1}{2}\theta^2\lambda^2 G''(\lambda) + (\rho - r + \theta^2)\lambda G'(\lambda) - rG(\lambda) + \lambda^{-\frac{1}{\gamma}} = 0, \tag{5.8}$$

for any $0 < \lambda < \widehat{\lambda}$, with the following boundary conditions:

$$G(\widehat{\lambda}) = (1-\nu)\frac{y}{r} \quad \text{and} \quad G'(\widehat{\lambda}) = 0, \tag{5.9}$$

which result from the constrained borrowing (2.1) implying that the first condition is obtained by (5.7) as the financial wealth w approaches its exogenously imposed liquidity limit $-\nu y/r$ and the second condition is obtained by the fact that $G(\lambda)$ is continuous, twice differentiable, and cannot have a maximum at any $\lambda > 0$ so that $G(\lambda)$ has a local minimum at $\lambda = \widehat{\lambda}$ with $G'(\widehat{\lambda}) = 0$ and $G''(\widehat{\lambda}) > 0$.

The general solution to the equation (5.8) is given by

$$G(\lambda) = A_\nu \lambda^{m_+} + \frac{1}{K}\lambda^{-\frac{1}{\gamma}}, \tag{5.10}$$

where A_ν is a constant to be determined with $\widehat{\lambda}$ according to the boundary conditions (5.9). Direct calculations of $G(\widehat{\lambda})$ and $G'(\widehat{\lambda})$ using the general solution (5.10) with (5.9) then determine A_ν and $\widehat{\lambda}$ as given in the theorem. The FOCs for consumption and investment given in (5.2) and (5.3), therefore, lead to the optimal consumption and investment strategies in the theorem.

Proof of Theorem 3.3

By standard invariant embedding arguments of dynamic programming, the problem (2.11) is equivalent to the Bellman equation as follows: for any $w > -\nu y/r$,

$$\max_{(c,\pi)} \left[\Big(rw + \pi(\mu - r) - c + y \Big) V'(w) + \frac{1}{2}\sigma^2\pi^2 V''(w) - \rho V(w) + \frac{(c+L)^{1-\gamma}}{1-\gamma} \right] = 0. \tag{5.11}$$

We conjecture that there exists a certain threshold of wealth $\widetilde{w}(L)$ below which the agent's optimal decision is to consume nothing. So, for any $-\nu y/r \leq w < \widetilde{w}(L)$, the Bellman equation (5.11) reduces to the following equation with $c = 0$:

$$\max_{\pi}\left[(rw+\pi(\mu-r)+y)V'(w)+\frac{1}{2}\sigma^2\pi^2V''(w)-\rho V(w)+\frac{L^{1-\gamma}}{1-\gamma}\right]=0. \tag{5.12}$$

The FOC with respect to the optimal investment π is given by

$$\pi^* = -\frac{\theta}{\sigma}\frac{V'(w)}{V''(w)}. \tag{5.13}$$

Substituting the FOC (5.13) into the equation (5.12), we obtain

$$(rw+y)V'(w)-\frac{1}{2}\theta^2\frac{V'(w)^2}{V''(w)}-\rho V(w)+\frac{L^{1-\gamma}}{1-\gamma}=0. \tag{5.14}$$

Taking the first derivative of the equation (5.14) with respect to w, we derive

$$\left(r-\rho-\theta^2\right)V'(w)+(rw+y)V''(w)+\frac{1}{2}\theta^2\frac{V'(w)^2V'''(w)}{V''(w)^2}=0. \tag{5.15}$$

We now apply the convex-duality approach of Bensoussan *et al.* (2016). We first introduce a dual variable defined as the first derivative of the value function:

$$\lambda_1 := \lambda_1(w) = V'(w). \tag{5.16}$$

The equation (5.15) can then be rewritten as

$$\left(r-\rho-\theta^2\right)\lambda_1(w)+(rw+y)\lambda_1'(w)+\frac{1}{2}\theta^2\frac{\lambda_1(w)^2\lambda_1''(w)}{\lambda_1'(w)^2}=0. \tag{5.17}$$

We next introduce a convex-dual function defined as the total wealth that is the sum of financial wealth and the present value of future income:

$$G_1(\lambda_1) := G_1(\lambda_1(w)) = w + \frac{y}{r}. \tag{5.18}$$

Notice the following relations:

$$G_1'(\lambda_1)\lambda_1'(w) = 1 \quad \text{and} \quad G_1''(\lambda_1)\lambda_1'(w)^2 + G_1'(\lambda_1)\lambda_1''(w) = 0. \tag{5.19}$$

With (5.18) and (5.19), we now rewrite (5.17) as follows:

$$\frac{1}{2}\theta^2\lambda_1^2G_1''(\lambda_1)+\left(\rho-r+\theta^2\right)\lambda_1G_1'(\lambda_1)-rG_1(\lambda_1)=0. \tag{5.20}$$

The general solution to the equation (5.20) is given as follows:

$$G_1(\lambda_1) = C_1\lambda_1^{m_+} + C_2\lambda_1^{m_-}, \tag{5.21}$$

where $m_+ > 0$ and $m_- < -1$ are two roots of the quadratic equation (3.2), and C_1 and C_2 are constants to be determined.

The Bellman equation (5.14) implies

$$\begin{aligned}
V(w) &= \frac{1}{\rho}(rw+y)V'(w) - \frac{1}{2\rho}\theta^2\frac{(V'(w))^2}{V''(w)} + \frac{L^{1-\gamma}}{\rho(1-\gamma)} \\
&= \frac{1}{\rho}\left[r\lambda_1 G_1(\lambda_1) - \frac{1}{2}\theta^2\lambda_1^2 G_1'(\lambda_1)\right] + \frac{L^{1-\gamma}}{\rho(1-\gamma)} \\
&= \frac{1}{\rho}\left[\left(r-\frac{1}{2}\theta^2 m_+\right)C_1\lambda_1^{m_++1} + \left(r-\frac{1}{2}\theta^2 m_-\right)C_2\lambda_1^{m_-+1}\right] + \frac{L^{1-\gamma}}{\rho(1-\gamma)} \\
&= \frac{m_+}{m_++1}C_1\lambda_1^{m_++1} + \frac{m_-}{m_-+1}C_2\lambda_1^{m_-+1} + \frac{L^{1-\gamma}}{\rho(1-\gamma)},
\end{aligned} \tag{5.22}$$

where the second equality is obtained from (5.16), (5.18), and (5.19), the third from (5.21), the last from

$$\frac{r-\frac{1}{2}\theta^2 m_\pm}{\rho} = \frac{m_\pm}{m_\pm+1}, \tag{5.23}$$

and λ_1 is the solution to the following algebraic equation by the relation (5.18) with the solution (5.21):

$$w = C_1\lambda_1^{m_+} + C_2\lambda_1^{m_-} - \frac{y}{r} := W_1(\lambda_1). \tag{5.24}$$

The constrained borrowing given in (2.1) imply that there exists $\widehat{\lambda} > 0$ such that

$$W_1(\widehat{\lambda}) = -\nu\frac{y}{r}, \tag{5.25}$$

where w goes down to its lower bound $-\nu y/r$ as λ goes up to its upper bound $\widehat{\lambda}$. The boundary condition (5.25) then implies that $W_1(\lambda_1)$ has the local minimum at $\lambda_1 = \widehat{\lambda}$ and hence,

$$W_1'(\widehat{\lambda}) = 0 \quad \text{and} \quad W_1''(\widehat{\lambda}) > 0, \tag{5.26}$$

The two boundary conditions (5.25) and (5.26) therefore determine the unknown constants C_1 and C_2 as

$$C_1 = -\frac{m_-}{m_+-m_-}\frac{(1-\nu)y}{r}\widehat{\lambda}^{-m_+} > 0, \quad C_2 = \frac{m_+}{m_+-m_-}\frac{(1-\nu)y}{r}\widehat{\lambda}^{-m_-} > 0, \tag{5.27}$$

where the constant $\widehat{\lambda}$ can be also determined by (5.24) with the substituted C_1 and C_2 as given in the theorem.

Next, for any $w \geq \widetilde{w}(L)$, we obtain with the FOC (5.13) the following Bellman equation:

$$(rw+y)V'(w) - \frac{1}{2}\theta^2\frac{(V'(w))^2}{V''(w)} - \rho V(w) + \max_{c\geq 0}\left[\frac{(c+L)^{1-\gamma}}{1-\gamma} - cV'(w)\right] = 0. \tag{5.28}$$

The FOC with respect to the optimal consumption c is given by

$$c = V'(w)^{-\frac{1}{\gamma}} - L. \tag{5.29}$$

By substituting the FOC above for consumption c in the equation (5.28) allows us to obtain the following equation:

$$(rw + y)V'(w) - \frac{1}{2}\theta^2 \frac{V'(w)^2}{V''(w)} - \rho V(w) + \frac{\gamma}{1-\gamma} V'(w)^{1-\frac{1}{\gamma}} + LV'(w) = 0. \tag{5.30}$$

Differentiating the equation (5.30) with respect to w, we obtain

$$(r - \rho - \theta^2)V'(w) + (rw + y)V''(w) + \frac{1}{2}\theta^2 \frac{V'(w)^2 V'''(w)}{V''(w)^2} - V'(w)^{-\frac{1}{\gamma}} V''(w) + LV''(w) = 0. \tag{5.31}$$

Applying the convex-duality approach of Bensoussan *et al.* (2016), we introduce a dual variable defined as the first derivative of the value function:

$$\lambda_2 := \lambda_2(w) = V'(w).$$

The equation (5.31) can then be restated as

$$(r - \rho - \theta^2)\lambda_2(w) + (rw + y)\lambda_2'(w) + \frac{1}{2}\theta^2 \frac{\lambda_2(w)^2 \lambda_2''(w)}{\lambda_2'(w)^2} - \lambda_2(w)^{-\frac{1}{\gamma}} \lambda_2'(w) = 0. \tag{5.32}$$

Next, we introduce a convex-dual function defined as the sum of financial wealth and the present value of future income plus living standard:

$$G_2(\lambda_2) := G_2(\lambda_2(w)) = w + \frac{y + L}{r} \tag{5.33}$$

with

$$G_2'(\lambda_2)\lambda_2'(w) = 1 \quad \text{and} \quad G_2''(\lambda_2)\lambda_2'(w)^2 + G_2'(\lambda_2)\lambda_2''(w) = 0.$$

We then rewrite (5.32) as

$$\frac{1}{2}\theta^2 \lambda_2^2 G_2''(\lambda_2) + (\rho - r + \theta^2)\lambda_2 G_2'(\lambda_2) - rG_2(\lambda_2) + \lambda_2^{-\frac{1}{\gamma}} = 0. \tag{5.34}$$

The general solution to the equation (5.34) is given as follows:

$$G_2(\lambda_2) = D_1 \lambda_2^{m+} + \frac{1}{K} \lambda_2^{-\frac{1}{\gamma}}, \tag{5.35}$$

where D_1 is a constant to be determined.

The Bellman equation (5.28) implies

$$
\begin{aligned}
V(w) &= \frac{1}{\rho}\Big[(rw+y)V'(w) - \frac{1}{2}\theta^2\frac{V'(w)^2}{V''(2)} + \frac{\gamma}{1-\gamma}V'(w)^{1-\frac{1}{\gamma}} + LV'(w)\Big] \\
&= \frac{1}{\rho}\Big[r\lambda_2 G_2(\lambda_2) - \frac{1}{2}\theta^2\lambda_2^2 G_2'(\lambda_2) + \frac{\gamma}{1-\gamma}\lambda_2^{1-\frac{1}{\gamma}}\Big] \\
&= \frac{1}{\rho}\Big[\Big(r-\frac{1}{2}\theta^2 m_+\Big)D_1\lambda_2^{m_++1} + \Big\{\frac{1}{K}\Big(r+\frac{\theta^2}{2\gamma}\Big) + \frac{\gamma}{1-\gamma}\Big\}\lambda_2^{1-\frac{1}{\gamma}}\Big] \\
&= \frac{m_+}{m_++1}D_1\lambda_2^{m_++1} + \frac{1}{(1-\gamma)K}\lambda_2^{1-\frac{1}{\gamma}},
\end{aligned}
\tag{5.36}
$$

where the last equality results from (5.23) and λ_2 is the solution to the following algebraic equation by (5.33) with (5.35):

$$
w = D_1\lambda_2^{m_+} + \frac{1}{K}\lambda_2^{-\frac{1}{\gamma}} - \frac{y+L}{r} := W_2(\lambda_2). \tag{5.37}
$$

By consumption constraints given in (2.12), optimal consumption c becomes zero as financial wealth w approaches the endogenously determined wealth level $\widetilde{w}(L)$. The FOC (5.29) then allows us to determine the dual variable $\tilde{\lambda}$ corresponding to $\widetilde{w}(L)$ at which optimal consumption c becomes zero:

$$
\tilde{\lambda} = L^{-\gamma}.
$$

By (5.24) and (5.37), we know from (Kim and Shin (2018)) that at $\lambda_1 = \lambda_2 = \tilde{\lambda}$,

$$
\widetilde{w}(L) = W_1(\tilde{\lambda}) = W_2(\tilde{\lambda}) \tag{5.38}
$$

or equivalently,

$$
C_1\tilde{\lambda}^{m_+} + C_2\tilde{\lambda}^{m_-} = D_1\tilde{\lambda}^{m_+} + \frac{1}{K}\tilde{\lambda}^{-\frac{1}{\gamma}} - \frac{L}{r}. \tag{5.39}
$$

Notice that the optimality of $\widetilde{w}(L)$ (or equivalently, $\tilde{\lambda}$) implies that the value function $V(w)$ is C^2 in w (Dumas (1991)). The value functions given in (5.22) and (5.36) with the C^2 property then have the following boundary conditions:

$$
\begin{aligned}
V\big(\widetilde{w}(L)-\big) &= V\big(\widetilde{w}(L)+\big), \\
V'\big(\widetilde{w}(L)-\big) &= V'\big(\widetilde{w}(L)+\big), \\
V''\big(\widetilde{w}(L)-\big) &= V''\big(\widetilde{w}(L)+\big)
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&\frac{m_+}{m_++1}C_1\tilde{\lambda}^{m_++1} + \frac{m_-}{m_-+1}C_2\tilde{\lambda}^{m_-+1} + \frac{L^{1-\gamma}}{\rho(1-\gamma)} \\
&\qquad = \frac{m_+}{m_++1}D_1\tilde{\lambda}^{m_++1} + \frac{1}{(1-\gamma)K}\tilde{\lambda}^{-\frac{1-\gamma}{\gamma}},
\end{aligned}
\tag{5.40}
$$

$$m_+C_1\tilde{\lambda}^{m_+} + m_-C_2\tilde{\lambda}^{m_-} = m_+D_1\tilde{\lambda}^{m_+} - \frac{1}{\gamma K}\tilde{\lambda}^{-\frac{1}{\gamma}},$$
$$m_+^2C_1\tilde{\lambda}^{m_+-1} + m_-^2C_2\tilde{\lambda}^{m_--1} = m_+^2D_1\tilde{\lambda}^{m_+-1} + \frac{1}{\gamma^2 K}\tilde{\lambda}^{-\frac{1}{\gamma}-1}.$$

Multiplying (5.39) by $m_+\tilde{\lambda}/(m_++1)$ with $\tilde{\lambda} = L^{-\gamma}$, we obtain

$$\begin{aligned}
&\frac{m_+}{m_++1}C_1\tilde{\lambda}^{m_++1} + \frac{m_+}{m_++1}C_2\tilde{\lambda}^{m_-+1}\\
&= \frac{m_+}{m_++1}D_1\tilde{\lambda}^{m_++1} + \frac{m_+}{m_++1}\frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}} - \frac{m_+}{m_++1}\frac{1}{r}L^{1-\gamma}\\
&= \frac{m_+}{m_++1}D_1\tilde{\lambda}^{m_++1} + \frac{m_+}{m_++1}\frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}} - \frac{m_+}{m_++1}\frac{1}{\rho}\frac{\rho}{r}L^{1-\gamma}\\
&= \frac{m_+}{m_++1}D_1\tilde{\lambda}^{m_++1} + \frac{m_+}{m_++1}\frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}} - \frac{m_+}{m_++1}\frac{1}{\rho}\frac{(m_++1)(m_-+1)}{m_+m_-}L^{1-\gamma}\\
&= \frac{m_+}{m_++1}D_1\tilde{\lambda}^{m_++1} + \frac{m_+}{m_++1}\frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}} - \frac{m_-+1}{m_-}\frac{L^{1-\gamma}}{\rho},
\end{aligned} \tag{5.41}$$

where third equality results from that

$$\rho = -\frac{(m_++1)(m_-+1)\theta^2}{2}, \quad r = -\frac{m_+m_-\theta^2}{2}. \tag{5.42}$$

Subtracting (5.40) from (5.41), we get

$$\begin{aligned}
&\Big(\frac{m_+}{m_++1} - \frac{m_-}{m_-+1}\Big)C_2\tilde{\lambda}^{m_-+1} - \frac{L^{1-\gamma}}{\rho}\frac{1}{1-\gamma}\\
&= \frac{m_+}{m_++1}\frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}} - \frac{m_-+1}{m_-}\frac{L^{1-\gamma}}{\rho} - \frac{1}{1-\gamma}\frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}}
\end{aligned}$$

and hence,

$$\begin{aligned}
&\frac{m_+-m_-}{(m_++1)(m_-+1)}C_2\tilde{\lambda}^{m_-+1}\\
&= -\frac{1}{(1-\gamma)m_-}\frac{L^{1-\gamma}}{\rho}\{-m_- + (1-\gamma)(m_-+1)\} + \Big\{\frac{m_+}{m_++1} - \frac{1}{1-\gamma}\Big\}\frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}}\\
&= -\frac{1}{(1-\gamma)m_-}\frac{L^{1-\gamma}}{\rho} + \frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}}\Big[\frac{\gamma(m_-+1)}{(1-\gamma)m_-}\frac{K}{\rho} - \frac{\gamma m_++1}{(1-\gamma)(m_++1)}\Big]\\
&= -\frac{1}{(1-\gamma)m_-}\frac{L^{1-\gamma}}{\rho} + \frac{1}{(1-\gamma)m_-K}\tilde{\lambda}^{1-\frac{1}{\gamma}}\Big[\gamma(m_-+1)\frac{K}{\rho} - \frac{m_-(\gamma m_++1)}{m_++1}\Big]\\
&= -\frac{1}{(1-\gamma)m_-}\frac{L^{1-\gamma}}{\rho} + \frac{1}{(1-\gamma)m_-K}\tilde{\lambda}^{1-\frac{1}{\gamma}}\Big[\gamma(m_-+1)\frac{(\gamma m_++1)(\gamma m_-+1)}{\gamma^2(m_++1)(m_-+1)} - \frac{m_-(\gamma m_++1)}{m_++1}\Big],
\end{aligned}$$

where the last equality results from (5.42) and that

$$K = -\frac{(\gamma m_++1)(\gamma m_-+1)\theta^2}{2\gamma^2} > 0. \tag{5.43}$$

We then obtain

$$\frac{m_+-m_-}{(m_++1)(m_-+1)}C_2\tilde{\lambda}^{m_-+1} = -\frac{1}{(1-\gamma)m_-}\frac{L^{1-\gamma}}{\rho} + \frac{\gamma m_++1}{\gamma(m_++1)(1-\gamma)m_-K}\tilde{\lambda}^{-\frac{1-\gamma}{\gamma}},$$

as a result, we determine C_2 as follows:

$$C_2 = -\frac{(m_+ + 1)(m_- + 1)}{(1-\gamma)m_-(m_+ - m_-)}\left[\frac{L^{1-\gamma}}{\rho} - \frac{\gamma m_+ + 1}{\gamma(m_+ + 1)K}\widetilde{\lambda}^{-\frac{1-\gamma}{\gamma}}\right]\widetilde{\lambda}^{-m_- - 1}$$
$$= -\frac{(m_+ + 1)(m_- + 1)}{(1-\gamma)m_-(m_+ - m_-)}\left\{\frac{1}{\rho} - \frac{\gamma m_+ + 1}{\gamma(m_+ + 1)K}\right\}L^{1+\gamma m_-} > 0.$$

With (5.27), we now determine $\widehat{\lambda}$ and C_1 as follows:

$$\begin{aligned}\widehat{\lambda} &= \left[-\frac{(m_+ + 1)(m_- + 1)}{(1-\gamma)m_+ m_-}\frac{r}{(1-\nu)y}\left\{\frac{K}{\rho} - \frac{\gamma m_+ + 1}{\gamma(m_+ + 1)}\right\}\frac{L}{K}\right]^{-\frac{1}{m_-}} L^{-\gamma} > 0,\\ C_1 &= -\frac{m_-}{m_+ - m_-}\cdot\frac{(1-\nu)y}{r}\widehat{\lambda}^{-m_+} > 0.\end{aligned} \tag{5.44}$$

Similar to (5.41), multiplying (5.39) by $m_-\tilde{\lambda}/(m_- + 1)$ with $\tilde{\lambda} = L^{-\gamma}$, we obtain

$$\begin{aligned}&\frac{m_-}{m_- + 1}C_1\tilde{\lambda}^{m_+ + 1} + \frac{m_-}{m_- + 1}C_2\tilde{\lambda}^{m_- + 1}\\ &= \frac{m_-}{m_- + 1}D_1\tilde{\lambda}^{m_+ + 1} + \frac{m_-}{m_- + 1}\frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}} - \frac{m_+ + 1}{m_+}\frac{L^{1-\gamma}}{\rho}.\end{aligned} \tag{5.45}$$

Subtracting (5.40) from (5.45), we get

$$\left(\frac{m_-}{m_- + 1} - \frac{m_+}{m_+ + 1}\right)(C_1 - D_1)\tilde{\lambda}^{m_+ + 1} - \frac{L^{1-\gamma}}{\rho}\frac{1}{1-\gamma}$$
$$= \frac{m_-}{m_- + 1}\frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}} - \frac{m_+ + 1}{m_+}\frac{L^{1-\gamma}}{\rho} - \frac{1}{1-\gamma}\frac{1}{K}\tilde{\lambda}^{1-\frac{1}{\gamma}}$$

and with rearrangement using (5.43), therefore

$$C_1 - D_1 = \frac{(m_+ + 1)(m_- + 1)}{(1-\gamma)m_+(m_+ - m_-)}\left[\frac{L^{1-\gamma}}{\rho} - \frac{\gamma m_- + 1}{\gamma(m_- + 1)K}\widetilde{\lambda}^{-\frac{1-\gamma}{\gamma}}\right]\widetilde{\lambda}^{-m_+ - 1}$$

or equivalently,

$$D_1 = C_1 - \frac{(m_+ + 1)(m_- + 1)}{(1-\gamma)m_+(m_+ - m_-)}\left\{\frac{1}{\rho} - \frac{\gamma m_- + 1}{\gamma(m_- + 1)K}\right\}L^{1+\gamma m_+}, \tag{5.46}$$

which is the determination of D_1 with C_1 determined in (5.44).

Notice that the following relations are obtained from the quadratic equation (3.2):

$$K = -\frac{(\gamma m_+ + 1)(\gamma m_- + 1)\theta^2}{2\gamma^2} > 0, \quad \rho = -\frac{(m_+ + 1)(m_- + 1)\theta^2}{2}, \quad r = -\frac{m_+ m_- \theta^2}{2}.$$

We then consider that

$$\begin{aligned}\frac{1}{\rho} - \frac{\gamma m_- + 1}{\gamma(m_- + 1)K} &= -\frac{2}{(m_+ + 1)(m_- + 1)\theta^2} + \frac{\gamma m_- + 1}{\gamma(m_- + 1)}\frac{2\gamma^2}{(\gamma m_+ + 1)(\gamma m_- + 1)\theta^2}\\ &= -\frac{2}{(m_+ + 1)(m_- + 1)\theta^2} + \frac{2\gamma}{(m_- + 1)(\gamma m_+ + 1)\theta^2}\\ &= -\frac{2}{(m_- + 1)\theta^2}\left[\frac{1}{(m_+ + 1)} - \frac{\gamma}{(\gamma m_+ + 1)}\right]\\ &= -\frac{2}{(m_- + 1)\theta^2}\frac{(\gamma m_+ + 1) - \gamma(m_+ + 1)}{(m_+ + 1)(\gamma m_+ + 1)}\\ &= -\frac{2(1-\gamma)}{(m_+ + 1)(m_- + 1)(\gamma m_+ + 1)\theta^2}.\end{aligned}$$

Hence,

$$
\begin{aligned}
&-\frac{(m_+ + 1)(m_- + 1)}{(1-\gamma)m_+(m_+ - m_-)}\left\{\frac{1}{\rho} - \frac{\gamma m_- + 1}{\gamma(m_- + 1)K}\right\}L^{1+\gamma m_+}\\
&= \frac{(m_+ + 1)(m_- + 1)}{(1-\gamma)m_+(m_+ - m_-)}\frac{2(1-\gamma)}{(m_+ + 1)(m_- + 1)(\gamma m_+ + 1)\theta^2}L^{1+\gamma m_+}\\
&= \frac{2}{m_+(m_+ - m_-)(\gamma m_+ + 1)\theta^2}L^{1+\gamma m_+} > 0
\end{aligned}
$$

and as a result, $D_1 > 0$ from (5.46) with $C_1 > 0$ in (5.44).

Having determined C_1 and C_2, we now determine by (5.24) the endogenous wealth level $\widetilde{w}(L)$ below which optimal decision is to consume nothing:

$$
\widetilde{w}(L) = C_1 L^{-\gamma m_+} + C_2 L^{-\gamma m_-} - \frac{y}{r} > -\frac{y}{r},
$$

which is the same as given in the theorem.

Having verified our conjecture with the endogenously determined wealth threshold $\widetilde{w}(L)$ below which $c_t = 0$, the FOC for investment given in (5.13) for any $-\nu y/r \le w < \widetilde{w}(L)$ lead to the optimal investment strategy as stated in the theorem. Also, the FOCs for consumption and investment given in (5.29) and (5.13), respectively, result in the optimal strategies for any $w \ge \widetilde{w}(L)$ as given in the theorem.

Proof of Proposition 3.1

From C_1 in (3.4), C_2 in (3.5) and $\widehat{\lambda}$ in (3.6) we obtain that

$$
\begin{aligned}
\widetilde{w}(L) &= C_1 L^{-\gamma m_+} + C_2 L^{-\gamma m_-} - \frac{y}{r}\\
&= -\frac{m_-}{m_+ - m_-}\cdot\frac{(1-\nu)y}{r}\left[-\frac{1}{(1-\nu)y(\gamma m_- + 1)}L\right]^{\frac{m_+}{m_-}} L^{\gamma m_+}L^{-\gamma m_+}\\
&\quad - \frac{m_+}{r(\gamma m_- + 1)(m_+ - m_-)}L^{1+\gamma m_-}L^{-\gamma m_-} - \frac{y}{r}\\
&= -\frac{m_-}{m_+ - m_-}\cdot\frac{(1-\nu)y}{r}\left[-\frac{1}{(1-\nu)y(\gamma m_- + 1)}\right]^{\frac{m_+}{m_-}} L^{\frac{m_+}{m_-}}\\
&\quad - \frac{m_+}{r(\gamma m_- + 1)(m_+ - m_-)}L - \frac{y}{r}.
\end{aligned}
$$

We now consider the first derivative

$$
\begin{aligned}
\frac{\partial \widetilde{w}(L)}{\partial L} &= -\frac{m_+}{m_+ - m_-}\cdot\frac{(1-\nu)y}{r}\left[-\frac{1}{(1-\nu)y(\gamma m_- + 1)}\right]^{\frac{m_+}{m_-}} L^{\frac{m_+ - m_-}{m_-}}\\
&\quad - \frac{m_+}{r(\gamma m_- + 1)(m_+ - m_-)}
\end{aligned}
$$

and the second derivative

$$
\frac{\partial^2 \widetilde{w}(L)}{\partial L^2} = -\frac{m_+}{m_-}\cdot\frac{(1-\nu)y}{r}\left[-\frac{1}{(1-\nu)y(\gamma m_- + 1)}\right]^{\frac{m_+}{m_-}} L^{\frac{m_+ - m_-}{m_-} - 1} > 0.
$$

We then see that

$$\left.\frac{\partial \widetilde{w}(L)}{\partial L}\right|_{L=L^*} = 0, \text{ where } L^* = -(\gamma m_- + 1)(1-\nu)y$$

and that the first derivative $\partial \widetilde{w}(L)/\partial L$ is an increasing function with respect to L. Since we consider this derivative when $L > L^*$, we obtain that

$$\left.\frac{\partial \widetilde{w}(L)}{\partial L}\right|_{L\to L^*+} = 0 \quad \Rightarrow \quad \frac{\partial \widetilde{w}(L)}{\partial L} > 0 \text{ for } L > L^*,$$

which implies that $\widetilde{w}(L)$ is an increasing function with respect to L.

Proof of Proposition 3.2

From $\widetilde{w}(L)$ in (3.7) with (3.4) and (3.5), we know that

$$\begin{aligned}\widetilde{w}(L) = &-\frac{m_-}{m_+ - m_-}\cdot\frac{(1-\nu)y}{r}\left[-\frac{1}{(1-\nu)y(\gamma m_- + 1)}\right]^{\frac{m_+}{m_-}} L^{\frac{m_+}{m_-}} \\ &- \frac{m_+}{r(\gamma m_- + 1)(m_+ - m_-)}L - \frac{y}{r}.\end{aligned}$$

Next, we consider the first derivative of $\widetilde{w}(L)$ with respect to ν as follows:

$$\begin{aligned}\frac{d\widetilde{w}}{dL} &= -\frac{m_-}{r(1-\nu)(m_+ - m_-)(\gamma m_- + 1)}\left[-\frac{1}{(1-\nu)y(\gamma m_- + 1)}\right]^{\frac{m_+}{m_-}-1} L^{\frac{m_+}{m_-}} \\ &\quad + \frac{m_+}{r(1-\nu)(m_+ - m_-)(\gamma m_- + 1)}\left[-\frac{1}{(1-\nu)y(\gamma m_- + 1)}\right]^{\frac{m_+}{m_-}-1} L^{\frac{m_+}{m_-}} \\ &= -\frac{y}{r}\left[-\frac{1}{(1-\nu)y(\gamma m_- + 1)}\right]^{\frac{m_+}{m_-}} L^{\frac{m_+}{m_-}} < 0.\end{aligned}$$

Thus, we now verify that $\widetilde{w}(L)$ decreases with respect to ν.