How to model bank competition: the case for Cournot

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Abstract

When firms choose their capacity and then compete à la Bertrand, the market equilibrium can correspond to the Cournot outcome (Kreps and Scheinkman, 1983). In the banking sector, a bank's lending capacity is determined by its capital structure due to regulatory capital requirements. This paper establishes the conditions under which the Bertrand-Cournot equivalence extends to banks. I treat capital as an imperfect capacity commitment, allowing banks to distribute dividends and raise additional capital at a short-term premium during the competition stage. I show that if the loan market is not severely affected by some types of frictions and the short-term premium is sufficiently large, the Cournot outcome is the unique equilibrium of the game. Such micro-foundations for Cournot competition in the loan market open new perspectives to the modelling of an elaborate, yet tractable, banking sector in macroeconomic models.

1 Introduction

In recent years, the macro-banking literature has received growing attention as researchers are introducing more elaborate banking sectors into macroeconomic models in order to capture the

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special role of banks in the economy. To this end, models include some key ingredients such as risk, limited liability, regulation, and asymmetric information. When several of these features are present, perfect competition can help maintain tractability. However, the banking sector is highly concentrated and banks have market power (Degryse and Ongena, 2008; Freixas and Rochet, 2008). Consequently, assuming perfect competition may result in outcomes or predictions that overlook important mechanisms driven by market power.

The tension between capturing key features of banking, incorporating imperfect competition, and maintaining tractability is a recurring challenge in the literature. Several papers incorporate micro-founded financial frictions into macroeconomic models, yet the lenders in these frameworks often lack the defining features of banks and are instead modeled as simple, risk-neutral agents (Bernanke and Gertler, 1986, 1990; Brunnermeier and Sannikov, 2014). Other contributions incorporate some important features of banks—such as credit and liquidity risk—but assume perfect competition and abstract from regulatory features (e.g.Christiano et al. (2010)). Finally, another set of papers includes many of the relevant bank characteristics, but run into tractability issues. Thakor (1996) and Begenau (2020) include capital requirements, but bypass the challenges associated with limited liability by proxing deposit insurance with a reduced-form subsidy from the government to banks. Abadi et al. (2022) allow banks to have market power, but they need to rely on a reduced-form cost function for banks, which is meant to capture agency costs and regulations. This paper proposes a solution to this modelling problem by micro-founding the Cournot competition in the banking sector under certain conditions. The Cournot approach effectively accounts for imperfect competition, improves tractability and, it enables researches to include many of the key ingredients in their models.

When combined with asymmetric information and regulation, imperfect competition can generate important economic mechanisms that would not arise under perfect competition Martinez-Miera and Repullo (2010); Schliephake (2016). Moreover, what matters is not only that competition is imperfect, but also the specific form it takes. Monopolistic competition is the main alternative to Cournot when researchers aim to capture bank market power. In the banking literature, the most common approaches are competition à la Salop and à la Dixit-Stiglitz.[1]. In a Salop model, borrowers are uniformly distributed on a circle and banks decide their location. Borrowers incur transportation costs to reach a bank. This type of competition can be narrowly interpreted as purely spatial, emphasasing that physical distance is an important factor in the lending market (Nguyen, 2019; Degryse and Ongena, 2005; Petersen and Rajan, 2002) or, in a broader sense, the unit circle can be seen as the space of products where banks offer loans with different features (e.g. customer service, payroll management) to gain market power. The standard interpretation of Dixit-Stiglitz also relies on product differentiation, but it implies that, *ceteris paribus*, borrowers are better off by having multiple loans with different banks rather than having one large loan with one bank. Ulate (2021) provides a plausible micro-foundation: the CES demand can be generated by a two step decision process in which first borrowers choose a bank through a taste shock and then decide on the loan quantity. These models rely on plausible non-monetary frictions to justify market segmentation. In contrast, the model proposed in this paper micro-founds Cournot competition using a simpler setting, with risk-neutral agents aiming to maximize cash flows.

The starting point of the micro-foundations of Cournot competition is Kreps and Scheinkman (1983)(hereinafter KS) who show that in a two-stage game in which firms first choose capacity and then compete à la Bertrand, the unique subgame perfect equilibrium is the Cournot outcome. Given this setting, the banking sector seems a natural application for two reasons: (1) given an amount of regulatory capital, capital requirements constrain a bank's lending capacity, hence bank capital choice can be interpreted as a capacity choice (Schliephake and Kirstein, 2013); (2) banks typically do not raise capital and issue loans simultaneously, but have medium-term capital targets (Couaillier, 2021). On the other hand, bank capital is different from physical capital. While physical capital investments, such as constructing a plant, are not feasible in the short term, bank capital can be a more flexible form of capacity. Maggi (1996) develops a model of capacity-price competition and allows firms to adjust their capacity in the second stage, but

[1]For competition à la Salop, examples include but are not limited to:Dell'Ariccia (2001); Chiappori et al. (1995); Andrés and Arce (2012); Andrés et al. (2013); for competition à la Dixit-Stiglitz:Gerali et al. (2010); Ulate (2021); Wang et al. (2022); Abadi et al. (2022).

restricts the analysis to differentiated goods and linear demand. Schliephake and Kirstein (2013) prove that Maggi (1996) can be extended to banks that issue risk-free loans, and show that if the cost of raising capital in the second stage is sufficiently high, the Cournot outcome is the unique subgame perfect equilibrium.

This paper generalizes that result. It identifies the conditions under which Cournot competition is the unique equilibrium in a richer banking environment, characterized by three key features. First, banks are protected by limited liability and deposits are insured by the government. Second, banks are subject to capital requirements and can adjust their capital in the competition stage. In line with Schliephake and Kirstein (2013), I find that the cost of recapitalising must be sufficiently high in order to sustain the Cournot equilibrium. If capacity constraints are not relevant, the competition stage reverts to a standard Bertrand game. Third, and most critically, loans are risky and their risk is endogenous - specifically, the probability of default to depend on the rate charged by the bank. This feature allows to incorporate asymmetric information as loan pricing can affect borrower selection and behaviour. Depending on the friction modeled, a higher loan rate may result in a safer portfolio (De Meza and Webb, 1987; Bernanke and Gertler, 1990) or a riskier one (Stiglitz and Weiss, 1981; Martinez-Miera and Repullo, 2010) or have an ambiguous effect on loan quality (House, 2006). The endogeneity of risk requires particular attention, as it threatens the Bertrand-Cournot equivalence. In KS, firms with small capacities have no incentive to undercut each other, as both operate at full capacity and price cuts do not increase sales. As a result, choosing capacity in stage one becomes equivalent to choosing quantity. However, in a banking context with asymmetric information, even a bank operating at full capacity may want to reduce its rate to improve its risk profile. For example, under moral hazard, a lower interest rate may induce borrowers to choose safer projects (Boyd and De Nicolo, 2005; Martinez-Miera and Repullo, 2010), improving expected returns despite fixed loan volume. Therefore, a second necessary condition for the Cournot outcome is that the expected residual cash flow— loan revenues net of deposit repayments—must be increasing in the loan rate. In practice, this condition limits how sensitive default risk can be to loan interest rate.

Finally, in settings where banks possess private information about borrower quality, price competition may trigger selection problems such as the winner's curse (Von Thadden, 2004), which eliminates pure-strategy equilibria and consequently break down the Bertrand-Cournot equivalence.

This paper contributes to the micro-foundation banking models that assume Cournot competition, but more importantly, provides a new tool which may open new perspectives for future research in macro-banking.

The rest of the paper is organised as follows: Section 2 sets up the model and find the equilibrium of the baseline game; Section 3 presents which frictions can be captured by the model; in Section 4 I allow banks to raise capital in the second stage and I discuss the assumptions on the dividends; Section 5 concludes.

2 The baseline model

The model builds on Kreps and Scheinkman (1983) and Martinez-Miera and Repullo (2010). Consider the following two-bank two-stage game. In stage 1, each bank $i \in \{1, 2\}$ chooses an initial capital level $k_i \in \mathbb{R}_+$. In stage 2, banks compete à la Bertrand in the loan market subject to capital requirements. At this stage, banks are allowed to adjust their capital position: they can either reduce capital by distributing dividends at a unit cost δ or raise more capital at a short-term premium κ. I assume δ to be positive and arbitrarily close to zero [2]. In the baseline model I assume $\kappa = +\infty$ as in KS, which implies that banks cannot raise capital in the short term. This assumption is relaxed in Section 4, where I derive the conditions under which the equilibrium of the game is unaltered. Capital regulation requires each bank to fund a fraction $\gamma \in (0, 1)$ of its loans l_i with capital, such that $k_i \geq \gamma l_i$. Loans can also be financed through deposits, d_i , which are supplied inelastically and are fully insured by the government. Let $L(r)$ denote the direct loan demand at the gross loan rate r. The second stage is a Bertrand competition with

[2] When $\delta = 0$, Cournot outcome is still a SPNE, a positive but arbitrarily close to zero cost is sufficient to rule out multiplicity of equilibria. This assumption and alternative setups will be discussed in Section 4

capacity constraintsFollowing KS, I assume that loan demand is rationed according to an efficient rule: borrowers willing to accept higher rates are served first:

$$l_i = \begin{cases} \min\left(\frac{k_i}{\gamma}, L(r_i)\right) & \text{if } r_i < r_j \\ \min\left(\frac{k_i}{\gamma}, \max\left(\frac{L(r_i)}{2}, L(r_i) - \frac{k_j}{\gamma}\right)\right) & \text{if } r_i = r_j \\ \min\left(\frac{k_i}{\gamma}, \max\left(0, L(r_i) - \frac{k_j}{\gamma}\right)\right) & \text{if } r_i > r_j \end{cases}$$

where r_i is the gross loan interest rate named by bank i. Each bank cannot exceed its regulatory capacity k_i/γ, determined by the capital requirement. If bank i sets the lowest rate, it serves the entire market up to its capacity. If both banks offer the same rate, demand is split equally unless one bank lacks the capacity to serve half the market, in which case the other bank serves the residual demand. Lastly, if bank i offers the higher rate, it is only able to serve any residual demand left by its competitor. Note that efficient rationing is not an inconsequential assumption; KS result does not hold under other types of rationing without further assumptions [3]. In a setting where borrower type is private information, caution is required when applying this rationing rule. Since borrowers are served based on their willingness to pay, it is essential that willingness to pay is independent of borrower type. Otherwise, the ordering implied by efficient rationing would contradict the assumption that banks cannot observe borrower type. Hence, in settings like Stiglitz and Weiss (1981) Cournot competition cannot be meaningfully applied[4].

Finally the inverse loan demand is given by $r(L)$, which is the gross interest rate on loans as a function of total loans $L = l_1 + l_2$.

Assumption 1. *The function $r(L)$ is strictly positive, twice continuously differentiable and strictly decreasing.*

Differently from Schliephake and Kirstein (2013), loans are allowed to be risky. Assume that all agents are risk neutral and the risk-free rate is normalised to one. The fraction of loans that

[3] See Lepore (2009) for more details

[4] See Section 3.2

default[5] is governed by the random variable x, which is distributed according to the cumulative distribution function $F(x|r_i)$ which has support $[0,1]$.

Assumption 2. *$F(x|r_i)$ is twice continuously differentiable in x and r_i, and it is strictly increasing in x over its support. Furthermore, bank i's distribution of defaults depends only on the interest rate set by bank i, that is $F_i(x|r_i, r_j) = F(x|r_i)$.*

This specification captures endogenous risk, where the default distribution depends on the loan rate set by bank i. Although general, it rules out models in which the competitor bank's rate r_j affects the fraction of defaults. This distinction is important for identifying the types of frictions that can micro-found the functional form $F(x|r_i)$. As I will extensively discuss in Section 3, this model can nest a series of moral hazard setups, but it is not suited to capture settings in which banks have different information sets. In a classic moral hazard framework, due to limited liability, borrowers choose riskier projects when facing a higher loan rate. This mechanism corresponds to $\frac{\partial F}{\partial r_i} < 0$, indicating that higher rates increase the probability of default. However, since the core results of the model do not hinge on the sign of this partial derivative, I do not impose additional restrictions on F.

2.1 Cournot game

Before turning to the two-stage game, I present the equivalent one-stage Cournot game, in which banks choose capital and loan quantities simultaneously. First, note that capital is endogenously more costly than deposits due to deposits insurance. A bank fails if it cannot repay its depositors in full, in which case shareholders receive nothing and insured deposits are repaid by the government. From the bank's perspective, deposits are repaid at the risk-free rate conditional on survival, whereas capital must be repaid at the risk-free rate in expectation over all states of the world. As a result, capital requirements are binding: $k_i = \gamma l_i$ and $d_i = (1-\gamma)l_i$ for $i \in \{1,2\}$. Taking l_j as

[5] I assume that if the loan defaults the recovery rate is zero, however the model can be easily extended to a positive recovery rate.

given, bank i solves the following problem:

$$\max_{l_i \geq 0} \left(\int_0^{\tilde{x}} ((1-x)r(L) - (1-\gamma))\, dF(x|r(L)) - \gamma \right) l_i$$

where $\tilde{x}$ is the maximum fraction of defaults that allows the bank to repay deposits, implicitly defined by:

$$(1-\tilde{x})r(L)l_i = (1-\gamma)l_i \quad \Rightarrow \quad \tilde{x} = \frac{r(L) - (1-\gamma)}{r(L)}$$

Define the function

$$Z(L) = \int_0^{\tilde{x}} ((1-x)r(L) - (1-\gamma))\, dF(x|r(L))$$

which represent the expected average residual cashflow, which is the cashflow accruing to shareholders once deposits are repaid. In this framework, $Z(L)$ plays the role of the inverse demand function in a standard Cournot model, therefore, in order to guarantee the existence and the uniqueness of the Cournot equilibrium, I assume the following:

Assumption 3. $Z'(L) < 0$ *and* $Z'(L) + Z''(L)L < 0$.

As I will discuss in the next section, the monotonicity is also crucial in the two-stage game.

Proposition 1. *Let*

$$b(l_j) = \arg\max_{l_i \geq 0} (Z(L) - \gamma) l_i$$

Under assumptions 1-3, the best response function $b(\cdot)$ has a unique fixed point $b(l^C) = l^C$. Therefore (l^C, l^C) and $r(2l^C)$ are respectively the equilibrium quantities and the equilibrium rate of the Cournot game.

Proof. See Appendix. □

Before turning to the two-stage game, I define an auxiliary Cournot game that helps partition the subgame space into three relevant regions, as shown in Figure 1. In other words, banks are assumed to have already raised sufficient capital and, must compare the profitability of using that capital to issue loans versus distributing dividends. Specifically, let:

$$\hat{b}(l_j) = \arg\max_{l_i \geq 0}(Z(L) - \gamma(1-\delta))l_i$$

It is straightforward to prove that $\hat{b}(\cdot)$ has the same properties of $b(\cdot)$ and that $\hat{b}(l_j) < b(l_j)$, with strict inequality when $b(l_j)$ is positive.

Now I proceed to the two-stage game, which has the following timeline: fist banks choose capital, second banks compete over rates in the loan market. Based on the resulting demand, each bank raises deposits and adjusts its capital position: if it holds excess capital relative to its loan volume, it pays out dividends; otherwise, it relies on deposits to meet funding needs. In the sections that follow, I solve the game by backward induction and show that, under certain conditions, the Cournot outcome characterized in the this section is the unique subgame perfect Nash equilibrium.

2.2 Second stage: Bertrand competition with capital requirements

In stage 2, every pair (k_1, k_2) is a different subgame which I denote by $\mathcal{H}(k_1, k_2)$. The strategy space of bank i, which I denote by $\mathcal{S}_{i,2}$, is the space of distributions over loan rates. Define $G_i(r) \in \mathcal{S}_{i,2}$ as $\Pr(r_i \leq r) = \Pr(r_i < r) + \alpha_i(r)$, where $\alpha_i(r)$ is the probability mass on r. Let $\text{Supp}(G_i) = [\underline{r}_i, \bar{r}_i]$ be the support of the distribution of the rates named by bank i. Taking G_j as given, if bank i names the rate r_i it expects to issue:

$$\begin{aligned} l_i(r_i, G_j) = \min\left(\frac{k_i}{\gamma}, \max\left(0, L(r_i) - \frac{k_j}{\gamma}\right)\right)(G_j(r_i) - \alpha_j(r_i)) + \\ + \min\left(\frac{k_i}{\gamma}, \max\left(\frac{L(r_i)}{2}, L(r_i) - \frac{k_j}{\gamma}\right)\right)\alpha_j(r_i) + \\ + \min\left(\frac{k_i}{\gamma}, L(r_i)\right)(1 - G_j(r_i)) \end{aligned}$$

$G_j(r_i) - \alpha_j(r_i)$ is the probability that $r_j < r_i$, hence bank i will serve the residual demand (if any); $\alpha_j(r_i)$ is the probability that $r_j = r_i$, hence bank i and bank j split the demand; finally $1 - G_j(r_i)$ is the probability that $r_j > r_i$, hence bank i will serve the entire demand up to its capacity. In the second, stage bank i aims at maximising[6]

$$\max_{G_i \in \mathcal{S}_{i,2}} \left\{ M(G_i, G_j) = \int_{\underline{r}_i}^{\bar{r}_i} \left(m(r_i) l_i(r_i, G_j) + (1-\delta)\left(k_i - \gamma l_i(r_i, G_j)\right)\right) dG_i(r_i) \right\}$$

where $m(r_i) = Z(L(r_i)) = \int_0^{\tilde{x}} ((1-x)r_i - (1-\gamma))\, dF(x|r_i)$. Note that $m'(r) = Z'(L)L'(r) > 0$. The assumption on the monotonicity of $Z(L)$, implies that the average residual cashflow is increasing in own rate. While for firms operating under constant returns to scale it is straightforward that an increase in price raises the average profit margin, for banks there are additional mechanisms at work[7]:

$$\frac{\partial m(r)}{\partial r} = \underbrace{\frac{\partial \tilde{x}}{\partial r} r F(\tilde{x}|r)}_{(+)\text{ Buffer}} + \int_0^{\tilde{x}} \underbrace{F(x|r)}_{(+)\text{ Margin}} + \underbrace{r\frac{\partial F}{\partial r}}_{(\pm)\text{ Distribution shifting}} dx \gtreqless 0$$

The first term reflects a buffer effect: as the loan rate increases, the default threshold $\tilde{x}$ rises, allowing the bank to remain solvent in a greater number of states. The second term captures a margin effect: higher loan rates increase returns on performing loans, much like standard pricing logic in firm theory. The final component, the distribution shifting effect, reflects how the distribution of defaults responds to changes in the loan rate. This effect is ambiguous in sign, as it depends on the specific frictions driving borrower behavior. For example, in a moral hazard setting (Boyd and De Nicolo, 2005; Martinez-Miera and Repullo, 2010; Schliephake, 2016) result in a negative distribution shifting effect. Assumption 3 ensures that the overall expression remains well-behaved: it rules out excessively strong negative distribution shifting effects by requiring that $Z(L)$ be decreasing in L. In practice, this limits the sensitivity of portfolio risk to changes in the loan rate.

The existence of an equilibrium in each subgame $\mathcal{H}(k_1, k_2)$ is guaranteed by Theorem 5 of Dasgupta and Maskin (1986) [8]. Define $\rho_\gamma(y)$ to be the loan rate that satisfies $m(\rho_\gamma(y)) = \gamma y$, and let

[6]Note that whenever $k_i > \gamma l_i$, bank i always prefer to pay out dividends to make the capital requirement binding. The reason is that the cost of paying dividends is assumed to be sufficiently small so that the bank is not willing to hold capital in excess of the regulatory minimum

[7]Note that $m(r) = \int_0^{\tilde{x}} rF(x|r)dx$ by integration by parts.

[8]by lowering its rate a bank always benefits from a (weak) increase in demand, hence bank i's payoff is weakly lower semi-continuous in r_i; whereas the sum of the payoffs is upper semi-continuous in both r_i and r_j.

$\Lambda_\gamma(y) = L(\rho_\gamma(y))$ denote the corresponding loan demand. Consider subgames $\mathcal{H}(k_1, k_2)$ such that $\min_i \frac{k_i}{\gamma} \geq \Lambda_\gamma(1-\delta)$. In this case, the capacity constraints are non biding at the zero-profit rate, hence the game reduces to a standard Bertrand competition. The unique equilibrium strategies (G_1^*, G_2^*) are such that $r_1 = r_2 = \rho_\gamma(1-\delta)$ with probability 1. Banks' equilibrium payoffs are equal to $M_i(G_i^*, G_j^*) = (1-\delta)k_i$. For the rest of the paper consider only subgames in which $\min_i \frac{k_i}{\gamma} < \Lambda_\gamma(1-\delta)$, i.e. at least one bank is constrained at the zero-profit rate.

Lemma 1. *In every subgame it must be that* $\underline{r}_i \geq r\left(\frac{k_1+k_2}{\gamma}\right) \equiv r^{FC}$ *for all* $i \in \{1,2\}$.

Proof. Fix any G_j, if a bank names a rate $r \leq r^{FC}$, then it is operating at full capacity with probability one. Given that $m(r)$ is an increasing function of r, any $r < r^{FC}$ is strictly dominated by r^{FC}. In other words, when a bank reaches maximum capacity has no incentive to undercut the opponent as it would decrease the expected residual cashflow without improving the quantity. Therefore any rate $r < r^{FC}$cannot be part of an equilibrium strategy and $\underline{r}_i \geq r^{FC}$. □

Lemma 2. *If* $\bar{r}_1 = \bar{r}_2 = \bar{r}$ *and* $\alpha_i(\bar{r}) > 0$ *for* $i \in \{1,2\}$, *then*

$$\underline{r}_i = \bar{r}_i = r^{FC} \quad \textit{and} \quad \frac{k_i}{\gamma} \leq \hat{b}\left(\frac{k_j}{\gamma}\right) \forall i \in \{1,2\}$$

Proof. See Appendix. □

Lemma 2 states that if there exists an equilibrium in which banks have the same supremum, this sumpremum must be smaller or equal to the full capacity rate. The intuition is the following: if $\bar{r} > r^{FC}$, then the bank with (weakly) more capacity has capital in excess with probability 1. Hence, the strategy $\bar{r}$ would be dominated by $\bar{r} - \epsilon$, with ϵ arbitrarily small, which keeps the expected average residual cashflow constant and increases the quantity. The second part of the lemma, $\frac{k_i}{\gamma} \leq \hat{b}\left(\frac{k_j}{\gamma}\right)$, ensures that each bank has no incentive to charge a rate that is higher than $\bar{r}$ and be the monopolist of the residual demand.

Lemma 3. *If* $\bar{r}_i > \bar{r}_j$ *or* $\bar{r}_i = \bar{r}_j$ *and* $\alpha_j(\bar{r}_j) = 0$, *then:*

(a) $\bar{r}_i = r\left(\hat{b}\left(\frac{k_j}{\gamma}\right) + \frac{k_j}{\gamma}\right)$ *and the equilibrium payoff of bank* i *is equal to*

$$M_i(G_i^*, G_j^*) = \underbrace{(m(\bar{r}_i) - \gamma(1-\delta))\,\hat{b}\left(\frac{k_j}{\gamma}\right)}_{P(k_j)} + (1-\delta)k_i$$

(b) $\frac{k_i}{\gamma} > \hat{b}\left(\frac{k_j}{\gamma}\right)$

(c) $\underline{r}_i = \underline{r}_j$ *and* $\alpha_i(\underline{r}_i) = 0$ *for all* $i \in \{1, 2\}$

(d) $k_i \geq k_j$

(e) the equilibrium payoff of bank j *is uniquely determined by* (k_1, k_2) *and*

$$P(k_j)\frac{k_j}{k_i} + (1-\delta)k_j \leq M_j(G_j^*, G_i^*) \leq P(k_j) + (1-\delta)k_i$$

Proof. See Appendix □

In any mixed strategy equilibrium, each bank must be indifferent over all loan rates within the support of its strategy. Suppose bank i charges the higher supremum $\bar{r}_i > \bar{r}_j$. Then bank i must be serving a positive residual demand at $\bar{r}_i$; otherwise, it would have an incentive to undercut its rival. Moreover, to prevent profitable deviations, $\bar{r}_i$ must be such that the bank issues exactly $\hat{b}\left(\frac{k_j}{\gamma}\right)$ - the best response under the auxiliary Cournot game when facing residual demand.

The lemma also implies that the bank with the higher supremum rate must have greater capital. The intuition is that the more capitalized bank must be sufficiently compensated—via a favorable rate and allocation—so that it restrains its lending capacity rather than undercutting the competitor and eliminating their profit opportunities.

Dasgupta and Maskin (1986) guarantee the existence of an equilibrium, therefore every $\mathcal{H}(k_1, k_2)$ has an equilibrium that must respect Lemmas 1-3. We can divide the subgames space into three relevant regions (see Figure 1 for reference) .

- Region 1 $\left\{\mathcal{H}(k_1, k_2) : \min_i \frac{k_i}{\gamma} > \Lambda_\gamma(1-\delta)\right\}$: in this region banks are so much capitalised that capacity constraints do not matter. The subgame equilibrium is $r_1 = r_2 = \rho_\gamma(1-\delta)$

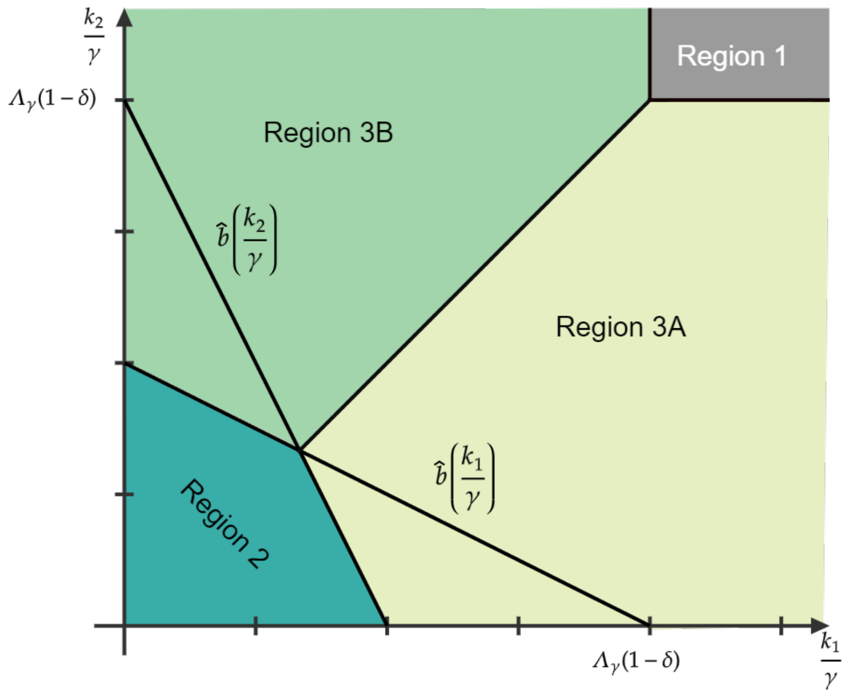

Figure 1: Equilibrium regions of the baseline model

with probability one and the equilibrium payoffs are $M_i(G_i^*, G_j^*) = (1-\delta)k_i$ for $i \in \{1,2\}$.

- Region 2 $\left\{\mathcal{H}(k_1,k_2) : \frac{k_i}{\gamma} \leq \hat{b}\left(\frac{k_j}{\gamma}\right) \forall i \in \{1,2\}\right\}$: in this region banks operate at full capacity. The subgame equilibrium is $r_1 = r_2 = r^{FC}$ with probability one and the equilibrium payoffs are $M_i(G_i^*, G_j^*) = m(r^{FC})\frac{k_i}{\gamma}$ for $i \in \{1,2\}$.

- Region 3A $\left\{\mathcal{H}(k_1,k_2) : k_1 \geq k_2 \text{ and } \frac{k_1}{\gamma} > \hat{b}\left(\frac{k_2}{\gamma}\right)\right\}$: in this region there is a mixed strategy equilibrium which has the characteristics described by Lemma 3. The equilibrium payoffs are $M_1(G_1^*, G_2^*) = P(k_2) + (1-\delta)k_1$ and $P(k_2)\frac{k_2}{k_1} + (1-\delta)k_2 \leq M_2(G_2^*, G_1^*) \leq P(k_2) + (1-\delta)k_1$

- Region 3B: symmetric to Region 3A

Note that stage 2 payoffs are continuous functions of k_1 and k_2.

2.3 First stage: capital choice

In the first stage, each bank chooses a capital level according to some probability measure $\mu_i(k)$ with support $[\underline{k}_i, \bar{k}_i] \subseteq \mathbb{R}_+$. Denote by $\mathcal{S}_{i,1}$ the strategy space of stage 1. Bank i aims to maximise its expected profits:

$$\max_{\mu_i(k_i)\in\mathcal{S}_{i,1}} \left\{ \pi(\mu_i,\mu_j) = \int_{\underline{k}_i}^{\bar{k}_i} \int_{\underline{k}_j}^{\bar{k_j}} \left(M_i(G_i^*, G_j^*) - k_i \right) d\mu_j(k_j) d\mu_i(k_i) \right\}$$

where μ_j is the opponent's strategy and (G_i^*, G_j^*) and the equilibrium strategies of the second-stage subgame $\mathcal{H}(k_1, k_2)$.

Proposition 2. *Under assumptions 1-3, the Cournot outcome, $k_1 = k_2 = \gamma l^C$ and $r_1 = r_2 = r^C$, is the unique subgame perfect Nash equilibrium (SPNE) of the two-stage game.*

Proof. See Appendix for a formal proof. Below I provide a sketch of the proof. □

To illustrate the logic, restrict the attention to pure strategies and partition the capital space into the three regions presented before:

- Region 1 (Overinvestment region): bank i profits are given by $\pi_i(k_i, k_j) = (1-\delta)k_i - k_i = -\delta k_i$. In this region the bank has raises too much capital. As paying dividends is costly, from a stage 1 perspective, the bank is better off by raising less capital. Hence any (k_1, k_2) that belong to region 1 cannot be a SPNE.

- Region 2 (Cournot region): bank i profits are given by $\pi_i(k_i, k_j) = m(r^{FC})\frac{k_i}{\gamma} - k_i = \left(Z\left(\frac{k_i+k_j}{\gamma}\right) - \gamma \right)\frac{k_i}{\gamma}$. In this region banks operate at full capacity and charge the full capacity rate, hence the stage 1 strategic choice of capacity is equivalent to the strategic choice of quantity. The only possible subgame perfect equilibrium in this region the Cournot equilibrium $(k_1^*, k_2^*) = (\gamma l^C, \gamma l^C)$[9].

- Region 3A (Asymmetric overinvestment region): bank 1 profits are equal to $\pi_1(k_1, k_2) = P(k_2) - \delta k_1$. Bank 1's profits are decreasing in k_1, hence bank 1 is better off by raising less capital. Note that this is true even at the border of the region when $k_1 = k_2$. The intuition is that in this region bank 1 has raised too much capital and has to pay dividends

[9]Note that the Cournot equilibrium belongs to this region because $\hat{b}(l) \leq b(l)$ for all l.

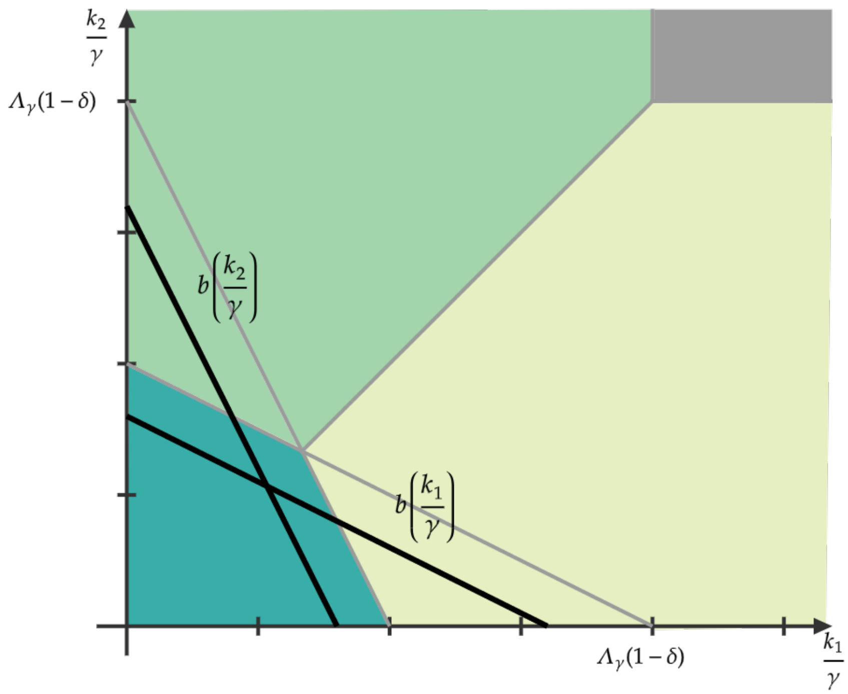

Figure 2: Full game equilibrium

in expectation. A positive cost for paying dividends rules out the possibility of having an equilibrium in this region and ensures the uniqueness of the SPNE.

- Region 3B: symmetric to 3A.

Therefore the only subagme perfect equilibrium is $(k_1^*, k_2^*) = (\gamma l^C, \gamma l^C)$, in stage 2 $r_1 = r_2 = r^{FC} = r(2l^C)$ with probability one.

3 Modelling the banking sector: When can we assume Cournot competition?

The key condition that must hold in order to assume Cournot competition is that the average expected residual cashflow must be increasing in own rate. Clearly, Cournot competition can be assumed when risk is exogenous $\left(\frac{\partial F}{\partial r} = 0\right)$ (e.g. Villa, 2023; Bahaj and Malherbe, 2020). In this section I go over the most common frameworks of asymmetric information and state whether is possible or not to micro-found Cournot competition.

3.1 Moral Hazard

Due to limited liability, when entrepreneurs face a higher loan rate, they choose a riskier project or exert less effort, hence the probability of default is increasing in loan rate. Consider a modified version of Boyd and De Nicolo (2005). Entrepreneurs choose among projects that require one unit of investment and have the following return function:

$$X = \begin{cases} \alpha(p) & \text{with prob. } (1-p) \\ 0 & \text{with prob. } p \end{cases}$$

Therefore entrepreneurs optimally pick a project by choosing the probability of default p. Assume $\alpha(p)$ to be continuous, increasing and strictly concave. Each entrepreneur t has her outside option $\bar{u}_t$ and solves the following problem

$$u(r) = \max_{p\in[0,1]} (1-p)(\alpha(p) - r)$$
$$\text{such that } u(r) \geq \bar{u}_t$$

In order to have an interior solution I further assume that $\alpha(0) - \alpha'(0) < r < \alpha(1)$. The first order condition is given by:

$$r - \alpha(p^*) + (1-p^*)\alpha'(p^*) = 0$$

By the implicit function theorem

$$\frac{dp^*}{dr} = \frac{1}{2\alpha'(p^*) - (1-p^*)\alpha''(p^*)} > 0$$

When charged a higher rate, entrepreneurs choose projects with higher probability of default. For simplicity assume that all loans are perfectly correlated. The expected average residual cashflow is given by:

$$m(r) = (1 - p^*(r))(r - (1 - \gamma))$$

The key condition is that $m(r)$ increasing, i.e.

$$\begin{aligned} m'(r) &= -\frac{dp^*}{dr}(r - (1 - \gamma)) + (1 - p^*(r)) > 0 \\ &\Leftrightarrow \frac{dp^*}{dr} < \frac{1 - p^*(r)}{r - (1 - \gamma)} \end{aligned}$$

This inequality implies that the probability of default must not respond too much to an increase in the loan rate. For instance, if $p(r) = a + br$, then:

$$\begin{aligned} m'(r) &= -b(r - (1 - \gamma)) + 1 - a - br > 0 \\ &\Leftrightarrow b < \frac{1 - a}{2r - (1 - \gamma)} \leq \frac{1 - a}{1 - \gamma} \end{aligned}$$

In conclusion, provided that the marginal residual cashflow is increasing in the loan rate, Cournot competition can be justified also in framework that entail moral hazard (Martinez-Miera and Repullo, 2010; Schliephake, 2016; Gasparini, 2023; Corbae and Levine, 2019).

3.2 Heterogeneous types of borrowers, adverse or favourable selection and screening

Recall the assumption of efficient rationing: when loan demand exceeds a bank's capacity, borrowers with higher reservation rates are served first. In environments where borrower types are private information, caution is required when applying this rule. Since rationing is based on willingness to pay, it must be the case that willingness to pay is independent of borrower type. Otherwise, the assumption of private information would be contradicted, as the bank would effectively be able to infer borrower types through observed reservation rates. This restriction rules out models in which the lending rate affects the composition of the borrower pool through selection effects, such as adverse selection or advantageous selection (Stiglitz and Weiss (1981); De Meza and Webb

(1987); House (2006)).

Finally, if the banks can screen borrowers through imperfect signals, the assumption $F(x|r_i, r_j) = F(x|r_i)$ might not be preserved. In particular, if banks observe uncorrelated private signals about borrower risk, strategic interaction becomes more complex. As shown in Marquez, 2002, the bank charging a higher rate will draw from a worse distribution, consisting of borrowers rejected by the lower-priced competitor. In such settings, banks have an incentive to undercut each other even when capacity constraints bind, which undermines the Cournot-Bertrand equivalence. One way to restore the equilibrium result is to assume that banks observe a common signal (e.g. open banking).

4 Adjusting capital in the second stage

This section relaxes the assumption on the short term capital premium ($\kappa = +\infty$) and discusses the conditions under which the Cournot equilibrium remains the unique SPNE of the two-stage game. It also revisits the assumptions on dividend payments and proposes an alternative formulation.

4.1 Raising additional capital

Consider the same two-stage game as in the baseline model, but now allow for $\kappa < \infty$. In this modified setup, when capital requirements are binding and loan demand exceeds capacity, banks may choose to raise additional capital during the second stage in order to serve unmet demand. This setting is conceptually similar to extensions of KS studied in the industrial organisation litarature. Boccard and Wauthy (2000; 2004) allow firms to build extra capacity in the competition stage at a premium cost. They show that when the Cournot price is below the short-term capacity cost, the Cournot outcome remains the unique SPNE. Otherwise, firms compete à la Bertrand and the price equals the premium. However, their analysis relies on implicit parametric restrictions, and a stronger assumption is actually needed.

Assumption 4. $\rho_\gamma(1+\kappa) > r(\hat{b}(0))$.

This assumes that the rate that makes the bank indifferent between raising or not extra capital is larger than the monopolist rate of the auxiliary game. Note that even when banks are allowed to adjust capital in stage two, the existence of equilibrium in every subgame remains guaranteed by Theorem 5 in Dasgupta and Maskin (1986).

Lemma 4. *In subgames* $\mathcal{H}(k_1, k_2)$ *where* $\frac{k_1+k_2}{\gamma} < \Lambda_\gamma(1+\kappa)$*, the unique subgame equilibrium is* $r_1 = r_2 = \rho_\gamma(1+\kappa)$ *with probability one.*

Proof. See Appendix for a formal proof. The intuition is that when $\frac{k_1+k_2}{\gamma} < \Lambda_\gamma(1+\kappa)$, then $r^{FC} > \rho_\gamma(1+\kappa)$, hence at the full capacity rate banks find it optimal undercut the opponent and expand their capacity. The typical demand-stealing mechanism of Bertrand competition is restored. Therefore, banks undercut each other until they make zero profits, i.e. $r_1 = r_2 = \rho_\gamma(1+\kappa)$. □

Lemma 5. *In subgames* $\mathcal{H}(k_1, k_2)$ *such that* $\frac{k_1+k_2}{\gamma} \geq \Lambda_\gamma(1+\kappa)$*, lemmas 1 to 3 hold.*

Proof. See Appendix. □

The intuition is that when $\frac{k_1+k_2}{\gamma} \geq \Lambda_\gamma(1+\kappa)$, which implies $r^{FC} \leq \rho_\gamma(1+\kappa)$, the possibility to raise more capital does not create any profitable deviation in the subgames equilibria found in the baseline game. Assumption 4 plays an important role for Lemma 3 as it ensures that the equilibrium payoffs are solely determined by (k_1, k_2) and are the same of the baseline game.

Now we can divide the subgames into four relevant regions. Regions 1 and 3A/B are the same of the baseline game with the same payoffs. Region 2 is now defined as $\left\{\mathcal{H}(k_1, k_2) : \frac{k_1+k_2}{\gamma} \geq \Lambda_\gamma(1+\kappa) \text{ and } \frac{k_i}{\gamma} \leq \hat{b}\left(\frac{k_j}{\gamma}\right) \forall i = 1, 2\right\}$. Finally in Region 4, defined as $\left\{\mathcal{H}(k_1, k_2) : \frac{k_1+k_2}{\gamma} < \Lambda_\gamma(1+\kappa)\right\}$,the equilibrium strategies are $r_1 = r_2 = \rho_\gamma(1+\kappa)$ and the equilibrium payoffs $M_i(G_i^*, G_j^*) = (1+\kappa)k_i$ for $i = 1, 2$. Also in this case, payoffs are continuous in (k_1, k_2).

Proposition 3. *Under assumptions 1-4, the Cournot outcome is the only SPNE of the two stage game.*

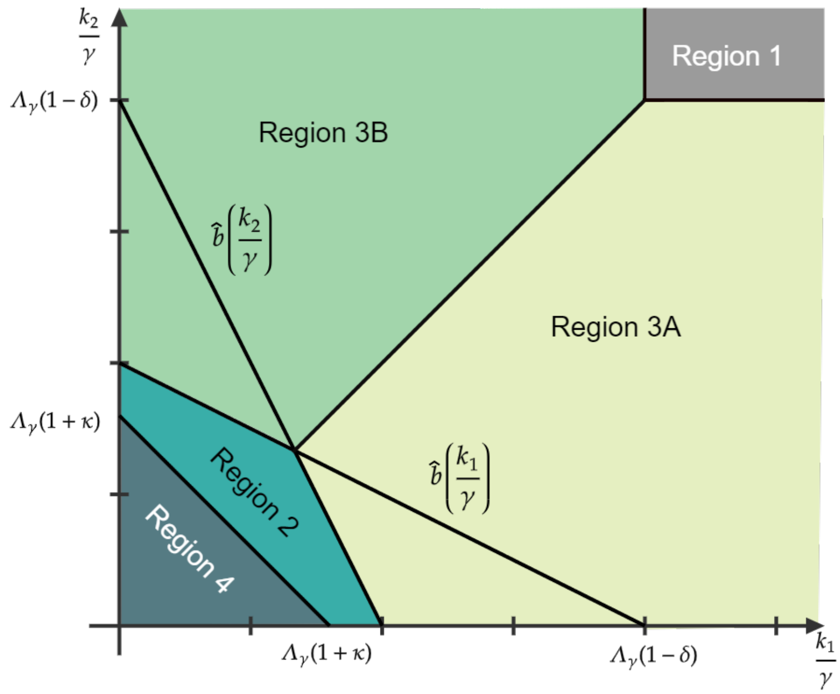

Figure 3: Equilibrium regions when $\rho_\gamma(1+\kappa) > r(\hat{b}(0))$

Proof. For a formal proof see Appendix. □

In Region 4, there cannot be a SPNE as $\pi_i(k_i, k_j) = \kappa k_i$, hence both banks have the incentive to increase capital. The intuition is that banks anticipate that they are going to expand capacity in the second stage. As raising capital in the short term is more costly, hence they are better-off by raising more capital in stage 1. In Regions 1 to 3A/B everything works as in the baseline game.

4.2 Paying dividends

In the baseline model I assume that the cost of paying dividends δ must be positive but sufficiently small. A positive cost is necessary to ensure equilibrium uniqueness, as it eliminates profitable deviations involving excess capital. Alternatively, it is possible to assume that in the first stage bank equity requires a premium $r_K > 0$, while dividend payments are costless. The essential modeling requirement is that adjusting capital in the second stage must be costly, to avoid banks deliberately overcapitalizing. However, this adjustment cost must remain low enough that banks

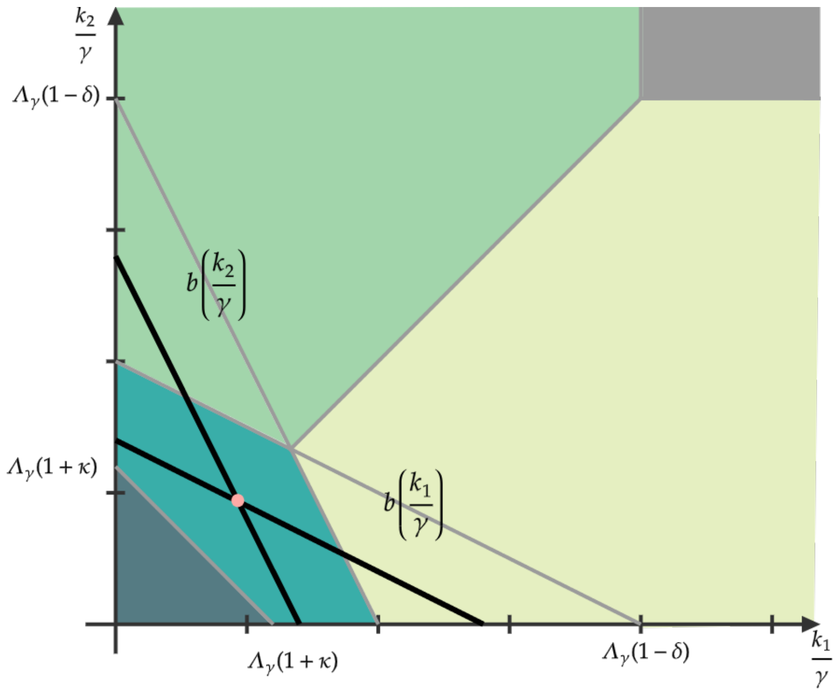

Figure 4: Subgame-perfect equilibrium with capital adjustment

do not hold excess capital relative to the regulatory constraint. If dividend payments are highly costly or outright prohibited ($\delta \to \infty$) the game needs simplifying assumptions in order to become tractable. When it is not possible to pay out dividends, the level of bank capital not only determines the lending capacity of the bank, but also the marginal funding costs. With fixed capital, a bank that issues fewer loans relative to its equity requires fewer deposits - or none at all if $l_i \leq k_i$. Moreover, as deposits are guaranteed by the government, the marginal cost of deposits depends on leverage (d_i/l_i): the more leveraged is a bank, the more likely it is to fail and hence the cheaper the deposits[10]. Schliephake and Kirstein (2013) show that the Cournot outcome is SPNE in a tractable model in which banks are not allowed to pay dividends, issue risk-free differentiated loans, and loan demand is linear. In contrast, when dividend payments are allowed at a small cost, leverage is fixed by the capital requirement, which implies that the marginal cost of deposits is constant across loan volumes.

[10]Recall: the cost of deposits is the risk-free rate times the probability of survival of the bank

δ	$\to \infty$	with no risk, bank loans as imperfect substitutes and linear demand Schliephake and Kirstein (2013)
	> 0 but close to zero*	this paper
	$= 0$	multiplicity of equilibria
κ	$\to \infty$	baseline game (for firms: Kreps and Scheinkman (1983))
	$< \infty$ but large**	this paper (for firms: Boccard and Wauthy (2000; 2004))
	$= 0$	standard Bertrand competition

Table 1: This table summarises how different values for the parameters of the model map to the literature. * δ is sufficiently small such that $(1-\delta)(k_i - \gamma l_i) \geq r_i l_i \int_{\tilde{x}}^{\check{x}} F(x|r_i)dx$, where $\check{x} = (r_i l_i - d_i)/r_i l_i$ is the default boundary when the bank does not distribute dividends; **κ is such that $\rho_\gamma(1+\kappa) \geq r(\hat{b}(0))$.

5 Conclusion

This paper shows that Cournot competition in the banking sector can be micro-founded through a two-stage game in which banks first choose capital levels and then compete à la Bertrand in the loan market, subject to capital requirements. For the Cournot outcome to arise as the unique subgame perfect equilibrium, two key conditions must be satisfied. First, the expected average residual cash flow must be increasing in the loan rate. This condition rules out settings where moral hazard effects are sufficiently strong to reverse the relationship between loan pricing and expected residual cashflow. Second, the cost of short-term recapitalization must be sufficiently high. This ensures that capital requirements act as effective capacity constraints. Together, these conditions ensure that capacity choices made in the first stage result into effective quantity choices in the second stage, thus recovering the logic of Cournot competition.

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Appendix

Proposition 1: assumption 3 guarantee that when $b(l_j)$ it is positive it must satisfy the following first order condition:

$$Z'(b(l_j)+l_j)b(l_j)+Z(b(l_j)+l_j)-\gamma=0$$

Given the equation above, the best response function has the following properties:

[a] $b(l_j)$ is strictly decreasing : by the implicit function theorem

$$\frac{db(l_j)}{dl_j}=-\frac{Z'(b(l_j)+l_j)(b(l_j)+1)}{2Z'(b(l_j)+l_j)+Z''(b(l_j)+l_j)b(l_j)}<0$$

[b] $b'(l_j)>-1$: increase l_j by ϵ and decrease $b(l_j)$ by the same amount. The FOC is equal to:

$$\begin{aligned}
&Z'(b(l_j)+l_j)(b(l_j)-\epsilon)+Z(b(l_j)+l_j)-\gamma\\
&=\underbrace{Z'(b(l_j)+l_j)b(l_j)+Z(b(l_j)+l_j)-\gamma}_{=0}-Z'(b(l_j)+l_j)\epsilon\\
&=-Z'(b(l_j)+l_j)\epsilon>0
\end{aligned}$$

Hence it must be that $b'(l_j)<-1$.

[c] If $l_j > b(l_j)$, then $b(b(l_j)) < l_j$: set $l_j = b(l_j)$ and $b(l_j) = l_j$ and evaluate the FOC:

$$Z'(b(l_j) + l_j)l_j + Z(b(l_j) + l_j) - \gamma$$
$$= Z'(b(l_j) + l_j)(l_j + b(l_j) - b(l_j)) + Z(b(l_j) + l_j) - \gamma$$
$$= Z'(b(l_j) + l_j)(l_j - b(l_j)) < 0$$

as $l_j > b(l_j)$ by hypothesis. This implies that the best response to $b(l_j)$ is smaller than l_j, i.e. $b(b(l_j)) < l_j$.

Property 3 ensures that $b(l_j)$ is a contraction and therefore has a unique fixed point.

Capital requirement binding in Stage 2. Given any triple (k_i, l_i, r_i) if $k_i > \gamma l_i$, bank i prefers to issue dividends and raise more deposits to make the capital requirement binding

$$\int_0^{\tilde{x}} ((1-x)r_i - (1-\gamma))\, dF(x|r_i)l_i + (1-\delta)(k_i - \gamma l_i) \geq \int_0^{\check{x}} \left((1-x)r_i l_i - (l_i - k_i)^+\right) dF(x|r_i)$$

where $\check{x} = \frac{r_i l_i - (l_i - k_i)^+}{r_i l_i}$ and $(y)^+ = \max\{0, y\}$. It is always possible to have a positive but arbitrarily close to zero δ that make the inequality above true. Re-arranging:

$$(1-\delta)(k_i - \gamma l_i) \geq r_i l_i \int_{\tilde{x}}^{\check{x}} F(x|r_i)dx$$

The RHS is *strictly* smaller than $r_i l_i(\check{x} - \tilde{x}) = (k_i - \gamma l_i)$, hence there exists a $\bar{\delta} > 0$, such that

$$(1-\bar{\delta})(k_i - \gamma l_i) = r_i l_i \int_{\tilde{x}}^{\check{x}} F(x|r_i)dx$$

Hence for any $\delta < \bar{\delta}$ the inequality holds.

Lemma 2: WLOG let $k_1 \geq k_2$. By hypothesis $\bar{r}_1 = \bar{r}_2 = \bar{r}$. Now suppose $\bar{r} > r^{FC}$. Bank 1 would have a profitable deviation to name a rate that is lower but arbitrarily close to $\bar{r}$

$$\lim_{\epsilon \downarrow 0} M_1(\bar{r}-\epsilon, G_j)-M_1(\bar{r}, G_j) = \alpha_2(\bar{r})(m(\bar{r})-\gamma(1-\delta))\left[\min\left(\frac{k_1}{\gamma}, L(\bar{r})\right) - \max\left(\frac{L(\bar{r})}{2}, L(\bar{r}) - \frac{k_j}{\gamma}\right)\right] > 0$$

Hence it must be that $\bar{r} \leq r^{FC}$. Now I prove the second part of the lemma. By lemma 1, it must be that $\underline{r}_i = \bar{r}_i = r^{FC}$ for all i. Then if bank i names a rate $r > r^{FC}$, its payoff must be equal to

$$(m(r) - \gamma(1-\delta))\left(L(r) - \frac{k_j}{\gamma}\right) + (1-\delta)k_i$$

Let $l_i = L(r) - \frac{k_j}{\gamma}$, then it is equivalent to maximise $\left(Z\left(l_i + \frac{k_j}{\gamma}\right) - \gamma(1-\delta)\right) l_i$. By definition it is maximised at $l_i = \hat{b}\left(\frac{k_j}{\gamma}\right)$, hence it must be that $\frac{k_i}{\gamma} \leq \hat{b}\left(\frac{k_j}{\gamma}\right)$, otherwise bank i would have a profitable deviation.

Lemma 3: WLOG assume $\bar{r}_1 > \bar{r}_2$. Before proceeding I must prove that $k_2 \geq \gamma\Lambda_\gamma(1-\delta)$ is incompatible with the hypotheses of the lemma. By hypothesis, $\min_i \frac{k_i}{\gamma} < \Lambda_\gamma(1-\delta)$, hence if $k_2 > \Lambda_\gamma(1-\delta)$, then $k_1 < \Lambda_\gamma(1-\delta)$. By naming $r \in \left(\rho_\gamma(1-\delta), r\left(\frac{k_1}{\gamma}\right)\right)$ bank 2 gets a payoff that is strictly higher than $(1-\delta)k_2$. Hence in equilibrium it must be that $\bar{r}_2 > \rho_\gamma(1-\delta)$. However if $\bar{r}_1 > \bar{r}_2$, it implies that when bank 1 names $\bar{r}_1$, the residual demand is always equal to zero and $M_1(\bar{r}_1, G_2) = (1-\delta)k_1$. However this cannot be part of an equilibrium as bank 1 has the profitable deviation to name any rate $r \in (\rho_\gamma(1-\delta), \bar{r}_2)$.

For (a) and (b): consider the function

$$\phi(r) = (m(r) - \gamma(1-\delta))\max\left(0, L(r) - \frac{k_j}{\gamma}\right)$$

By naming any rate $r \geq \bar{r}_1$, bank 1 gets $M_1(r, G_2) = \phi(r) + (1-\delta)k_1$, hence it must be that $\phi(r)$ is maximised at $\bar{r}_1$. In order to maximise $\phi(r)$, bank 1 should choose r such that $\frac{k_2}{\gamma} \leq L(r) \leq \frac{k_1+k_2}{\gamma}$. For any level of r there is a loan quantity, namely $l(r) = L(r) - \frac{k_2}{\gamma}$, such that

$\phi(r) = \left(Z\left(l(r) + \frac{k_2}{\gamma}\right) - \gamma(1-\delta)\right) l(r)$. Picking r to maximise $\phi(r)$ is equivalent to maximise:

$$\max_{l \in \left[0, \frac{k_1}{\gamma}\right]} \left(Z\left(l + \frac{k_2}{\gamma}\right) - \gamma(1-\delta)\right) l$$

By Assumption 3, this is maximised at $\min\left(\frac{k_1}{\gamma}, \hat{b}\left(\frac{k_2}{\gamma}\right)\right)$, if the capital requirement binds we are in the case of Lemma 2, which is incompatible with the hypothesis of this lemma, hence it must be that $\frac{k_1}{\gamma} > \hat{b}\left(\frac{k_2}{\gamma}\right)$ and $\bar{r}_1 = r\left(\hat{b}\left(\frac{k_2}{\gamma}\right) + \frac{k_2}{\gamma}\right)$.

(c) Suppose that $\underline{r}_i < \underline{r}_j$. By naming $\underline{r}_i$ bank i gets $M_i(\underline{r}_i, G_j) = (m(\underline{r}_i) - \gamma(1-\delta)) \min\left(\frac{k_i}{\gamma}, L(\underline{r}_i)\right) + (1-\delta)k_i$. Clearly if $L(\underline{r}_i) > \frac{k_i}{\gamma}$, then the payoff is strictly increasing in $r$and bank i would have the profitable deviation to name $\underline{r}_i + \epsilon$; if $L(\underline{r}_i) < \frac{k_i}{\gamma}$, it must be that $\underline{r}_i = r(\hat{b}(0))$ otherwise bank i would have a profitable deviation. However $\underline{r}_i \leq \bar{r}_1 = r\left(\hat{b}\left(\frac{k_2}{\gamma}\right) + \frac{k_2}{\gamma}\right) < r(\hat{b}(0))$, therefore it cannot be an equilibrium. It must be that $\underline{r}_1 = \underline{r}_2 = \underline{r}$. Note that $\underline{r} > r^{FC}$, otherwise for bank 1 would be profitable to deviate and name $r\left(\hat{b}\left(\frac{k_2}{\gamma}\right) + \frac{k_2}{\gamma}\right)$. Now I prove that $\alpha_i(\underline{r}) = 0$ for all $i \in \{1, 2\}$. Let i denote the bank that has (weakly) more capital[11] and bank j the bank that has (weakly) less capital. Suppose bank j names $\underline{r}$ with positive probability. Then bank i prefers to name a rate that is smaller but arbitrarily close to $\underline{r}$

$$\lim_{\epsilon \downarrow 0} M_i(\underline{r}-\epsilon, G_j) - M_i(\underline{r}, G_j) = \alpha_j(\underline{r})(m(\underline{r}) - \gamma(1-\delta)) \underbrace{\left(\min\left(\frac{k_i}{\gamma}, L(\underline{r})\right) - \max\left(\frac{L(\underline{r})}{2}, L(\underline{r}) - \frac{k_j}{\gamma}\right)\right)}_{>0}$$

Therefore it must be that $\alpha_j(\underline{r}) = 0$. Bank j names $\underline{r}$ with zero probability, however $\underline{r}$ is the infimum of the support, hence it must be that bank j names a rate that is arbitrarily close and above $\underline{r}$ but not exactly $\underline{r}$

$$M_j(\underline{r}, G_i) - \lim_{\epsilon \downarrow 0} M_j(\underline{r}+\epsilon, G_i) = \alpha_i(\underline{r})(m(\underline{r}) - \gamma(1-\delta)) \underbrace{\left(\min\left(\frac{k_j}{\gamma}, \frac{L(\underline{r})}{2}\right) - \max\left(0, L(\underline{r}) - \frac{k_i}{\gamma}\right)\right)}_{>0}$$

Hence it must be that $\alpha_i(\underline{r}) = 0$.

[11] I still have to prove that this is bank 1.

(d) $\underline{r} \leq \bar{r}_1 = r\left(\hat{b}\left(\frac{k_2}{\gamma}\right) + \frac{k_2}{\gamma}\right)$, implies $L(\underline{r}) \geq \hat{b}\left(\frac{k_2}{\gamma}\right) + \frac{k_2}{\gamma} > \frac{k_2}{\gamma}$. Hence the equilibrium payoff of bank 2 must be equal to $m(\underline{r})\frac{k_2}{\gamma}$. Now suppose $k_2 > k_1$, it must be that the equilibrium payoff of bank 1 is equal to $m(\underline{r})\frac{k_1}{\gamma}$. By part (a) we also know that the equilibrium payoff of bank 1 is equal to $P(k_2) + (1-\delta)k_1$, which implies that $m(\underline{r}) = P(k_2)\frac{\gamma}{k_1} + \gamma(1-\delta)$. The payoff of bank 2 can be re-written as $P(k_2)\frac{k_2}{k_1} + (1-\delta)k_2$. If bank 2 names $r = r\left(\hat{b}\left(\frac{k_1}{\gamma}\right) + \frac{k_1}{\gamma}\right) > \bar{r}_1$, it gets $P(k_1) + (1-\delta)k_2$. Therefore if $P(k_1) + (1-\delta)k_2 > P(k_2)\frac{k_2}{k_1} + (1-\delta)k_2$, which can be re-written as $k_1 P(k_1) > k_2 P(k_2)$, bank 2 has a profitable deviation and $k_2 > k_1$ contradicts the hypotheses of the lemma. Define the function $\psi(k) = kP(k) = k\left(Z\left(\hat{b}\left(\frac{k}{\gamma}\right) + \frac{k}{\gamma}\right) - \gamma(1-\delta)\right)\hat{b}\left(\frac{k}{\gamma}\right)$ and compute the derivative[12]

$$\psi'(k) = \left(Z\left(\hat{b}\left(\frac{k}{\gamma}\right) + \frac{k}{\gamma}\right) - \gamma(1-\delta)\right)\left(\hat{b}\left(\frac{k}{\gamma}\right) - \frac{k}{\gamma}\right)$$

Hence:

$$\psi(k_2) - \psi(k_1) = \int_{k_1}^{k_2} \left(Z\left(\hat{b}\left(\frac{k}{\gamma}\right) + \frac{k}{\gamma}\right) - \gamma(1-\delta)\right)\left(\hat{b}\left(\frac{k}{\gamma}\right) - \frac{k}{\gamma}\right) dk$$

Bank 2 has a profitable deviation if the expression above is negative. As $\hat{b}(\cdot)$ is decreasing, this integral is more likely to be positive when k_2 is as small as possible. From (b) we know that $k_2 > \gamma\hat{b}^{-1}\left(\frac{k_1}{\gamma}\right)$, hence:

$$\psi(k_2) - \psi(k_1) < \psi\left(\gamma\hat{b}^{-1}\left(\frac{k_1}{\gamma}\right)\right) - \psi(k_1)$$

$$= k_1\left(\left(Z\left(\hat{b}^{-1}\left(\frac{k_1}{\gamma}\right) + \frac{k_1}{\gamma}\right) - \gamma(1-\delta)\right)\hat{b}^{-1}\left(\frac{k_1}{\gamma}\right) - \left(Z\left(\hat{b}\left(\frac{k_1}{\gamma}\right) + \frac{k_1}{\gamma}\right) - \gamma(1-\delta)\right)\hat{b}\left(\frac{k_1}{\gamma}\right)\right) \leq 0$$

The term above is negative because by definition $\left(Z\left(l + \frac{k_1}{\gamma}\right) - \gamma(1-\delta)\right) l$ is maximised at $\hat{b}\left(\frac{k_1}{\gamma}\right)$. Therefore it must be that $k_1 \geq k_2$.

Finally (e): from part (a), (c) and (d) we know that

$$m(\underline{r})\frac{k_2}{\gamma} \leq (m(\underline{r}) - \gamma(1-\delta))\min\left(\frac{k_1}{\gamma}, L(\underline{r})\right) + (1-\delta)k_1 = P(k_2) + (1-\delta)k_1$$

[12]See KS for details on how to derive the expression of $\psi'(k)$.

Hence in equilibrium bank 2 can get at most $P(k_2)+(1-\delta)k_1$. We also know that $P(k_2)+(1-\delta)k_1 = (m(\underline{r}) - \gamma(1-\delta)) \min\left(\frac{k_1}{\gamma}, L(\underline{r})\right) + (1-\delta)k_1 \leq m(\underline{r})\frac{k_1}{\gamma}$, which implies $m(\underline{r}) \geq P(k_2)\frac{\gamma}{k_1} + \gamma$. Hence $m(\underline{r})\frac{k_2}{\gamma} \geq P(k_2)\frac{k_2}{k_1} + (1-\delta)k_2$.

Proposition 2: WLOG let $\bar{k}_1 \geq \bar{k}_2$.

- (Step 1) In equilibrium it must be that $\bar{k}_1/\gamma \geq b(\underline{k}_2/\gamma)$. Suppose not: $\bar{k}_1/\gamma < b(\underline{k}_2/\gamma)$, which implies that $b(\bar{k}_1/\gamma) > b(b(\underline{k}_2/\gamma))$. As $\underline{k}_2 \leq \bar{k}_1$, it must be that $\underline{k}_2/\gamma < b(\underline{k}_2/\gamma)$. Then it must be that $b(b(\underline{k}_2/\gamma)) > \underline{k}_2/\gamma$[13]. By transitivity, $\underline{k}_2/\gamma < b(\bar{k}_1/\gamma)$. Therefore when bank 2 raises $\underline{k}_2$ is for sure in Region 2:

$$\pi(\underline{k}_2, \mu_1) = \int_{\underline{k}_1}^{\bar{k}_1} \left(Z\left(\frac{k_1 + \underline{k}_2}{\gamma}\right) - \gamma\right)\frac{\underline{k}_2}{\gamma} d\mu_1(k_1)$$

  The profits are strictly increasing in $\underline{k}_2$ as $\underline{k}_2 < b(k_1/\gamma)$ for all $k_1 \in [\underline{k}_1, \bar{k}_1]$, hence bank 2 can profitably deviate and name $\underline{k}_2 + \epsilon$. Therefore it must be that $\bar{k}_1/\gamma \geq b(\underline{k}_2/\gamma)$.

- (Step 2) $\bar{k}_1/\gamma \leq b(\bar{k}_2/\gamma)$. Suppose not: $\bar{k}_1/\gamma > b(\bar{k}_2/\gamma)$. Then when bank 1 raises $\bar{k}_1$ is either in region 2 or in region 3A:

$$\pi(\bar{k}_1, \mu_2) = \int_{\underline{k}_2}^{\xi(\bar{k}_1)} \left(Z\left(\frac{\bar{k}_1 + k_2}{\gamma}\right) - \gamma\right)\frac{\bar{k}_1}{\gamma} d\mu_2(k_2) + \int_{\xi(\bar{k}_1)}^{\bar{k}_2} (P(k_2) - \delta k_1) d\mu(k_2)$$

  where $\xi(k) = \frac{1}{\gamma}\hat{b}^{-1}\left(\frac{k}{\gamma}\right)$. The profits are strictly decreasing in $\bar{k}_1$, in particular the first term is decreasing because $\bar{k}_1/\gamma \geq b(\underline{k}_2/\gamma) \geq b(k_2/\gamma)$ for all k_2 in the support. Therefore bank 1 would have the profitable deviation to raise $\bar{k}_1 - \epsilon$. Hence it must be that $\bar{k}_1/\gamma \leq b(\bar{k}_2/\gamma)$.

- (Step 3) The previous step imply that $\bar{k}_1/\gamma = b(\bar{k}_2/\gamma) = b(\underline{k}_2/\gamma)$, which implies that bank 2's equilibrium strategy is a pure strategy k_2; bank 1 best response to the pure strategy k_2 is $b\left(\frac{k_2}{\gamma}\right)$. In turn, bank 2 must best respond to that hence $\frac{k_2}{\gamma} = b\left(\frac{k_1}{\gamma}\right) = b\left(b\left(\frac{k_2}{\gamma}\right)\right)$. The unique solution is $k_1 = k_2 = \gamma l^C$, and in the second stage banks name $r^{FC} = r(2l^C) = r^C$ with probability one.

[13]See proof of Proposition 1 for the properties of $b(\cdot)$

Lemma 4: Recall that demand is rationed according to the efficient rule as in the baseline game. In particular, when banks name the same rate, they can raise more capital only if they cannot serve the entire demand collectively, i.e. $r_1 = r_2 = r$ and $L(r) > \frac{k_1+k_2}{\gamma}$.

(Step 1) $\underline{r}_i \geq \rho_\gamma(1+\kappa)$ for all $i \in \{1,2\}$. Suppose not and let $\underline{r}_i < \rho_\gamma(1+\kappa)$ for some i. By hypothesis $\rho_\gamma(1+\kappa) < r^{FC}$, hence when bank i names $\underline{r}_i$it gets:

$$M_i(\underline{r}_i, G_j) = m(\underline{r}_i)\frac{k_i}{\gamma}$$

Bank i is operating at full capacity and does not find it profitable to raise more capital as $\underline{r}_i < \rho_\gamma(1+\kappa)$. However $m(\cdot)$ is an increasing function, hence bank i would be better off by naming $\underline{r}_i + \epsilon$. Hence, this cannot be an equilibrium and it must be that $\underline{r}_i \geq \rho_\gamma(1+\kappa)$ for all $i \in \{1,2\}$.

(Step 2) $\bar{r}_i \leq \rho_\gamma(1+\kappa)$ for all $i \in \{1,2\}$. Suppose not and let $\bar{r}_i > \rho_\gamma(1+\kappa)$. WLOG divide the proof into two cases:

- $\bar{r}_i > \bar{r}_j$ or $\bar{r}_i = \bar{r}_j$ and $\alpha_j(\bar{r}_j) = 0$. By naming $\bar{r}_i$ bank i gets $M_i(\bar{r}_i, G_j) = (1-\delta)k_i$. As $\underline{r}_j \geq \rho_\gamma(1+\kappa)$, bank j will always find it profitable to raise more capital and supply the entire market, therefore bank i has no residual demand to serve. Bank i is better off by naming $\rho_\gamma(1+\kappa)$ and getting $(1+\kappa)k_i$.

- $\bar{r}_i = \bar{r}_j = \bar{r}$ and $\alpha_i(\bar{r}) > 0$ for all $i = 1,2$. If $\rho_\gamma(1+\kappa) < \bar{r} < r^{FC}$:

$$M_i(\bar{r}, G_j) = \alpha_j(\bar{r})\left[\left(m(\bar{r}) - \gamma\eta\left(\frac{L(\bar{r})}{2}\right)\right)\frac{L(\bar{r})}{2} + \eta\left(\frac{L(\bar{r})}{2}\right)k_i\right] + (1-\alpha_j(\bar{r}))(1-\delta)k_i$$

  If bank i instead names $\bar{r} - \epsilon$:

$$\lim_{\epsilon \downarrow 0} M_i(\bar{r}-\epsilon) = \alpha_j(\bar{r})\left[\left(m(\bar{r}) - \gamma\eta(L(\bar{r}))\right)L(\bar{r}) + \eta(L(\bar{r}))k_i\right] + (1-\alpha_j(\bar{r}))(1-\delta)k_i$$

Hence:

$$\lim_{\epsilon \downarrow 0} M_i(\bar{r}-\epsilon) - M_i(\bar{r}, G_j) =$$
$$\alpha_j(\bar{r}) \left[(m(\bar{r}) - \gamma\eta(L(\bar{r}))) \frac{L(\bar{r})}{2} + \left(\eta\left(\frac{L(\bar{r})}{2}\right) - \eta(L(\bar{r})) \right) \left(\gamma \frac{L(\bar{r})}{2} - k_i \right) \right] > 0$$

where

$$\eta(l) = \begin{cases} (1-\delta) & \text{if } l \leq \frac{k_i}{\gamma} \\ (1+\kappa) & \text{if } l > \frac{k_i}{\gamma} \end{cases}$$

The first term is larger than zero because $m(\bar{r}) - \gamma\eta(L(\bar{r})) \geq m(\bar{r}) - \gamma(1+\kappa) > 0$ as $\bar{r} > \rho_\gamma(1+\kappa)$. The second term is non-negative because either $\eta\left(\frac{L(\bar{r})}{2}\right) = \eta(L(\bar{r}))$; or $\eta(L(\bar{r})) = 1+\kappa$ and $\eta\left(\frac{L(\bar{r})}{2}\right) = 1-\delta$, which implies $\frac{k_i}{\gamma} > \frac{L(\bar{r})}{2}$. Finally if $\bar{r} \geq r^{FC}$ both banks have incentives to undercut as in every standard Bertrand game.

Therefore it must be that $\bar{r}_i \leq \rho_\gamma(1+\kappa)$ for all $i \in \{1,2\}$.

(Step 3) $\rho_\gamma(1+\kappa) \leq \underline{r}_i \leq \bar{r}_i \leq \rho_\gamma(1+\kappa)$, then that $r_i = \rho_\gamma(1+\kappa)$ with probability 1 for all $i \in \{1,2\}$ is the only possible equilibrium (existence is guaranteed by Dasgupta and Maskin (1986), however is immediate to show that given the opponent's strategy there are no profitable deviations).

Lemma 5: Start with Lemma 1. The proof is the same of the baseline model as we are working under the hypothesis that $\frac{k_1+k_2}{\gamma} \geq \Lambda_\gamma(1+\kappa)$, which implies $r^{FC} \leq \rho_\gamma(1+\kappa)$. Lemma 2 follows exactly.

the possibility of expanding does not alter the first part of the proof. Hence, if $\bar{r}_1 = \bar{r}_2 = \bar{r}$ and $\alpha_i(\bar{r}) > 0$ for all $i \in \{1,2\}$ it must be that $\bar{r} = \underline{r} = r^{FC}$. Banks do not have incentives to undercut the opponent, so we must check that there are no incentives to charge a higher rate. Given the opponent strategy $\alpha_j(r^{FC}) = 1$, bank i maximises its payoff:

$$\max_r G_j(\rho_\gamma(1+\kappa)) \left((m(r) - \gamma(1-\delta)) \min\left(\frac{k_i}{\gamma}, L(r) - \frac{k_j}{\gamma} \right) \right) + (1-\delta)k_i$$

the payoff is multiplied by $G_j(\rho_\gamma(1+\kappa))$ because if the opponent charges a rate higher than

$\rho_\gamma(1+\kappa)$ it will serve the entire market. However $\alpha_j(r^{FC}) = 1$ and $\frac{k_1+k_2}{\gamma} \geq \Lambda_\gamma(1+\kappa)$, imply $G_j(\rho_\gamma(1+\kappa)) = 1$. The rest of the proof follows.

Finally. Lemma 3. WLOG of generality let $\bar{r}_1 > \bar{r}_2$. Before proceeding I must prove that $k_2 \geq \gamma\Lambda(\gamma(1-\delta))$ is incompatible with the hypotheses of the lemma. By hypothesis, $\min_i \frac{k_i}{\gamma} < \Lambda_\gamma(1-\delta)$, hence if $k_2 > \Lambda_\gamma(1-\delta)$, then $k_1 < \Lambda_\gamma(1-\delta)$. By naming $r \in \left(\rho_\gamma(1-\delta), \min\left(r\left(\frac{k_1}{\gamma}\right), \rho_\gamma(1+\kappa)\right)\right)$ bank 2 gets a payoff that is strictly higher than $(1-\delta)k_2$. Hence in equilibrium it must be that $\bar{r}_2 > \rho_\gamma(1-\delta)$. However if $\bar{r}_1 > \bar{r}_2$, it implies that when bank 1 names $\bar{r}_1$, the residual demand is always equal to zero and $M_1(\bar{r}_1, G_2) = (1-\delta)k_1$. However this cannot be part of an equilibrium as bank 1 has the profitable deviation to name any rate $r \in (\rho_\gamma(1-\delta), \bar{r}_2)$. By naming any rate $r \geq \bar{r}_1$, bank 1 gets $M_1(r, G_2) = G_2(\rho_\gamma(1+\kappa))\phi(r) + (1-\delta)k_1$, hence it must be that $\phi(r)$ is maximised at $\bar{r}_1$. The optimisation problem is equivalent to the one of the baseline model, hence $\bar{r}_1 = r\left(\hat{b}\left(\frac{k_2}{\gamma}\right) + \frac{k_2}{\gamma}\right)$. By Assumption 4, $r(\hat{b}(0)) < \rho_\gamma(1+\kappa)$, hence $r\left(\hat{b}\left(\frac{k_2}{\gamma}\right) + \frac{k_2}{\gamma}\right) < \rho_\gamma(1+\kappa)$ and $\bar{r}_2 \leq \bar{r}_1 < \rho_\gamma(1+\kappa)$. This implies that $G_2(\rho_\gamma(1+\kappa)) = 1$ and the rest of the proof follows.

Proposition 3. WLOG let $\bar{k}_1 \geq \bar{k}_2$. The proof follows the steps of Proposition 2, but we need to add a preliminary step:

- (Step 1) $\underline{k}_1/\gamma \geq \Lambda_\gamma(1+\kappa) - \bar{k}_2/\gamma$ for all $i \in \{1, 2\}$. Suppose not and $\underline{k}_1/\gamma < \Lambda_\gamma(1+\kappa) - \bar{k}_2/\gamma$. When bank i raises $\underline{k}_1$, it is for sure in Region 4 and gets profits equal to

$$\pi_1(\underline{k}_1, \mu_j) = (1+\kappa)\underline{k}_1 - \underline{k}_1 = \kappa\underline{k}_1$$

  this is clearly increasing in $\underline{k}_1$, hence bank i would have the profitable deviation to name $\underline{k}_1 + \epsilon$. This inequality implies that $\bar{k}_1/\gamma \geq \underline{k}_1/\gamma \geq \Lambda_\gamma(1+\kappa) - \underline{k}_2/\gamma$.

- (Step 2) In equilibrium it must be that $\bar{k}_1/\gamma \geq b(\underline{k}_2/\gamma)$. Suppose not: $\bar{k}_1/\gamma < b(\underline{k}_2/\gamma)$, which implies $\underline{k}_2/\gamma < b(\bar{k}_1/\gamma)$. Therefore when bank 2 raises $\underline{k}_2$ is either in Region 4 or in Region 2:

$$\pi(\underline{k}_2, \mu_1) = \int_{\underline{k}_1}^{\beta(\underline{k}_2,\kappa)} \kappa\underline{k}_2 d\mu_1(k_1) + \int_{\beta(\underline{k}_2,\kappa)}^{\bar{k}_1} \left(Z\left(\frac{k_1+\underline{k}_2}{\gamma}\right) - \gamma\right)\frac{\underline{k}_2}{\gamma} d\mu_1(k_1)$$

  where $\beta(\underline{k}_2, \kappa) = \gamma\Lambda_\gamma(1+\kappa) - \underline{k}_2$. Profits are strictly increasing in $\underline{k}_2$ for all $k_1 \in [\underline{k}_1, \bar{k}_1]$, hence bank 2 can profitably deviate and name $\underline{k}_2 + \epsilon$. Therefore, putting together step 0 and step 1 it must be that $\bar{k}_1/\gamma \geq \max\{b(\underline{k}_2/\gamma), \Lambda_\gamma(1+\kappa) - \underline{k}_2/\gamma\}$.

- (Step 3) $\bar{k}_1/\gamma \leq b(\bar{k}_2/\gamma)$. Suppose not: $\bar{k}_1/\gamma > b(\bar{k}_2/\gamma)$ and $\bar{k}_1/\gamma \geq \Lambda_\gamma(1+\kappa) - \underline{k}_2/\gamma$. Then when bank 1 raises $\bar{k}_1$ is either in region 2 or in region 3A:

$$\pi(\bar{k}_1, \mu_2) = \int_{\underline{k}_2}^{\xi(\bar{k}_1)} \left(Z\left(\frac{\bar{k}_1 + k_2}{\gamma}\right) - \gamma \right) \frac{\bar{k}_1}{\gamma} d\mu_2(k_2) + \int_{\xi(\bar{k}_1)}^{\bar{k}_2} (P(k_2) - \delta k_1) d\mu(k_2)$$

where $\xi(k) = \frac{1}{\gamma}\hat{b}^{-1}\left(\frac{k}{\gamma}\right)$. The profits are strictly decreasing in $\bar{k}_1$, in particular the first term is decreasing because $\bar{k}_1/\gamma \geq b(\underline{k}_2/\gamma) \geq b(k_2/\gamma)$ for all k_2 in the support. Therefore bank 1 would have the profitable deviation to raise $\bar{k}_1 - \epsilon$.

- (Step 4) Putting together the previous steps it must be that $\bar{k}_1/\gamma \leq b(\bar{k}_2/\gamma)$ and $\bar{k}_1 \geq \max\{b(\underline{k}_2/\gamma), \Lambda_\gamma(1+\kappa) - \underline{k}_2/\gamma\}$, which implies that bank 2 is playing a pure strategy k_2. Bank 1, must best respond to the pure strategy k_2, hence Bank 1 will solve:

$$\max_{k_1 \geq 0} \pi_1(k_1, k_2)$$

where

$$\pi_1(k_1, k_2) = \begin{cases} \kappa k_1 & \text{if } \frac{k_1}{\gamma} \leq \Lambda_\gamma(1+\kappa) - \frac{k_2}{\gamma} \\ \left(Z\left(\frac{k_1+k_2}{\gamma}\right) - \gamma\right)\frac{k_1}{\gamma} & \text{if } \Lambda_\gamma(1+\kappa) < \frac{k_1}{\gamma} \leq \hat{b}\left(\frac{k_2}{\gamma}\right) \\ \tilde{P}(k_2) - k_1 & \text{if } \frac{k_1}{\gamma} > \hat{b}\left(\frac{k_2}{\gamma}\right) \end{cases}$$

Hence $\frac{k_1^*}{\gamma} = \max\left(b\left(\frac{k_2^*}{\gamma}\right), \Lambda_\gamma(1+\kappa) - \frac{k_2^*}{\gamma}\right)$. At the same time bank 2 will have to best respond to that and similarly $\frac{k_2^*}{\gamma} = \max\left(b\left(\frac{k_1^*}{\gamma}\right), \Lambda_\gamma(1+\kappa) - \frac{k_1^*}{\gamma}\right)$. As δ is arbitralily small $r^C < r(\hat{b}(0)) < \rho_\gamma(1+\kappa)$. Therefore $b\left(\frac{k_i^*}{\gamma}\right) > \Lambda_\gamma(1+\kappa) - \frac{k_i^*}{\gamma}$, which imply $\frac{k_i^*}{\gamma} = b\left(b\left(\frac{k_i^*}{\gamma}\right)\right) = l^C$.