Sets of Indistinguishable Models for Robust Optimisation

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Abstract

Models can be wrong and recognising their limitations is important in financial and economic decision making under uncertainty. Finding the explicit specification of the uncertainty set has been difficult so far. We develop a method that provides a plausible set of models to use in robust decision making. The choice of the specific size of the uncertainty region is what we will focus on. We use the Neyman-Pearson Lemma to characterise a set of models that cannot be distinguished statistically from a baseline model. The set of indistinguishable models can explicitly be obtained for a given probability for the Type I and II error.

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1 Introduction

Models can be wrong and recognising their limitations is important in financial and economic decision making. In asset pricing, model uncertainty has implications for the valuation of derivatives and long-dated contracts. In such situations the value of an asset is not unique but falls in a range, requiring extensions of standard pricing methods. We develop a method that provides a plausible set of models surrounding the baseline model to use in robust decision making. Making financial decisions robust calls for applying worst case scenarios. Therefore even if an optimisation problem is considered under the allowance of model uncertainty, still the question remains how to determine the set of alternative models. The choice of the uncertainty set is what we will focus on, contrary to most literature that solves a robust control problem for a given set of alternatives. The shared intuition is that an agent is concerned about model misspecification, consequently the robust control decisions are based on the worst case scenario. This is obtained by evaluating the control function for every plausible model. Intuitively, these models come from a set surrounding the baseline model. Our objective is to determine explicitly the set that incorporates all plausible alternative models using statistical testing theory. We use the Neyman-Pearson Lemma and impose a Type I and II error to construct the set of indistinguishable models.

The outline is as follows: in Section 1 we continue the introductory section with a literature review whereafter we describe the intuition underlying our contribution. In Section 2 we introduce the model and derive the indistinguishable set of models for deterministic alternatives that serves as an intuitive illustration. Then we show an example of a stochastic alternative that fundamentally changes the probability distribution. In Section 3 our main contribution is presented where we create the set of indistinguishable models for stochastic time-consistent alternatives ex ante. The link to a numerical bound on several divergences is provided in Section 4. Before the conclusion in Section 6, we apply the quantification of the set of indistinguishable models to the robust Merton problem.

1.1 Literature Review

We would like to highlight the difference between risk and uncertainty. First of all, in this paper we use ambiguity and uncertainty interchangeably. Risk stands for the uncertainty that a certain event might happen while knowing the probability, whereas for uncertainty one does not know the probability. The Ellsberg paradox (Ellsberg, 1961) states that people prefer known risk over unknown risk. This phenomenon is called ambiguity aversion. People differ among their risk attitudes. If one is indifferent between the bet and the certainty equivalent, he is risk-neutral. If the risk premium, the difference between the expected payoff and the certainty equivalent, is positive, he is risk-loving. Lastly, a risk-averse person accepts a certain payment lower than the expected payoff of the gamble. Since we focus on the quantification of model ambiguity we indirectly assume a risk attitude that is linear, i.e. a risk-neutral agent, to be able to concentrate on one uncertainty in specific. The numerical robust Merton problem we present as application deals with several different risk-aversion parameters.

Ben-Tal et al. (2013) focus on ϕ-divergences in robust optimisation. General optimisation problems, examples ranging from finance to operations research, are solved robustly over an uncertainty region $\mathscr{U}$. The uncertainty region is identified by the confidence set using a specific ϕ-divergence function. The described divergence functions are Kullback-Leibler, Burg

Entropy, J-divergence, χ^2-distance, Hellinger distance, Variation distance and Cressie-Read. The method proposed by Ben-Tal et al. (2013) is merely a procedure how a set could be created with multiple choices that have to be made rather than that they specify the characterisation of the set of indistinguishable models. The choices include the selection of the divergence function and the bounds on these. In section 4 we will quantify the bounds on these different measures.

Breuer and Csiszár (2013) base stress tests on plausible sets, that do not consider scenarios that are too implausible and do incorporate the dangerous ones. These scenarios are obtained by considering mixed scenarios, known as risk factor distributions. The Kullback-Leibler divergence from these distributions has to fall within an uncertainty ball, where the radius is assumed to coincide with the 1% TVaR or the bound can be calculated based on historical data. However, for all divergences the critical value that distinguishes a plausible from an implausible model is not quantified. Merely the optimisation problem that incorporates an ambiguity measure is the focus of these papers.

Hansen and Sargent (2008) (hereafter H&S) motivate their approach of uncertainty to robust optimisation by choosing models surrounding the baseline model with bounded entropy (i.e. with bounded Kullback-Leibler divergence). However, when implementing their method they make a subtle switch: they replace the endogenous Lagrange multiplier of the entropy bound by a fixed entropy penalty. On the one hand this leads to a fundamentally different class of optimisation problems on finite horizons. On the other hand, by "giving up" the explicit entropy constraint H&S obtain a time-consistent operator, which is a desirable property for dynamic optimisation problems. However, the explicit connection with the characteristics of the uncertainty set is lost, and the determination how to choose the fixed penalty parameter remains an uninvestigated aspect.

Specifically, H&S start with an optimisation problem, then they pick a Lagrange multiplier and calculate the worst case path which depends on the multiplier and the specific optimisation problem. Next they calculate the detection error probability for the specific multiplier and the associated worst case path. If the probability of the average of the two incorrect rejections is too high, then the worst case choice from Mother Nature is too extreme, with other words too far from the approximating model. Therefore it is deemed unlikely that these two models cannot be distinguished from each other. Hence this multiplier is rejected and will not belong to the set of alternatives. By this procedure one plausible worst-case based multiplier is selected rather than a set. The main difference with our approach is the order of the procedure. H&S start with an explicit optimisation problem whereas we focus on the creation of the set of plausible alternative models that can be applied to and is independent of the particular choice of the optimisation problem. In Section 5 we solve the robust Merton problem both from the constraint and the penalty approach for an risk-averse (power utility) agent who is uncertain about the underlying model.

Hansen et al. (2011) introduce the concept of Model Confidence Set (MCS). This method, or actually algorithm, is a sequential method that starts with a collection of possible models and ends with a subset of these that contain the best models with a given level of confidence. Best is in terms of a *chosen* test statistic. The MCS method is based on an equivalence test and an elimination rule. If the equivalence test indicates that the set of models at hand are not equivalent, in our terms are not indistinguishable, then at least one of the models performs worse than the others and should be removed. The elimination rule performs the execution step repeatedly until the set consists of only those models that are equivalently good for a

given confidence level.In comparison with the indistinguishable method discussed, the MCS method starts with a collection of competing models. Rather than introducing a discrete number of both plausible and implausible models, the indistinguishable method is exempted from this initial input.

It is possible to simply construct a confidence interval based on the estimated parameter around the baseline model. However, the limitation is that only a very specific class of models are considered. Namely only those models with parameters that are supposed to be constant over the observation period. The alternative models are models with other constant parameter values. In this paper we would like to consider alternative models with different structures explicitly, specifically time-dependent and stochastic parameters, such that in this manner the set of alternative models incorporates as much classes of models as possible. Moreover, the confidence interval approach performs the test *ex post* and imposes a Type I error to construct the set of alternative models. We emphasis on the construction of a set of models *ex ante* by imposing both the probability on the Type I and Type II errors.

1.2 Intuition

Our goal is to obtain the explicit characterisation of the set of indistinguishable models based on statistical testing theory. We use the Neyman-Pearson Lemma (Neyman and Pearson, 1933) to characterise a set of models that cannot be distinguished statistically from a baseline model. Therefore the set of indistinguishable models can explicitly be obtained *ex ante*, for a given Type I and II error.
Suppose we have an optimisation problem which extends over the time interval $[0,T]$. With other words, suppose there would be T years of extra information available. Our baseline model is specified on a filtered probability space $\left(\Omega,\{\mathscr{F}_t\}_{0\le t\le T},\mathbb{P}\right)$, where $\mathbb{P}$ denotes the probability measure that corresponds to our baseline model. The idea is that we define the plausible set of alternative models, as those models that cannot be distinguished statistically from the baseline model if one would take the observations accumulated over $[0,T]$ into consideration. In other words, we want to exclude ex ante (at time 0) those models which could possibly be rejected by a statistical test procedure at time T with a reasonable level of confidence. We want to use the most powerful tests possible, which are likelihood ratio tests, as stated by the Neyman-Pearson Lemma.

2 Statistically Indistinguishable Models

Let us make our model set-up more specific. We assume that we are considering models that can be described by diffusion processes. This means that we are considering stochastic processes X that are described by stochastic differential equations of the form

$$dX(t) = \mu(t,\omega)\,dt + \sigma(t,\omega)\,dW(t) \tag{2.1}$$

For the specification of possible alternative models, we consider Brownian Motion with a (stochastic) drift process $dW(t) + \lambda(t,\omega)\,dt$. Such an alternative model specification of the Brownian Motion can be captured as a change in probability measure from $\mathbb{P}$ to a new probability measure $\mathbb{A}$. With slight abuse of notation we will denote both the alternative model and the alternative probability measure by $\mathbb{A}$.

The likelihood ratio $H_0 : \mathbb{P}$ versus $H_A : \mathbb{A}$ (based on the information over the interval $[0, T]$) is given by the value of the Radon-Nikodym derivative $R(T)$ at time T. In our diffusion model setting, we know from Girsanov's Theorem that the likelihood ratio (i.e. the inverse of the Radon-Nikodym derivative) is a stochastic process $R(t) = \frac{d\mathbb{A}}{d\mathbb{P}}$ which is given by the stochastic differential equation

$$dR(t) = \lambda(t,\omega) R(t)\, dW^{\mathbb{P}}(t) \tag{2.2}$$

The superscript $\mathbb{P}$ denotes the probability measure we are considering. The solution to the stochastic differential equation (2.2) can be represented as

$$R(T) = \exp\left\{-\frac{1}{2}\int_0^T \lambda(t,\omega)^2\, dt + \int_0^T \lambda(t,\omega)\, dW^{\mathbb{P}}(t,\omega)\right\} \tag{2.3}$$

Hence, the value at time T of the likelihood ratio $R(T,\omega)$ is completely determined by the realisation ω of a path of the Brownian Motion $\left\{W^{\mathbb{P}}(t,\omega)\right\}_{0\le t\le T}$ and the specification $\{\lambda(t,\omega)\}_{0\le t\le T}$ of the alternative model $\mathbb{A}$ along this path.

Based on the realised path of the Brownian Model at time T we could test if model $\mathbb{P}$ should be rejected in favour of model $\mathbb{A}$. We are testing two simple hypotheses, and the Neyman-Pearson Lemma tells us that the most powerful test is a likelihood ratio test. The form of the optimal test procedure is that we reject model $\mathbb{P}$ if $R(T)$ is larger than the critical value γ. The critical value γ is determined by the equation

$$\mathbb{P}[R(T) \ge \gamma] = \alpha \tag{2.4}$$

We set the critical value γ such that probability of incorrectly rejecting model $\mathbb{P}$ when model $\mathbb{P}$ is the true model is equal to α. This is known as the Type I error. The probability α is the significance level of the test, and is typically set at 0.05. [1]

We should also be worried about the Type II error: this is the error of incorrectly rejecting model $\mathbb{A}$ when model $\mathbb{A}$ is the true model. This probability is typically denoted by β and can be computed as

$$\mathbb{A}[R(T) < \gamma] = \beta \tag{2.5}$$

The complement of the Type II error is the probability of accepting model $\mathbb{A}$ when model $\mathbb{A}$ is the true model. This is known as the *power* of the statistical test. The power can be computed as

$$\mathbb{A}[R(T) \ge \gamma] = 1 - \beta \tag{2.6}$$

A typical value for β is 0.20, leading to a statistical power of 0.80.

The Type I and Type II probabilities can be computed ex ante at time 0 for a given alternative model $\mathbb{A}$. The model selection procedure we propose is based on the Type II error of the likelihood ratio test. The intuition is a follows. For small values of $\lambda(t,\omega)$ we will have a model $\mathbb{A}$ that is "close" to the baseline model $\mathbb{P}$. This closeness can be identified by the fact that the likelihood ratio $R(T)$ will be a random variable with a probability distribution tightly concentrated around the value $R(T) = 1$. Even though we can define a critical value γ for any model $\mathbb{A}$, for models that are "close" there will be almost no difference between the $\mathbb{P}$-probability and the $\mathbb{A}$-probability of the event $R(T) \ge \gamma$. Hence, the power of the statistical

[1] Inclusion/exclusion of the equality sign in the expression $R(T) \ge \gamma$ is potentially relevant when there are point-masses in the probability distribution of $R(T)$. For ease of exposition, we assume this is not the case. When we do have point-masses we can still handle this mathematically, but then we must consider randomised tests.

test will be very low. In the limiting case when $\mathbb{P} = \mathbb{A}$ the power of the (randomised) likelihood ratio test will be as low as α.

Hence, our model selection criterion will include all models for which the statistical power $\mathbb{A}[R(T) \geq \gamma]$ is below $1-\beta$. We consider these models to be *statistically indistinguishable* from the baseline model $\mathbb{P}$. By imposing the α, for each deviation λ the critical value γ is defined. If the associated power is too high the λ is excluded from the set of indistinguishable models and vice versa.

We can express $\mathbb{A}[R(T) \geq \gamma]$ also as $\mathbb{E}^{\mathbb{P}}[R(T)\mathbb{1}(R(T) \geq \gamma)]$, and we obtain an interpretation for the $\mathbb{P}$-expectation as the Tail-Value-at-Risk (TVaR) or Conditional-Value-at-Risk (CVaR) of the random variable $R(T)$ with a confidence level of α. Hence, if we put an upper bound of $1-\beta$ on the power of the likelihood ratio test, this is equivalent to restricting the TVaR of the Radon-Nikodym derivative $R(T)$ to $1-\beta$. Tail-Value-at-Risk is a *coherent risk measure* (see, Artzner et al. (1999) and Rockafellar and Uryasev (2000, 2002)), which has attractive properties. For example, the acceptance set (that is the set of all $R(T)$ for which $\text{TVaR}(R(T)) \leq 1-\beta$) is a closed convex set. Hence, we obtain immediately that our set of indistinguishable models is also a closed convex set.

It can be challenging to compute the statistics in full generality. For deterministic $\lambda(t)$ we can compute everything explicitly, though our main goal is to find the set of indistinguishable models for stochastic $\lambda(t,\omega)$. To illustrate our general idea we first consider the deterministic case.

2.1 Deterministic Drift Term

When Mother Nature is only allowed to use alternative models with a deterministic drift term $\lambda(t)$, we can compute the probability distribution of the Radon-Nikodym derivative defined in equation (2.2) explicitly. For this case we obtain

$$R(T) = \exp\left\{-\tfrac{1}{2}\int_0^T \lambda(t)^2\,dt + \int_0^T \lambda(t)\,dW^{\mathbb{P}}(t)\right\} \tag{2.7}$$

In particular, $\ln R(T)$ has a normal distribution with mean $-\frac{1}{2}\int_0^T \lambda(t)^2\,dt$ and variance $\int_0^T \lambda(t)^2\,dt$. The likelihood ratio test procedure $R(T) > \gamma$ is equivalent to performing a test on the statistic

$$r(T) = \int_0^T \lambda(t)\,dW^{\mathbb{P}}(t) \tag{2.8}$$

This test statistic is intuitively appealing: we compute the inner product between the model-drift $\lambda(t)$ and the realised changes in the Brownian Motion $dW(t)$ along the whole path $[0,T]$. Every time $\lambda(t)$ and $dW(t)$ have the same sign, this increases the value of $r(T)$. Hence, if model $\mathbb{A}$ is true, then $r(T)$ will on average have a positive value. *Ex post*, the realisation of the path of $W(T)$ is observed at indicates the likelihood whether it was generated by model $\mathbb{P}$ or $\mathbb{A}$. However, *ex ante* the test will not be conducted but rather serves as a hypothetical test. Note that before time T the test statistic is a random variable.

The test statistic $r(T)$ (under model $\mathbb{P}$) has a normal distribution with mean 0 and variance $\int_0^T \lambda(t)^2\,dt$. The hypothesis $\mathbb{P}$ is rejected if $r(T) \geq \gamma$. By imposing a significance level α the critical value γ can be derived analytically. Under the alternative model $\mathbb{A}$ the test statistic has a normal distribution with mean $\int_0^T \lambda(t)^2\,dt$ and variance $\int_0^T \lambda(t)^2\,dt$. The power of the

test can be computed explicitly as

$$\mathbb{A}[r(T) > \gamma] = \Phi\left(\Phi^{-1}(\alpha) + \left(\int_0^T \lambda(t)^2\,dt\right)^{\frac{1}{2}}\right) \tag{2.9}$$

For the case $\lambda(t) \equiv 0$ we see that the power is $\Phi\big((\Phi^{-1}(\alpha)\big) = \alpha$. For non-zero values of $\lambda(t)$ the expression $(\int_0^T \lambda(t)^2\,dt)^{\frac{1}{2}}$ is strictly positive and therefore the power will be larger than α. If we consider all models with a power below $1-\beta$ as indistinguishable, then the class of indistinguishable models (with deterministic $\lambda(t)$) is given by all models for which the L_2-norm $(\int_0^T \lambda(t)^2\,dt)^{\frac{1}{2}}$ is below a certain threshold.

If we take for example $\alpha = 0.05$, then $\Phi^{-1}(\alpha) = -1.64$. If we take $\beta = 0.20$ then the power is 0.80 and we have $\Phi^{-1}(0.80) = 0.84$. Hence, the class of all indistinguishable models is then given by all models that satisfy $(\int_0^T \lambda(t)^2\,dt)^{\frac{1}{2}} \leq 0.84 - (-1.64) = 2.48$, if one would have T years of extra data.

The deterministic example we have formulated can be generalised easily to the *multi-dimensional case*. For a vector-valued Brownian Motion all alternative models are specified by the deterministic vector-valued process $\lambda(t)$. The test statistic $r(t)$ is then given by

$$r(T) = \int_0^T \lambda(t)\cdot dW^{\mathbb{P}}(t) \tag{2.10}$$

This is also a random variable with mean 0 and variance $\int_0^T |\lambda(t)|_2^2\,dt$, where $|\lambda(t)|_2$ denotes the L_2-norm of the vector $\lambda(t)$. Hence, in the multi-dimensional case the set of indistinguishable models is given by all models for which $(\int_0^T |\lambda(t)|_2^2\,dt)^{\frac{1}{2}}$ is below the same threshold as in the one-dimensional case (e.g. $2.48/\sqrt{T}$). We may choose the power dependent on the number of dimensions.

2.2 Stochastic Drift Term

The deterministic $\lambda(t)$ serves as an intuitive illustration, but our ambition is to consider a much larger class of alternative models: $\lambda(t,\omega)$. If we allow for stochastic $\lambda(t,\omega)$ then a very large class of alternative models is accessible over an interval $[0,T]$. By the Martingale Representation Theorem *any* probability distribution (with support on whole $\Re$ can be attained over an interval $[t, t+\varepsilon]$ with $\varepsilon > 0$.

Let us consider a model with stochastic $\lambda(t,\omega)$. Suppose we consider the random variable

$$R(T) := e^{-\frac{1}{2}a^2T}\cosh(aW(T)) = \tfrac{1}{2}\left(e^{-\frac{1}{2}a^2T + aW(T)} + e^{-\frac{1}{2}a^2T - aW(T)}\right) \tag{2.11}$$

which is strictly positive and has expectation $\mathbb{E}[R(T)] = 1$, hence this is a valid Radon-Nikodym derivative. This $R(T)$ corresponds to a $\mathbb{A}$-model where the probability distribution of $W(T)$ at time T is given by a mixture distribution of two normal distributions with mean $+aT$ and $-aT$, the same variance T, and mixing probabilities $\frac{1}{2}$. Note that this mixture distribution is not a normal distribution, and has mean 0 and variance equal to $T + (aT)^2$ (see Appendix A), which is larger than the variance T under the $\mathbb{P}$-model.

Although this is a very simple example, it shows explicitly that with a stochastic $\lambda(t,\omega)$ we can fundamentally alter the properties of the probability distribution of $W(T)$, beyond only changing the mean of the normal distribution.

2.3 General Case

The deterministic case and the stochastic hyperbolic cosine served as an illustration of our approach. However, we are interested in the generalisation of this method. With other words, we would like to allow for a wide class of alternative models. As such we have shown in the previous subsection that stochastic deviations can lead to fundamentally different models. Therefore we would like to generate the set of stochastic alternative models surrounding the baseline model that are indistinguishable based on an insufficient power and sufficient probability on the Type I error.

For general $\lambda(t,\omega)$ the power calculation is based on (2.3) from which the distribution is unknown.

$$R(T) = \exp\left\{-\tfrac{1}{2}\int_0^T \lambda(t,\omega)^2\,dt + \int_0^T \lambda(t,\omega)\,dW^{\mathbb{P}}(t,\omega)\right\}$$

Because of the difficulty of the power calculation for general $\lambda(t,\omega)$, we impose time-consistency on the set of indistinguishable models (with insufficient power). We search for the distribution of the Radon-Nikodym derivative that generates the maximum power possible in Section 3.1. All the models that imply a power lower than a maximum of $1-\beta$ are defined to be indistinguishable.

3 Stochastic and Time-Consistent Set of Indistinguishable Models

3.1 Time Consistency

One of the main motivations for studying the set of statistically indistinguishable sets are robust solutions to stochastic optimal control problems in economics and in financial markets.

When we are solving optimal control problems, we want to consider solutions that are *time-consistent*. This means that the optimal solution at any time-point $0 < t < T$ does not depend on the history of the process between $[0, t]$. In other words, the optimal policy devised at time 0 for the interval $[0, T]$ is still valid at time t given the information $\mathscr{F}_t$.

The set of indistinguishable models we have defined thus far is *not* time-consistent: the set is defined as those models that have sufficiently low power at time T using the information over the whole path $[0, T]$. We have established in Section 2 that the set of indistinguishable models defines a coherent risk measure. But, this risk measure is "static" at time 0 and not time-consistent.

We can however look at a smaller class of risk measures: the class of time-consistent risk (dynamic) measures. This class has been extensively studied in recent years, and we know how to characterise this class of risk measures. Delbaen (2006) proves that time-consistent (coherent and convex) risk measures are generated by m-stable sets of probability measures. A similar structure (albeit with less mathematical rigour) was already proposed by Epstein and Schneider (2003). An alternative characterisation of time-consistent risk measures has been provided by Rosazza Gianin (2006). She proves that every time-consistent risk measure is equivalent to a g-expectation $\mathscr{E}^g[]$. These non-linear g-expectations can be computed as the solution of a backward stochastic differential equation (BSDE) with a driver $g(t, Y, Z)$. A further characterisation has been provided by Barrieu and El Karoui (2007): they prove that time-consistent coherent risk measures are generated by drivers $g(t, Z)$ that satisfy a Lipschitz

growth constraint in Z, and time-consistent convex risk measures are generated by drivers that satisfy a quadratic growth constraint in Z.

Hence, we propose to intersect the class of indistinguishable models (which are coherent, but not time-consistent) with the collection of time-consistent coherent risk measures. Since the objective can be interpreted as the coherent risk measure TVaR/CVaR. We then obtain the set of time-consistent indistinguishable models. The question is now: how can we obtain an explicit characterisation of this intersection?

3.2 Maximum Power Calculation

We obtain an explicit characterisation in the following way. The class of time-consistent coherent risk measures are generated by BSDE's with drivers that satisfy the Lipschitz growth condition $g(t, Y, Z) \leq k|Z|$. This is equivalent to the class of Radon-Nikodym derivatives with kernels $|\lambda(t,\omega)| \leq k$.

We want to investigate the maximum power that can be achieved within the class of Radon-Nikodym derivatives with $|\lambda(t,\omega)| \leq k$, such that the Type-I error is equal to α. We can formulate this as a stochastic optimisation problem of the form

$$\begin{aligned} \max_{\gamma, |\lambda(t,\omega)|\leq k} \ & \mathbb{E}\left[R(T)\mathbb{1}(R(T)\geq\gamma)\right] \\ \text{s.t. } & \mathbb{E}\left[\mathbb{1}(R(T)\geq\gamma)\right] = \alpha \\ & dR = \lambda(t,\omega)R\,dW, R_0 = 1 \end{aligned} \tag{MP}$$

where γ is defined by $\mathbb{E}\left[\mathbb{1}(R(T)\geq\gamma)\right] = \alpha$.

The objective function is the power of the test $R(T) \geq \gamma$ formulated as a $\mathbb{P}$-expectation. The second line gives the Type-I error (also formulated as a $\mathbb{P}$-expectation), the third line describes the stochastic process for the Radon-Nikodym derivative given the control variable $\lambda(t,\omega)$. The optimisation problem is non-convex due to the indicator function. Since it is easier to work with convex functions, we introduce the auxiliary function

$$\begin{aligned} F_\alpha(R,\gamma) &= \alpha\gamma + \mathbb{E}\left[(R-\gamma)^+\right] \\ &= \alpha\gamma + \mathbb{E}\left[R\mathbb{1}(R(T)\geq\gamma)\right] - \gamma\mathbb{E}\left[\mathbb{1}(R(T)\geq\gamma)\right] \\ &= \mathbb{E}\left[R\mathbb{1}(R(T)\geq\gamma)\right] + \left(\alpha - \mathbb{E}\left[\mathbb{1}(R(T)\geq\gamma)\right]\right)\gamma \end{aligned} \tag{3.1}$$

where $(R-\gamma)^+$ denotes $\max(R-\gamma,0)$. The functional $F_\alpha(R,\gamma)$ is convex in and continuous as a function in $\gamma \in \mathbb{R}$ and $R(T) \in L_2(T) : \mathbb{E}[R(T)^2] < \infty$. This is also shown by Rockafellar and Uryasev (2000, 2002) who introduce a similar auxiliary function to minimise the Conditional-Value-at-Risk (CVaR).

In order to solve the constrained optimisation problem (MP) we solve

$$\begin{aligned} \max_{|\lambda(t,\omega)|\leq k} \ & \min_{\gamma} F_\alpha(R,\gamma) \\ \text{s.t. } & dR = \lambda(t,\omega)R\,dW \end{aligned} \tag{MaMi}$$

The optimisation (MaMi) is equivalent to (MP), this is proven by Rockafellar and Uryasev (2000, 2002). They prove that the CVaR can be obtained by rewriting the optimisation problem

in terms of a convex auxiliary function.

$$\begin{aligned}\frac{\partial F_\alpha(R,\gamma)}{\partial\gamma} &= \alpha-\mathbb{E}\left[\mathbb{1}(R(T)\geq\gamma)\right] \\ &= 0\end{aligned} \tag{3.2}$$

This implies the Type I error constraint to hold. Since $F_\alpha(R,\gamma)$ is convex in γ, the extreme value is a minimum. In particular, after minimisation we have

$$\begin{aligned}\min_\gamma F_\alpha(R,\gamma) &= \min_\gamma \mathbb{E}\left[R(T)\mathbb{1}(R(T)\geq\gamma)\right]+(\alpha-\alpha)\gamma \\ &= \mathbb{E}\left[R(T)\mathbb{1}(R(T)\geq\gamma)\right]\end{aligned} \tag{3.3}$$

This is the power that we would like to maximise. Henceforth we continue by maximising the minimised $F_\alpha(R,\gamma)$. However, because this is nontrivial we would prefer to change the order of the optimisation. Solving the initial non-convex optimisation problem is identical with solving the max min. The power from the (MaMi) is always lower than or equal to the reversed order optimisation by the max min inequality. Let (MiMa) be

$$\begin{gathered}\min_\gamma \max_{|\lambda(t,\omega)|\leq k} F_\alpha(R,\gamma) \\ \text{s.t. } dR=\lambda(t,\omega)R\,dW\end{gathered} \tag{MiMa}$$

then, we can summarise the relations between the different optimisation formulations by

$$\text{(MP)} \overbrace{=}^{\text{CVaR}} \text{(MaMi)} \underbrace{\overbrace{\leq}^{\text{Max Min}}}_{\text{Inequality}} \text{(MiMa)} \tag{3.4}$$

If we solve the (MiMa) we find an upperbound on (MaMi) and (MP). First we solve the (MiMa), in specific we start with the inner maximisation problem

$$\begin{gathered}\max_{|\lambda(t,\omega)|\leq k} F_\alpha(R,\gamma) \\ \text{s.t. } dR=\lambda(t,\omega)R\,dW\end{gathered} \tag{3.5}$$

The inner maximisation is solved by formulating it as a HJB problem. Note that for every fixed $\gamma\in\mathbb{R}$ the function $F_\alpha(R,\gamma)$ is convex in R. We introduce the value function $V(t,r,\gamma)=\mathbb{E}[F_\alpha(R(T),\gamma)|R(t)=r]$ with boundary condition $V(T,r,\gamma)=F_\alpha(R(T),\gamma)$.

To maximise $V(t,r,\gamma)$, for any value of α and γ the optimised V function w.r.t. λ for $t\leq T$ solves the HJB-equation

$$V_t+\max_{|\lambda(t,r)|\leq k}\tfrac{1}{2}\lambda(t,r)^2r^2V_{rr}=0. \tag{3.6}$$

Optimal control problems of this sort have been studied in the literature in the context of uncertain volatility models, see Avellaneda et al. (1995) and Vanden (2006), and are called *Black-Scholes-Barenblatt* equations.

We propose $|\lambda(t,\omega)|=k$ as candidate solution. The analytical expression for the value function is

$$V(t,r,\gamma) = \alpha\gamma+rN(d_1)-\gamma N(d_2) \tag{3.7}$$

where $d_1 = \frac{1}{k\sqrt{T-t}}\left(\ln\left(\frac{r}{\gamma}\right) + \frac{1}{2}k^2(T-t)\right)$ and $d_2 = d_1 - k\sqrt{T-t}$. This function solves the HJB equation for the boundary condition. Moreover,

$$\left|\lambda^*(t,r)\right| = \begin{cases} k & \text{for } V_{rr}(t,r,\gamma) \geq 0 \\ 0 & \text{for } V_{rr}(t,r,\gamma) < 0 \end{cases} \tag{3.8}$$

The value function is convex as

$$V_{rr}(t,r,\gamma) \quad = \quad \frac{n(d_1)}{rk\sqrt{T-t}} \geq 0 \ \forall \ r,t < T \tag{3.9}$$

where $n(\cdot)$ is the standard normal density function.

The Verification Theorem 11.2.2 of Øksendal (2003) states that if $V(t,r,\gamma)$ is uniformly integrable then a solution of the HJB is an optimal control. The value function $V(t,r,\gamma)$ is indeed uniformly integrable. Since $R(T)$ itself is uniformly integrable for all $\lambda(t,\omega)$, consequently also the partial moment $V(t,r,\gamma) = \mathbb{E}[F_\alpha(R,\gamma)|R(t) = r]$ is uniformly integrable. The Radon-Nikodym derivative is uniformly integrable because the expectation $\mathbb{E}[R(T)^2] \leq e^{k^2T} < \infty$ as $|\lambda(t,\omega)| \leq k$. Thus $|\lambda(t,\omega)| = k$ is an optimal control that leads to the maximum power.

The optimal value function is $V(0,1,\gamma) = \mathbb{E}\left[F_\alpha(R(T),\gamma)|\mathscr{F}_0\right] = \alpha\gamma +$
$\mathbb{E}\left[(R(T)-\gamma)^+|t=0, R_0 = 1\right]$, hence this is the optimal power and can be interpreted as a lognormal expectation equal to a constant plus the expected value of a Black-Scholes call option. The remaining part is the outer minimisation. In specific,

$$\min_\gamma V(0,1,\gamma) \tag{3.10}$$

This simple calculation leads to the optimal γ^* to be the $(1-\alpha)$-quantile of the lognormal $R(T)$. This proves that the Radon-Nikodym derivative with $|\lambda(t,\omega)| \equiv k$ for all $0 \leq t \leq T$ achieves the highest possible power within the constraint on the Type-I error. Note that the lognormal Radon-Nikodym derivative is not the unique solution that leads to the unique optimum. Though all that is needed is the optimal value of the normal of $\lambda(t,\omega)$ which is unique.

We can conclude that the upperbound on (MaMi) is the lognormal power of (MiMa). The original problem (MP) is equivalent (MaMi). Since $|\lambda^*(t,R^*)| = k$ is a feasible solution of the original problem, the upperbound on the power is reached. Hence, the inequality between (MaMi) and (MiMa) becomes an equality and the maximum power possible is obtained for $|\lambda^*(t,R^*)| = k$. ■

Given that the optimal $R^*(T)$ is a lognormal martingale with volatility k, then the optimal value for γ is equal to the $(1-\alpha)$-quantile of $R^*(T)$. The optimised power at time $t=0$ and $R(0) = 1$ is therefore equal to

$$\mathbb{E}\left[R^*(T)\mathbb{1}\left(R^*(T) > \gamma^*\right)\right] = \mathbb{A}\left[R^*(T) > \gamma^*\right] = \Phi\left(\Phi^{-1}(\alpha) + k\sqrt{T}\right). \tag{3.11}$$

If we want to construct a class of time-consistent coherent risk measures with stochastic $\lambda(t,\omega)$ such that every element in the class is statistically indistinguishable, then a sufficient condition is to choose a k such that the "worst case" power (3.11) does not exceed $(1-\beta)$. For example, if we take $\alpha = 0.05$ and $\beta = 0.20$, we find $k = 2.48/\sqrt{T}$.

All previous derivations also hold for vector Brownian Motions and vector $\lambda(t,\omega)$. The multivariate equivalent is a bound on the L2-norm; $|\lambda(t,\omega)| \leq \frac{2.48}{\sqrt{T}}$.

3.3 Sufficiency

The lognormal Radon-Nikodym derivative with maximal power for stochastic deviations is a sufficient rather than a necessary condition. This implies that the bound k might be larger and still admit a power below $1-\beta$. For deterministic deviations the lognormal solution is both sufficient and necessary. The intuition that some power might be lost due to stochasticity is as follows; imagine $\lambda(t,\omega)$ to switch stochastically between $+k$ and $-k$. The Radon-Nikodym will be lower than for constant sign, as the signs drop the second term in (2.3). This implies a lower power which means that it becomes more difficult to distinguish between the models. Henceforth, larger deviations are allowed in the set of indistinguishable models.

The stochastic example of section 2.2 illustrates that the bound on the norm of the stochastic deviations is a sufficient condition to determine the set of indistinguishable models. For hyperbolic cosine as choice for $R(T)$ we can explicitly compute the conditional expectation $R(t)=\mathbb{E}^{\mathbb{P}}[R(T)|\mathscr{F}_t]=e^{-\frac{1}{2}a^2 t}\cosh(aW(t))$. If we apply Itô's Lemma to $R(t)$ we obtain the stochastic differential equation

$$dR(t)=ae^{-\frac{1}{2}a^2 t}\sinh(aW(t))dW(t)=a\tanh\big(aW(t)\big)R(t)dW(t). \tag{3.12}$$

Hence, this Radon-Nikodym derivative corresponds to a model where $\lambda(t,\omega)=a\tanh(aW(t))$. For positive values of $W(t)$ we have a positive and increasing drift, and the drift is bounded by $+a$. For negative values of $W(t)$ we have a negative and increasing drift, and the drift is bounded by $-a$. Hence, the alternative model $\mathbb{A}$ is a "mean repelling" process, which will increase the variance of $W(T)$ under model $\mathbb{A}$.

The likelihood ratio test will reject model $\mathbb{P}$ if $R(T)>\gamma$. As $\cosh(aW(T))$ is symmetric around $W(T)=0$ and strictly increasing in $|W(T)/\sqrt{T}|$, the rejection set defined by $R(T)>\gamma$ is equivalent to the rejection set $|W(T)/\sqrt{T}|>\gamma'$. If we want to test at a significance level of $\alpha=0.05$ then $\gamma'=-\Phi^{-1}(\alpha/2)=1.96$.

The power of the likelihood ratio test can be computed as $\mathbb{A}[|W(T)/\sqrt{T}|>\gamma']$ which can be expressed as $\mathbb{E}^{\mathbb{Q}}[\mathbb{1}(|W(T)/\sqrt{T}|>\gamma')]=\mathbb{E}^{\mathbb{P}}[R(T)\mathbb{1}(W(T)/\sqrt{T}>\gamma')]+\mathbb{E}^{\mathbb{P}}[R(T)\mathbb{1}(W(T)/\sqrt{T}<-\gamma')]$. A direct computation of the expectations yields

$$\mathbb{A}\left[|W(T)/\sqrt{T}|>\gamma'\right]=\Phi\big(\Phi^{-1}(\alpha/2)+a\sqrt{T}\big)+\Phi\big(\Phi^{-1}(\alpha/2)-a\sqrt{T}\big) \tag{3.13}$$

see Appendix B for the full derivation. If we solve this last equation (numerically) for $a\sqrt{T}$ with $\alpha=0.05$ and $1-\beta=0.80$ then we find $a\sqrt{T}=2.80$. Hence, for the tanh example we find the result that all models that are indistinguishable from the baseline model $a=0$ are given by $|a\sqrt{T}|<2.80$. This set is larger than the "constant lambda" set $|\lambda\sqrt{T}|<2.48$. Conclusion for this example: the bound $|\lambda\sqrt{T}|<2.48$ gives a sufficient condition for statistical indistinguishability.

4 Bounds on Divergences

Ben-Tal et al. (2013) discuss several possible divergences (non-symmetric distance measures) to generate robust results in optimisation problems. ϕ-Divergence (or f-divergence) functions measure the distance between two probability distributions weighted by the specific function. The choice which measure should be picked is an unanswered issue in their and many other papers, plus the question when the distance is too far is rarely investigated. In

this paper we explicitly focus on the size of the set of alternatives. We can directly link this to a critical value for each measure. Note that Ben-Tal et al. (2013) consider the discrete versions, (4.1), whereas we consider the continuous divergences, (4.2) and (4.3), as a function of the Radon-Nikodym derivatives. The numerical value for the set of time-consistent models that cannot be distinguished between with a Type I error of 5% and a power less than 80% is displayed in the last column of Table 1. The derivations can be found in the Appendix C.

The general discrete ϕ-divergence is defined as

$$D_\phi(p,q) \quad = \quad \sum_{i=1}^{m} q_i \phi\left(\frac{p_i}{q_i}\right) \tag{4.1}$$

We use the continuous version

$$D_\phi(p,q) \quad = \quad \mathbb{E}^{\mathbb{A}}\left[\phi\left(\frac{1}{R(T)}\right)\right] \tag{4.2}$$

$$\quad = \quad \mathbb{E}^{\mathbb{P}}\left[R(T)\phi\left(\frac{1}{R(T)}\right)\right] \tag{4.3}$$

where the functions for $\phi(\cdot)$ are given for each measure and by definition $\phi(\cdot)$ is convex. Note that we use $\tilde{\phi}(t) = t\phi\left(\frac{1}{t}\right)$ which are for the divergences considered convex as well as every $\tilde{\phi}$ coincides with a ϕ. The variable we consider is $t = R(T)$.

The lay-out of the proof for each divergence function is similar as the proof of the maximum power calculation. First we postulate the lognormal Radon-Nikodym derivative as candidate solution. Then the analytical expression for the value function is used to prove its convexity, that implies that the proposed solution $|\lambda(t,\omega)| = k$ solves the HJB. By the Verification Theorem 11.2.2 of Øksendal (2003) and the uniform integrability of the value function the optimal control is obtained. ■

Table 1: ϕ-Divergences

Divergence	$\phi(t)$	for $\lvert\lambda(t,\omega)\rvert = k$	$k\sqrt{T} = 2.48$
Kullback-Leibler	$t\ln t - t + 1$	$\frac{1}{2}k^2T$	3.08
Burg entropy	$-\ln t + t - 1$	$\frac{1}{2}k^2T$	3.08
J-divergence	$(t-1)\ln t$	k^2T	6.15
χ^2-divergence	$\frac{1}{t}(t-1)^2$	$e^{k^2T} - 1$	467.90
Modified χ^2-divergence	$(t-1)^2$	$e^{k^2T} - 1$	467.90
Hellinger distance	$(\sqrt{t}-1)^2$	$2 - 2e^{-\frac{1}{8}k^2T}$	1.07
Variation distance	$\lvert t-1\rvert$	$4N(\frac{1}{2}k\sqrt{T}) - 2$	1.57
χ-divergence of order $\theta > 1$	$\lvert t-1\rvert^\theta$	$-$	see Table 2
Cressie-Read $\theta \neq 0, 1$	$\frac{1-\theta+\theta t - t^\theta}{\theta(1-\theta)}$	see below	see Table 2

The first column shows the divergence function, the second column the optimal maximum divergence and the last column the numerical bound on the divergence for a Type I error of 5% and the complement of the Type II error equal to 80%.

Both the χ-divergence of order $\theta > 1$ and the Cressie-Read depend on the additional parameter θ. The Cressie-Read can analytically be expressed by

$$\frac{1}{\theta(1-\theta)}\left(1 - e^{-\frac{1}{2}k^2\theta(1-\theta)T}\right) \tag{4.4}$$

and the χ-divergence can be derived analytically for integer values of θ as well. For $k\sqrt{T} = 2.48$ we get the bounds on both measures displayed in Table 2. An optimal control problem where the agent is worried about possible misspecification of the baseline model, could be described by an additional constraint that the Kullback-Leibler divergence between the baseline model and all alternatives should be bounded by a cutoff point to differentiate plausible alternatives from implausible ones. If one would impose a probability of incorrectly rejecting the baseline model of 5% and if one would need at least a power of 80% to correctly accept the alternative. Then for all models that yield a lower power, the information is insufficient to distinguish between $\mathbb{P}$ and $\mathbb{A}$. Hence the cutoff point on the Kullback-Leibler divergence equals 3.08. The same reasoning holds for the other divergences.

Table 2: Numerical bounds

Divergence θ	1.5	2.0	2.5	3.0
χ-divergence of order θ	10.40	467.90	1.02×10^6	1.03×10^8
Cressie-Read	12.05	233.95	2.72×10^4	1.72×10^7

For a given θ we calculated the maximum divergence possible by imposing time-consistency, a Type I error of 5% and Type II error of 20%.

5 Investment Problem

Whereas others concentrate on the functional pricing rule for a given cutoff point in the robustness literature. We derive an intuition behind the choice of the cutoff point that distinguishes plausible from implausible models. The investment problem illustrates a possible application of our previously derived ideas. However, the applicability of the set of indistinguishable models is much more general.

The fact that risky assets have a relatively high return compared with riskfree returns would imply high investment strategies. However, practice learns us that this is not the case. This puzzling phenomenon is known as the equity premium puzzle. Several hypotheses see the acknowledgement of model uncertainty as explanatory variable, i.e. additional to risk aversion agents might be concerned about the underlying model specification. In the classical Merton problem risk aversion is considered by means of the power utility. However, it is known that first moments are considerably hard to estimate. Therefore we apply the set of indistinguishable models to the Merton investment problem to investigate the effect of dealing with uncertainty aversion on top of risk aversion. We compare the classical investment problem with the robust version based on a constraint and a penalty formulation.

The Merton problem is an optimisation problem where an agent's objective is to maximise the utility from his wealth by allocating some of his wealth to the risky assets and the rest to the riskfree bank account. Contrary to the standard set-up of Merton, we add the agent's uncertainty about the underlying processes. Merton (1969) and the extensions of his work all assume power utility (i.e. the isoelastic function for utility, CRRA utility function), so do we. We derive the robust Merton problem by the dynamic and time-consistent optimisation, similarly as Biagini and Pinar (2015).

H&S and co-authors (Hansen et al., 2006; Anderson et al., 2003; Hansen et al., 2002a,b, 1999) have several papers on robustness in which they motivate the constraint approach based on the maxmin framework of Gilboa and Schmeidler (1989). However, they transform

the constraint into a penalty term. The bound on the constraint, our k, is indirectly linked to the constant θ which is the Lagrange multiplier. By solving the adjusted objective, one is optimising a fundamentally different problem. The solution of an unknown objective is derived. Besides, Maenhout (2004) starts with the entropy penalty of H&S and transforms the multiplier into a function of the value function. The motivation behind this step is that the homothetic condition increases the analytical tractability. However, due to this step the connection with the initial motivation of ellipsoid uncertainty is lost. Pathak et al. (2002) has an interesting discussion on the implications of these changes. H&S do not consider the investment problem, instead they allow for general utility and stochastic alternatives. Therefore the distribution of the entropy is unknown which they tackle by imposing a detection error probability for possible values of the Lagrange multiplier and solve these numerically. They consider linear/quadratic problems with examples in the field of macroeconomics. Maenhout focusses on the investment problem itself and fixes the functional form of the Lagrange multiplier to ensure the solution of the PDE and a time-independent investment strategy. Still the Lagrange multiplier depends on a constant that has to be chosen, either calibrated to data or quantified by a similar idea as the insufficient power or detection error probability. Both approaches are combined and implemented by Maenhout. Namely, he uses the data to calculate the detection error probability which is supposed to be an ex ante derivation.

5.1 Classic and Constraint

Let $A(t)$ be the wealth at time t and let there be n risky assets which can be described by the stochastic differential equation

$$d\boldsymbol{S}(t) = \boldsymbol{S}(t)\left(\boldsymbol{\mu}dt + \boldsymbol{\sigma}d\boldsymbol{W}(t)\right) \tag{5.1}$$

where $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}'$ is invertible. Let $\pi_i(t)$ denote the fraction of wealth invested in the risky asset i, then by the self-financing condition the total wealth evolves as

$$dA(t) \quad = \quad \left(\boldsymbol{\pi}'(t)(\boldsymbol{\mu} - r\boldsymbol{I})A(t) + rA(t)\right)dt + A(t)\boldsymbol{\pi}'(t)\boldsymbol{\Sigma}^{1/2}d\boldsymbol{W}(t) \tag{5.2}$$

The robust version of the Merton problem is

$$\begin{aligned} dA(t) &= \left(\boldsymbol{\pi}'(t)(\boldsymbol{\mu} - r\boldsymbol{I})A(t) + rA(t)\right)dt + A(t)\boldsymbol{\pi}'(t)\boldsymbol{\Sigma}^{1/2}\left(d\boldsymbol{W}(t) + \boldsymbol{\lambda}(t,\boldsymbol{\omega})dt\right) \\ &= \left(\boldsymbol{\pi}'(t)(\boldsymbol{\mu} - r\boldsymbol{I})A(t) + rA(t) + A(t)\boldsymbol{\pi}'(t)\boldsymbol{\Sigma}^{1/2}\boldsymbol{\lambda}(t,\boldsymbol{\omega})\right)dt + \\ &\quad A(t)\boldsymbol{\pi}'(t)\boldsymbol{\Sigma}^{1/2}d\boldsymbol{W}(t) \end{aligned} \tag{5.3}$$

where we consider the wealth process under an alternative measure to indicate the possibility that the measure might be misspecified. By Girsanov theorem this implies that the agent is uncertain about the drift of the wealth process,

$$d\boldsymbol{W}(t) + \boldsymbol{\lambda}(t,\boldsymbol{\omega})dt \tag{5.4}$$

The robust solution is obtained as the investment strategy that is least sensitive to perturbations of the model. Strategically this is ensured by giving a so called Mother Nature the power to minimise the objective that the agent tries to maximise. She can do this by picking the worst case probability measure among the set of plausible alternatives. In this section

we assume that the agent wants to maximise the utility from his terminal wealth at the finite horizon T. Thus, we exclude consumption from the Merton problem. The uncertainty set or equivalently the set of indistinguishable models are those models surrounding the baseline model that one cannot distinguish between with enough power for a given probability on the Type I error. This takes the form of a bounded ellipsoid where we quantified a plausible cutoff point in the previous sections.

The robust objective is

$$\max_{\boldsymbol{\pi}(t)} \min_{\boldsymbol{\lambda}(t,\boldsymbol{\omega})} \quad \mathbb{E}[u(T, A(T))] \tag{5.5}$$

and we assume power/CRRA utility

$$u(t, A(t)) \quad = \quad c(t)\frac{A^{1-p}}{1-p} \tag{5.6}$$

The stochastic differential equation of the wealth process is given by (5.3) and the uncertainty region is given by

$$\boldsymbol{\lambda}(t,\boldsymbol{\omega})'\boldsymbol{\lambda}(t,\boldsymbol{\omega}) \leq k^2 \quad \Leftrightarrow \quad |\boldsymbol{\lambda}(t,\boldsymbol{\omega})| \leq k \tag{5.7}$$

where $\boldsymbol{\lambda}(t,\boldsymbol{\omega})$ is allowed to be both deterministic and stochastic (for the stochasticity allowance we use the $\boldsymbol{\omega}$-notation). The optimisation problem summarises as

$$\begin{aligned}
\max_{\boldsymbol{\pi}(t)} \min_{\boldsymbol{\lambda}(t,\boldsymbol{\omega})} \quad & \mathbb{E}\left[c(T)\frac{A(T)^{1-p}}{1-p}\right] \\
\text{s. t.} \quad & dA(t) = \left(\boldsymbol{\pi}'(t)\tilde{\boldsymbol{\mu}} + r\right)A(t)dt + \boldsymbol{\pi}'(t)\boldsymbol{\Sigma}^{1/2}A(t)d\boldsymbol{W}(t) \\
& \boldsymbol{\lambda}(t,\boldsymbol{\omega})'\boldsymbol{\lambda}(t,\boldsymbol{\omega}) \leq k^2
\end{aligned} \tag{5.8}$$

where $\tilde{\boldsymbol{\mu}} = \left((\boldsymbol{\mu} - r\boldsymbol{I}) + \boldsymbol{\Sigma}^{1/2}\boldsymbol{\lambda}(t,\boldsymbol{\omega})\right)$. The ellipsoid constraint is equivalent to having a constraint on the Kullback-Leibler divergence. In the previous section we derived the connection between the bounds. Moreover, since

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\ln R(T)^{-1}\right] \quad &= \quad \mathbb{E}^{\mathbb{P}}\left[\tfrac{1}{2}\int_0^T \lambda(t)^2\,dt - \int_0^T \lambda(t,)\,dW^{\mathbb{P}}(t)\right] \\
&= \quad \tfrac{1}{2}\int_0^T \lambda(t)^2\,dt
\end{aligned} \tag{5.9}$$

it follows that $(\int_0^T \lambda(t)^2 dt)^{\frac{1}{2}} \leq 2.48$ corresponds to $\frac{1}{2}\int_0^T \lambda(t)^2\,dt \leq 3.08$.
The HJB formulation for the value function $v(t,A) = \mathbb{E}\left[u(T,A(T))|\mathscr{F}_t\right]$ is

$$\max_{\boldsymbol{\pi}} \min_{\boldsymbol{\lambda}(t,\boldsymbol{\omega})} \mathscr{L} \quad := \quad v_t + \left(\boldsymbol{\pi}'(t)\tilde{\boldsymbol{\mu}}A(t) + rA(t)\right)v_A + \tfrac{1}{2}A(t)^2\boldsymbol{\pi}'(t)\boldsymbol{\Sigma}\boldsymbol{\pi}(t)v_{AA} + \tfrac{1}{2}g\left(k^2 - \boldsymbol{\lambda}'\boldsymbol{\lambda}\right)$$

We first solve the inner minimisation, by the FOC with respect to $\boldsymbol{\lambda}$ and g

$$\boldsymbol{\lambda}^* \quad = \quad \pm\frac{k\boldsymbol{\Sigma}^{1/2}\boldsymbol{\pi}}{\sqrt{\boldsymbol{\pi}'\boldsymbol{\Sigma}\boldsymbol{\pi}}} \tag{5.10}$$

where the negative root minimises the objective. If we plug this into the HJB

$$\max_{\boldsymbol{\pi}} \quad v_t + \left(\boldsymbol{\pi}'(t)\tilde{\boldsymbol{\mu}}A(t) + rA(t)\right)v_A + \tfrac{1}{2}A(t)^2\boldsymbol{\pi}'(t)\boldsymbol{\Sigma}\boldsymbol{\pi}(t)v_{AA}$$

remains where $\tilde{\boldsymbol{\mu}} = (\boldsymbol{\mu} - r\boldsymbol{I}) - k\frac{\boldsymbol{\Sigma}\boldsymbol{\pi}}{\sqrt{\boldsymbol{\pi}'\boldsymbol{\Sigma}\boldsymbol{\pi}}}$. The FOC with respect to the outer maximisation leads to

$$\boldsymbol{\pi}^*(t) \quad = \quad \boldsymbol{\Sigma}^{-1}\left((\boldsymbol{\mu} - r\boldsymbol{I}) + \boldsymbol{\Sigma}^{1/2}\boldsymbol{\lambda}(t, \boldsymbol{\omega})\right)\frac{v_A}{A(t)v_{AA}} \tag{5.11}$$

For simplicity we skip the full derivation where we include the inner derivative of the multivariate $\boldsymbol{\lambda}$ explicitly. In combination with the positive condition on the root $\sqrt{\boldsymbol{\pi}'(t)\boldsymbol{\Sigma}\boldsymbol{\pi}(t)}$ to ensure the maximum, this leads to the constraint on k which should be smaller than the maximum Sharpe ratio (the market price of risk). See Biagini and Pinar (2015) for the full derivation.

We propose the functional form of $v(\cdot)$ to be a power utility, since at terminal time we know that $v(T, A) = u(T, A)$.

$$v(t, A) \quad = \quad c(t)\frac{A(t)^{1-p}}{1-p} \tag{5.12}$$

Then

$$\boldsymbol{\pi}^*(t) \quad = \quad \boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}\frac{1}{p} \tag{5.13}$$

We can now plug in the optimal $\boldsymbol{\lambda}^*$ in the optimal hedging position

$$\boldsymbol{\pi}^*(t) \quad = \quad \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{I})\left(1 - \frac{k}{\sqrt{(\boldsymbol{\mu} - r\boldsymbol{I})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{I})}}\right)\left(\frac{1}{p}\right) \tag{5.14}$$

The optimal investment strategy is between 0 and the classical Merton solution $\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{I})\left(\frac{1}{p}\right)$. The lower bound is mathematically derived by the fact that we need the FOC for the maximum $\boldsymbol{\pi}$. Thus (5.14) holds if $0 \leq \frac{k}{\sqrt{(\boldsymbol{\mu} - r\boldsymbol{I})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{I})}} \leq 1$. To incorporate these conditions we state

$$\boldsymbol{\pi}^*(t) \quad = \quad \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{I})\left(\frac{\max\left(\sqrt{(\boldsymbol{\mu} - r\boldsymbol{I})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{I})} - k, 0\right)}{\sqrt{(\boldsymbol{\mu} - r\boldsymbol{I})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{I})}}\right)\left(\frac{1}{p}\right) \tag{5.15}$$

If there is no uncertainty, $k = 0$ then we are back in the classical Merton setting. If the agent is uncertain he will invest less in the risky assets, consequently he invests more on the bank account. This behaviour goes on until he is so uncertain that he shall not invest in the risky assets anymore. Hence, if on the one hand the agent becomes certain about the specification of the baseline model, $k \to 0$, then his optimal investment strategy equals to standard Merton solution without uncertainty, $\boldsymbol{\pi}^*(t) \to \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r\boldsymbol{I})\left(\frac{1}{p}\right)$. On the other hand, if the agent becomes extremely worried about possible misspecification, $k \to \infty$, then he plays safe by putting all his wealth on the riskfree bank account, $\boldsymbol{\pi}^*(t) \to 0$.

For ease of exposition, we assume for the remainder of the robust investment problem that there is one risky asset ($n = 1$). The optimal wealth for the constraint case is

$$A^*(T)_{\mathrm{C}} \quad = \quad A_0 \exp\left\{\left(\pi\tilde{\mu} + r - \tfrac{1}{2}\pi^2\sigma^2\right)T + \pi\sigma W(T)\right\} \tag{5.16}$$

Table 3: Robust Optimal Investments

$p \setminus T$	1	100	200	π^*_{M}
0.1	0%	12.5%	466.5%	1562.5%
0.2	0%	6.25%	233.2%	781.25%
0.5	0%	2.5%	93.3%	312.5%
1	0%	1.25%	46.65%	156.25%
3	0%	0.42%	15.55%	52.1%
5	0%	0.25%	9.33%	31.25%

The optimal robust investment strategy by the addition of a constraint is shown, π^*_{C} for several values of $T = \{1, 100, 200\}$ with $k = \frac{2.48}{\sqrt{T}}$. The last column shows the classical Merton solution without model uncertainty.

where $\tilde{\mu} = \mu - r - \sigma k$ and $\pi = \frac{\tilde{\mu}}{\sigma^2 p}$. Under the classical Merton problem without uncertainty the optimal wealth is

$$A^*(T)_{\mathrm{M}} = A_0 \exp\left\{\left(\pi(\mu - r) + r - \tfrac{1}{2}\pi^2\sigma^2\right) T + \pi\sigma W(T)\right\} \tag{5.17}$$

where $\pi = \frac{\mu - r}{\sigma^2 p}$.

For a market price of risk of 0.25 implied by $\mu = 8\%$, $r = 4\%$ and $\sigma = 16\%$, the uncertainty parameter equal to $k = 2.48/\sqrt{100}$ and $p = 0.5$, the optimal investment strategy of the robust agent is $\pi^*_{\mathrm{C}} = 0.025$. This shows that if he would take 100 year of extra data into account the agent's optimal investment strategy is to invest 2.5% of his wealth in the risky asset S, and to invest the rest, 97.5%, on the savings account. Table 3 and Figure 1 show the optimal investment strategy against the denominator of the uncertainty parameter. The classical Merton solution is 312.5% implying that the agent goes short of 212.5% of his wealth on the bank account.

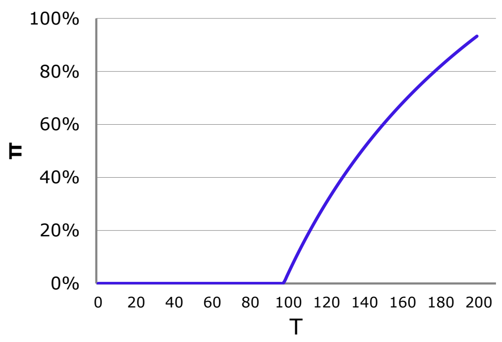
The classical Merton problem considers risk uncertainty by the parameter p. The larger p the more risk-averse the agent is and the less he invests in the risky assets. If $p \to 0$, the agent becomes risk-neutral and the optimal investment strategy is a long position in the risky assets that goes to infinity and minus infinity as a short position on the bank account. However, if the risk-neutral agent is assumed to be uncertain about the underlying model we see that he invests 12.5% in the risky asset, for the case with minor risk-aversion and taking into account 100 years of future information. A different explanation is that the model has the same functional form for 100 years whereafter the dynamic process changes. Hence, for small risk aversion the uncertainty aversion explains financial plausible investment strategies.
By simplifying the optimal wealth in terms of the Brownian motions, we get

$$A^*(T)_{\mathrm{C}} = A_0 \exp\left\{\frac{\mu - r - \sigma k}{\sigma p} W(T)\right\} c_{\mathrm{C}} \tag{5.18}$$

$$A^*(T)_{\mathrm{M}} = A_0 \exp\left\{\frac{\mu - r}{\sigma p} W(T)\right\} c_{\mathrm{M}} \tag{5.19}$$

where $c_{\mathrm{C}} = \exp\left\{\left(\frac{(\mu - r - \sigma k)^2}{\sigma^2 p}\left(1 - \frac{1}{2p}\right) + r\right) T\right\}$ and $c_{\mathrm{M}} = \exp\left\{\left(\frac{(\mu - r)^2}{\sigma^2 p}\left(1 - \frac{1}{2p}\right) + r\right) T\right\}$. Note that for ease of notation we suppressed the condition that $0 \le k \le \frac{\mu - r}{\sigma}$. On the one hand, if the cutoff

Figure 1: Risk-Averse Robust Optimal Investment



The figure displays the amount invested in the risky assets for the above specified parameters, with $p = \frac{1}{2}$ against the extra amount of years T of data taken into account. The classical Merton solution corresponding to this risk parameter is $\pi_{\mathrm{M}} = 312.5\%$.

point becomes large, thus the uncertainty increases $k \to \frac{\mu-r}{\sigma}$, then the prudent risk premium $\mu - r - \sigma k \to 0$ vanishes. Therefore the optimal wealth position moves towards putting all the wealth on the bank account, $A^*(T)_{\mathrm{C}} \to A_0 e^{rT}$. On the other hand if the uncertainty vanishes $k \to 0$, then the optimal wealth implied by the constraint case becomes obsolete and equals the classical non robust Merton solution, $A^*(T)_{\mathrm{C}} \to A^*(T)_{\mathrm{M}}$. So if $0 < k < \frac{\mu-r}{\sigma}$ then the optimal wealth will lie in between $A_0 e^{rT}$ and the Merton solution $A^*(T)_{\mathrm{M}}$.

5.2 Penalty

The addition of a penalty entropy to the Merton problem is what Maenhout (2004) and Anderson et al. (2003) investigate. On top of that, Maenhout assumes homotheticity, i.e. that the optimal portfolio weight is independent from the initial wealth position. In Maenhout (2006) he extends this by a mean-reverting risk premium. First of all, Maenhout makes the same subtle switch as H&S, to incorporate the entropy term by a penalty on the utility premultiplied by the strength of preference for robustness. They express the dependence of the objective and penalty on the Radon-Nikodym derivative in terms of an endogenous drift $u(A_t)$. We use $\lambda_m(t)$ to prevent misunderstandings with the utility function $u(\cdot)$. Note the difference with our $\lambda(t,\omega)$ throughout this paper, which is now multiplied by the term in front of the Brownian motion.

The penalty formulation of the optimisation problem becomes

$$\max_{\pi(t)} \min_{\lambda(t)} \quad \mathbb{E}[u(A(T))] + \frac{1}{2\theta}\lambda_m(T)^2\pi(T)^2\sigma(T)^2A(T)^2 \tag{5.20}$$

$$\text{s. t.} \quad dA_t = A(r + \pi_t(\mu - r) + \sigma^2\pi_t^2 A\lambda_m)dt + \pi_t\sigma A dW_t \tag{5.21}$$

The expression of the Kullback-Leibler divergence in terms of $\lambda_m(t)$ is obtained by (5.10), where $\lambda(t) = \lambda_m(t)\pi(t)\sigma(t)A(t)$. In the original paper of Maenhout, the utility comes from consumption. To remain in the similar setting throughout the comparison we assume again that there is no utility of intertemporary consumption but solely from terminal wealth. One could actually combine the consumption benefits with the terminal wealth by accumulating

the consumption on the bank account up to time T. The HJB formulation for the value function $v(t, A) = \mathbb{E}\left[u(T, A(T)) + \frac{1}{2\theta}\lambda_m(T)^2\pi(T)^2\sigma(T)^2A(T)^2 \middle| \mathscr{F}_t\right]$ is

$$\max_{\pi}\min_{\lambda_m}\mathscr{L} \;:=\; v_t + \left(\pi(t)(\mu - r) + r + \lambda_m(t)A(t)\pi(t)^2\sigma(t)^2\right)A(t)v_A + \tfrac{1}{2}A(t)^2\pi^2(t)\sigma^2(t)v_{AA} + \frac{1}{2\theta}\lambda_m(t)^2\sigma(t)^2\pi(t)^2A(t)^2$$

The inner minimisation solves for

$$\lambda_m^*(t) = -v_A\theta \tag{5.22}$$

The outer maximisation solves for

$$\pi^*(t) = \frac{-v_A}{\left(v_{AA} - \theta v_A^2\right)A(t)}\frac{\mu - r}{\sigma^2} \tag{5.23}$$

Maenhout (2004) fixes the functional form of θ (he uses Ψ) as a function of $v(A, t)$. Due to this transformation the link with the multiplier and constraint is lost. Therefore we solve the investment problem for a constant multiplier proposed by H&S. The PDE becomes nonlinear for power utility, therefore we assume a power utility of order 1, hence log utility

$$v(A, t) = c(t)\ln(A(t)) \tag{5.24}$$

Then

$$\lambda_m^*(t) = -c(t)\frac{1}{A(t)}\theta \tag{5.25}$$

$$\pi^*(t) = \frac{1}{1+\theta}\frac{\mu - r}{\sigma^2} \tag{5.26}$$

The optimal wealth position comparable with the two previous subsections boils down to

$$A^*(T)_{\mathrm{P}} = A_0\exp\left\{\left(\pi(\mu - r) + r + \sigma^2\pi^2A\lambda_m - \tfrac{1}{2}\pi^2\sigma^2\right)T + \pi\sigma W(T)\right\} \tag{5.27}$$

$$A^*(T)_{\mathrm{P}} = A_0\exp\left\{\frac{1}{1+\theta}\frac{\mu - r}{\sigma}W(T)\right\}c_{\mathrm{P}} \tag{5.28}$$

$$c_{\mathrm{P}} = \exp\left\{\left(\frac{(\mu - r)^2}{\sigma^2}\frac{1}{2(1+\theta)^2} + r\right)T\right\}$$

Thus for $p = 1$, the Maenhout result holds without transformation. In Appendix B of Maenhout (2004) they explicitly derive the log case.

5.3 Comparison

The classical Merton solution is purely based on risk-aversion without any model uncertainty, i.e. the agent completely trusts the model. As we have seen, a low risk-aversion yields an tremendously long-short position. This equity puzzle might be split and argued from both a risk-aversion and model uncertainty-aversion behaviour. By incorporating, complementary to the power utility, also model uncertainty the amount invested in the risky assets goes down rather fast. However, when the risk aversion becomes small a plausible investment

Table 4: Input Variables

A_0	μ	r	σ	k	T	p	θ
100	8%	4%	16%	24.8%	100	1	1

All figures and tables in this paper are based on the input variables set equalt to the values depicted above. The initial wealth position is A_0, the return on the risky asset is μ, the riskfree rate is r, the volatility of the risky asset is σ, the uncertainty parameter is k and depends on the future time point T, the risk aversion of the power utility is p and the strength of preference for robustness is θ.

strategy remains when accounted for model uncertainty rather than in the classical Merton setting. The acknowledgement of the model uncertainty by an additional penalty term on the utility is what we explored lastly. The robust Merton problem with constraint serves often as an intuitive explanation whereafter the penalty approach is implemented. The constraint solution can be interpreted as an adjustment of the return on the risky assets and the penalty solution can be interpreted as an adjustment of the volatility of the risky assets. Where we derived the latter only properly for log utility, in its general set-up the objective that belongs to the solution is nontrivial. To summarise, the optimal wealth positions at time T are for the classical Merton solution, the robust solution by constraint and by penalty respectively,

$$A^*(T)_{\mathrm{M}} = A_0 \exp\left\{\frac{\mu - r}{\sigma p} W(T)\right\} c_{\mathrm{M}} \tag{5.29}$$

$$A^*(T)_{\mathrm{C}} = A_0 \exp\left\{\frac{\mu - r - \sigma k}{\sigma p} W(T)\right\} c_{\mathrm{C}} \tag{5.30}$$

$$A^*(T)_{\mathrm{P}} = A_0 \exp\left\{\frac{1}{1+\theta}\frac{\mu - r}{\sigma} W(T)\right\} c_{\mathrm{P}} \tag{5.31}$$

where

$$c_{\mathrm{M}} = \exp\left\{\left(\frac{(\mu-r)^2}{\sigma^2 p}\left(1-\frac{1}{2p}\right)+r\right)T\right\}$$

$$c_{\mathrm{C}} = \exp\left\{\left(\frac{(\mu-r-\sigma k)^2}{\sigma^2 p}\left(1-\frac{1}{2p}\right)+r\right)T\right\}$$

$$c_{\mathrm{P}} = \exp\left\{\left(\frac{(\mu-r)^2}{\sigma^2(1+\theta)}\left(\frac{1}{2(1+\theta)}\right)+r\right)T\right\}$$

For $k = \frac{2.48}{\sqrt{100}}$ the agent is almost too uncertain about the underlying process such that he will only invest in the bank account, i.e. he invests about 0.42% in S. Therefore the wealth position becomes independent of the Brownian motion that represents the stochastic and uncertain future of the risky assets. This is visible by the horizontal line ($\mu - r - \sigma k = 0.032\%$), $\theta \to \infty$ corresponds with this strategy. If $\theta = 100\%$, the agent assumes the volatility to be twice as large as without model uncertainty. The k that matches this result is the one with $k = \frac{2.48}{\sqrt{100}}\frac{1}{4}$, implying the process to remain constant for the next 1600 years. In Maenhout (2004) a θ of 14 and 237 are obtained based on two different data sets, both converging to the conservative strategy when $k = \frac{2.48}{\sqrt{100}}$. If the agent puts his wealth for 100% on the bank account then his final wealth at T is €5,459.82.

To conclude, changing the drift or volatility have the same effect as one alters the nominator or denominator respectively. However, with the indistinguishable method we can indicate the set of plausible alternatives *ex ante*. Whereas the detection error probability is calculated *ex post*. Moreover, the interpretation of uncertainty and the time horizon of data that one would take into consideration or the time horizon during which the model would stay stable is established. The equity premium puzzle can be explained when the risk aversion is $p = 7$ and $\theta = 14$, and in the second and shorter sample investigated in Maenhout (2004), $p = 10$ and $\theta = 237$. This corresponds to $T = \{347, 107\}$ respectively for a Type II error of 80% and Type I error of 5%. The relation between k and θ is

$$\theta = \frac{p(\mu - r)}{\mu - r - \sigma k} - 1 \tag{5.32}$$

by setting $A^*(T)_{\mathrm{C}} = A^*(T)_{\mathrm{P}}$. If we denote k by a function that depends on the Type I and II error divided by the point in time at which one would hypothetically perform the likelihood ratio test we get

$$\theta = \frac{\sigma f(\alpha, \beta)}{\sqrt{T}(\mu - r) - \sigma f(\alpha, \beta)} \tag{5.33}$$

for $p = 1$ and where $f(\alpha, \beta) = \Phi^{-1}(1-\beta) - \Phi^{-1}(\alpha)$. In general, θ is decreasing in T. If $T \to \infty$, then the agent's uncertainty vanishes and his investment strategy converges to the classical Merton solution. Thus $k \to 0$ and $\theta \to 0$. If the agent becomes extremely uncertain, $T \to 0$, and hence the investment strategy goes to minus infinity. However this short position is ruled out by the condition that $\mu - r - \sigma k > 0$ and $\theta > 0$. Therefore both $k \to \infty$ and $\theta \to \infty$.

If we set the variables as in Table 4 then the associated θ equals 124 (note that this is highly sensitive with respect to changes in $f(\alpha, \beta)$, such as including the decimals). If $T = \{200, 400\}$, then $\theta = \{2.35, 0.98\}$. However, to make a proper comparison we use the same input as in Maenhout (2004), where we take the stock process S only. Thus we use (5). The k implied by

Table 5: Input Variables Maenhout (2004)

Period	$\mu_S - r$	r	σ_S	$\gamma(= p)$	θ
1891-1994	6.258%	1.955%	0.18534	7	14
1947.2-1996.3	7.852%	0.7852%	0.15218	10	237

Parameters used in Maenhout (2004). Originally estimated by Campbell (1999).

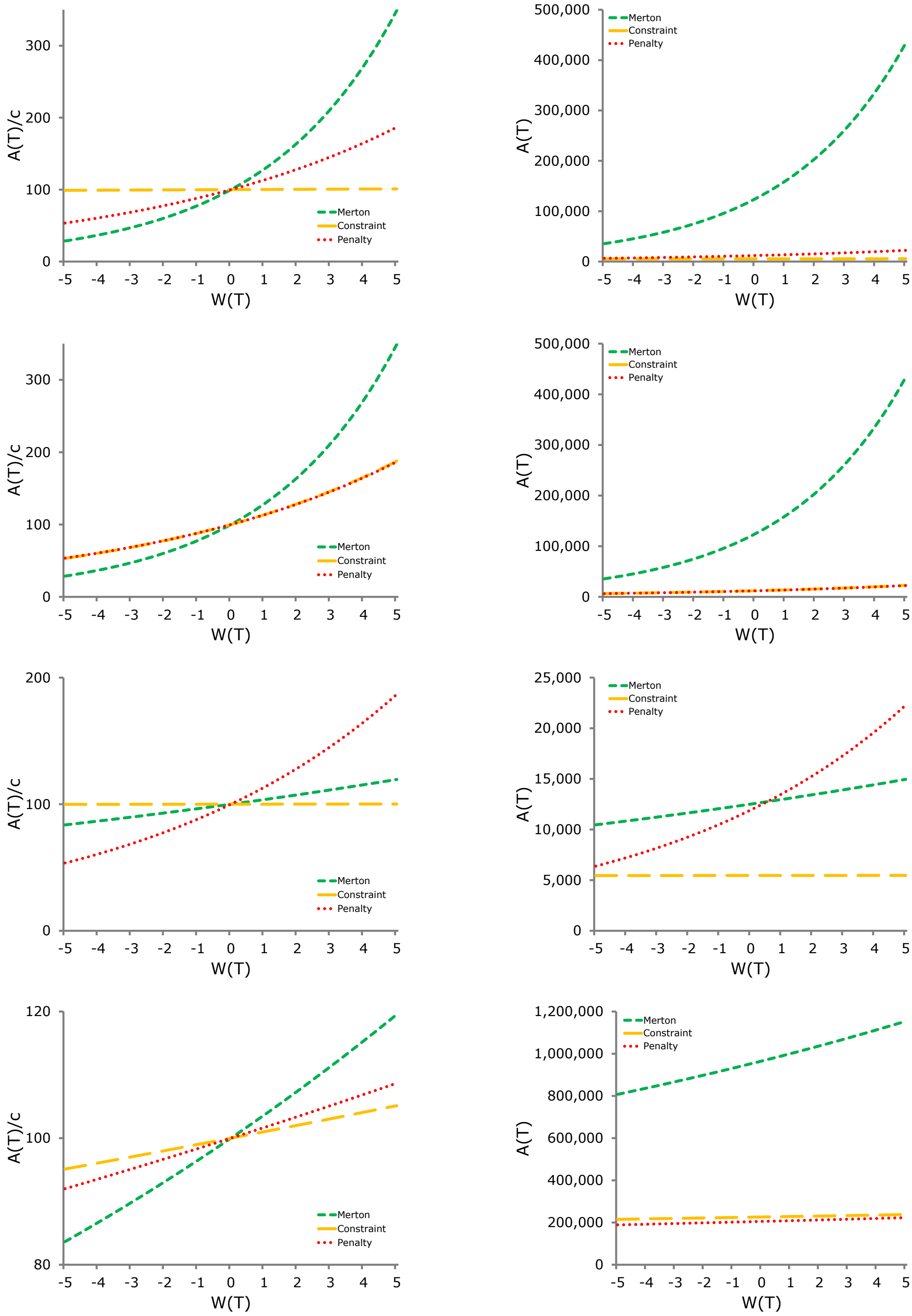
the specification as in Table 5 is 0.2251. Implying a T of 190 years for $\alpha = 5\%$ and $1 - \beta = 80\%$. Or a power of 56%, and $T = 100$ and $\alpha = 5\%$. Thus the θ shown in Maenhout (2004) is based on having 190 of additional information. Or put it differently, the power at which one is able to distinguish between models is lower than the 80%, hence the set of alternative models is a little smaller.

The second and much smaller sampler Maenhout considers yields only 25 years of data that one would need to have this risk and uncertainty aversion. The k is larger than in the standard case where we consider 100 year of extra data. Hence the agent is extremely uncertain. The power at which one would have enough strength to distinguish between models 99%.

In the first sample, the detection error probability implied by the stock process without dividend would be 24.50% versus $\frac{1}{2} \cdot 5\% + \frac{1}{2} \cdot 80\% = 12.5\%$. The detection error probability indicates the difficulty to be able to distinguish properly between the baseline and alternative model. This has a similar intuition as our indistinguishable set by usage of the Type I and II error. However, we derived the distribution of the Radon-Nikodym derivative in its complete general setting. Moreover, we create the set *ex ante*. Henceforth, we use the insufficient power directly instead of calibrating the multiplier. The more difficult it is to distinguish between models, the higher the probability that the likelihood ratio test is incorrect. With other words, the higher the detection error probability the more similar the models are. Therefore H&S suggest a minimum of 10%. The detection error probability is the average of the Type I and Type II error. For an $\alpha = 5\%$ and $\beta = 20\%$, the detection error probability is $\epsilon_N = 0.5p_1 + 0.5p_2 = 12.5\%$. Or reversed, for an α of 5%, the implied power for a detection error probability as suggested by H&S is 85%.

In Maenhout (2004) the Radon-Nikodym derivative is lognormal distributed. Based on experimental data, the θ is calibrated and the associated detection error probability is obtained. However, in Maenhout (2006) they argue that finding the distribution of the Radon-Nikodym derivative is nontrivial because λ is time-varying, therefore they derive the characteristic function. In this paper we are able to find an upperbound on the power for time-consistent Radon-Nikodym derivatives.

Figure 2: Merton Optimal Investment



The figure displays the optimal wealth for the above specified parameters (Table (4)), against possible realisations of the Brownian motion at time T. The classical, robust by constraint and robust by penalty Merton solutions are depicted without $c_{\mathrm{M}}, c_{\mathrm{C}}, c_{\mathrm{P}}$. The graph on the right includes these constants. The figures in the middle are based on an indirect $T = 400$ (constants based on adjusted k, but T kept 100 to visualise differences easier) and thus $k = 12.4\%$ and $\theta = 1$. For the figures in the third row the risk aversion is changed from $p = 1$ to $p = 7$. And for the bottom figures, we set the variables as in Table (5) for the period 1891-1994, with $T = 190$ and $\theta = 14$.

6 Conclusion

To conclude, by imposing probabilities on the Type I and II error of the Radon-Nikodym derivative we are able to quantify uncertainty in an intuitive way. Hence if an agent acknowledges that his model might be misspecified, he would like to evaluate the optimal decision rule against all plausible alternative models to incorporate robustness. Applications that build upon the uncertainty need the specific set of indistinguishable models. Examples can be found on a wide range to price and hedge in incomplete markets, for instance long-dated insurance contracts or illiquid assets.

We used the Neyman-Pearson Lemma to characterise a set of models that cannot be distinguished statistically from a baseline model. Both deterministic and time-consistent stochastic deviations are proven to have maximal power for a lognormal Radon-Nikodym derivative with bounded volatility. Allowing for stochastic alternatives yields a tremendous enlargement of the class of alternative models that will be considered to be indistinguishable. The set of indistinguishable models can explicitly be obtained *ex ante*, for given Type I and II probabilities. The result can be linked to quantify cutoff points on ϕ-divergences such as the Kullback-Leibler divergence. Lastly, we showed the impact on the robust investment problem.

A Variance Hyperbolic Cosine

$$\begin{aligned} H_0 : W(t) \quad &\text{versus} \quad H_A : W(t) + \int_0^t \lambda(s,\omega)ds \\ H_0 : W(T) \quad &\text{versus} \quad H_A : W(T) + \int_0^T \lambda(s,\omega)ds \end{aligned} \tag{A.1}$$

Where $W(T) \sim N(0,T)$. The moment generating function is

$$\begin{aligned} M_{\mathbb{P}}(t) &= \mathbb{E}^{\mathbb{P}}\left[e^{tW(T)}\right] \\ &= e^{\frac{1}{2}Tt^2} \\ \mathbb{E}[W(T)] = \frac{\partial M}{\partial t}(0) &= Tte^{\frac{1}{2}Tt^2}\Big|_{t=0} = 0 \\ \mathbb{E}\left[W(T)^2\right] = \frac{\partial^2 M}{\partial t^2}(0) &= T^2t^2e^{\frac{1}{2}Tt^2} + Te^{\frac{1}{2}Tt^2}\Big|_{t=0} = T \end{aligned} \tag{A.2}$$

And under $\mathbb{Q}$

$$\begin{aligned} M_{\mathbb{Q}}(t) &= \mathbb{E}^{\mathbb{P}}\left[e^{tW(T)}R(T)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[e^{tW(T)}\tfrac{1}{2}\left(e^{-\frac{1}{2}a^2T+aW(T)} + e^{-\frac{1}{2}a^2T-aW(T)}\right)\right] \\ &= \tfrac{1}{2}\left(e^{-\frac{1}{2}a^2T+(a+t)^2\frac{1}{2}T} + e^{-\frac{1}{2}a^2T+(-a+t)^2\frac{1}{2}T}\right) \\ \mathbb{E}[W(T)] = \frac{\partial M}{\partial t}(0) &= \tfrac{1}{2}\left((a+t)Te^{-\frac{1}{2}a^2T+(a+t)^2\frac{1}{2}T}\right) + \\ &\quad \tfrac{1}{2}\left((-a+t)Te^{-\frac{1}{2}a^2T+(-a+t)^2\frac{1}{2}T}\right) \\ &= 0 \\ \mathbb{E}\left[W(T)^2\right] = \frac{\partial^2 M}{\partial t^2}(0) &= \tfrac{1}{2}\left((a(a+t)T^2+T)e^{-\frac{1}{2}a^2T+(a+t)^2\frac{1}{2}T}\right) + \\ &\quad \tfrac{1}{2}\left((-a(-a+t)T^2+T)e^{-\frac{1}{2}a^2T+(-a+t)^2\frac{1}{2}T}\right) \\ &= (aT)^2 + T \end{aligned} \tag{A.3}$$

B Power Hyperbolic Cosine

Under $\mathbb{A}$ the probability distribution of $W(T)$ is $N(aT,T)$. Hence $W(T)/\sqrt{T} \sim \frac{1}{2}N(a\sqrt{T},1) + \frac{1}{2}N(-a\sqrt{T},1)$. Then

$$
\begin{aligned}
\mathbb{A}\left[|W(T)/\sqrt{T}| > \gamma'\right] &= \mathbb{A}\left[W(T)/\sqrt{T} > \gamma'\right] + \mathbb{A}\left[W(T)/\sqrt{T} < -\gamma'\right] \\
\mathbb{A}\left[W(T)/\sqrt{T} > \gamma'\right] &= \tfrac{1}{2}\mathbb{A}\left[W(T)/\sqrt{T} - a\sqrt{T} > \gamma' - a\sqrt{T}\right] + \\
&\quad \tfrac{1}{2}\mathbb{A}\left[W(T)/\sqrt{T} + a\sqrt{T} > \gamma' - a\sqrt{T}\right] \\
&= \tfrac{1}{2}\mathbb{A}\left[W(T)/\sqrt{T} - a\sqrt{T} < -\gamma' + a\sqrt{T}\right] + \\
&\quad \tfrac{1}{2}\mathbb{A}\left[W(T)/\sqrt{T} + a\sqrt{T} < -\gamma' + a\sqrt{T}\right] \\
&= \tfrac{1}{2}\Phi(-\gamma' + a\sqrt{T}) + \tfrac{1}{2}\Phi(-\gamma' - a\sqrt{T}) \\
\mathbb{A}\left[W(T)/\sqrt{T} < -\gamma'\right] &= \tfrac{1}{2}\mathbb{A}\left[W(T)/\sqrt{T} - a\sqrt{T} < -\gamma' - a\sqrt{T}\right] + \\
&\quad \tfrac{1}{2}\mathbb{A}\left[W(T)/\sqrt{T} + a\sqrt{T} < -\gamma' - a\sqrt{T}\right] \\
&= \tfrac{1}{2}\Phi(-\gamma' - a\sqrt{T}) + \tfrac{1}{2}\Phi(-\gamma' + a\sqrt{T})
\end{aligned}
$$

Since $\lambda(t,\omega) = a\tanh(aW(t))$ and $|a\sqrt{T}| < 2.80 \Rightarrow |a| < 2.80/\sqrt{T}$ this can be one-to-one compared with $|\lambda\sqrt{T}| < 2.48 \Rightarrow |\lambda| < 2.80/\sqrt{T} \Rightarrow |a\tanh(aW(t))| < 2.80/\sqrt{T}$. Because $-a < a\tanh(aW(t)) < a$.

C Divergences

C.1 Kullback-Leibler

The Kullback-Leibler divergence (also known as *entropy*) is defined as

$$
D(\mathbb{P}\|\mathbb{A}) = \mathbb{E}^{\mathbb{P}}[-\ln R(T) - 1 + R(T)] = \mathbb{E}^{\mathbb{A}}\left[-\frac{\ln R(T) + 1 - R(T)}{R(T)}\right] \tag{C.1}
$$

We can arrive at a similar conclusion for a maximum entropy (= max K-L) calculation. We want to investigate the maximum entropy that can be achieved within the class of Radon-Nikodym derivatives with $|\lambda(t,\omega)| \le k$. We can formulate this as a stochastic optimisation problem of the form

$$
\begin{aligned}
&\max_{|\lambda(t,\omega)|\le k} \mathbb{E}[-\ln R(T) - 1 + R(T)] \qquad \text{(C.2)}\\
&\text{s.t. } dR = \lambda(t,\omega) R\, dW \\
&\quad R > 0
\end{aligned}
$$

This optimisation problem admits the following HJB representation. If we set $V(t,r) := \mathbb{E}[-\ln R(T) - 1 + R(T) \mid R(t) = r]$, then the optimised value function $V(t,r)$ for $t \le T$ is given by the HJB-equation

$$
V_t + \max_{|\lambda(t,r)|\le k} \tfrac{1}{2}\lambda(t,r)^2 r^2 V_{rr} = 0 \tag{C.3}
$$

The terminal condition $V(T,R) = -\ln R - 1 + R$ is a convex payoff in R. Hence we propose $|\lambda(t,\omega)| = k$ to solve the HJB. The implied value function for the candidate solution is

$$V(t,r) \quad = \quad -\ln r + \tfrac{1}{2}k^2(T-t) - 1 + r \tag{C.4}$$

This function solves the HJB equation for the boundary condition and maximises the objective in (C.3) as V is convex. Similarly as in the maximum power calculation in section 3.2, the value function $V(t,r)$ is uniformly integrable and thus the optimal control is $|\lambda(t,\omega)| = k$. We arrive at the conclusion that the lognormal Radon-Nikodym derivative with $|\lambda(t,\omega)| \equiv k$ for all $0 \le t \le T$ achieves the maximal entropy of $V(0,1) = \frac{1}{2}k^2T$. Hence, we can alternatively characterise the class of time-consistent indistinguishable models with $|\lambda(t,\omega)| \le 2.48/\sqrt{T}$, which implies a maximum attainable entropy of $\frac{1}{2}(2.48)^2 = 3.08$.

C.2 Burg Entropy

Burg Entropy or also called Minimum Discrimination Information is defined as

$$D(\mathbb{A}\|\mathbb{P}) = \mathbb{E}^{\mathbb{A}}[\ln R(T) + \frac{1}{R(T)} - 1] = \mathbb{E}^{\mathbb{P}}[R(T)\ln R(T) + 1 - R(T)] \tag{C.5}$$

We can formulate this as a stochastic optimisation problem of the form

$$\begin{aligned} &\max_{|\lambda(t,\omega)|\le k} \mathbb{E}[R(T)\ln R(T) + 1 - R(T)] \\ &\text{s.t. } dR = \lambda(t,\omega) R\, dW \end{aligned} \tag{C.6}$$

This optimisation problem admits the following HJB representation. If we set $V(t,r) := \mathbb{E}[R(T)\ln R(T) + 1 - R(T) \mid R(t) = r]$, then the optimised value function $V(t,r)$ for $t \le T$ is given by the HJB-equation

$$V_t + \max_{|\lambda(t,R)|\le k} \tfrac{1}{2}\lambda(t,r)^2 r^2 V_{rr} = 0 \tag{C.7}$$

As the terminal condition $V(T,R) = R\ln R + 1 - R$ is a strictly convex function in R, we arrive at the conclusion that the lognormal Radon-Nikodym derivative with $|\lambda(t,\omega)| \equiv k$ for all $0 \le t \le T$ achieves the maximal Burg entropy of $V(0,1) = \frac{1}{2}k^2T$ since for $q \in \mathbb{R}$

$$\begin{aligned} \mathbb{E}\left[R(T)^q \,\middle|\, \mathscr{F}_t\right] &= \mathbb{E}\left[r^q e^{-\frac{1}{2}qk^2(T-t)+qk(W^{\mathbb{P}}(T)-W^{\mathbb{P}}(t))} \,\middle|\, \mathscr{F}_t\right] \\ &= r^q e^{\frac{1}{2}k^2(T-t)(q^2-q)} \end{aligned} \tag{C.8}$$

The derivative with respect to q on both sides leads to

$$\begin{aligned} \frac{\partial \mathbb{E}\left[R(T)^q \,\middle|\, \mathscr{F}_t\right]}{\partial q} &= \mathbb{E}\left[R(T)^q \ln R(T) \,\middle|\, \mathscr{F}_t\right] \\ \frac{\partial r^q e^{\frac{1}{2}k^2(T-t)(q^2-q)}}{\partial q} &= r^q \ln r e^{\frac{1}{2}k^2(T-t)(q^2-q)} + r^q e^{\frac{1}{2}k^2(T-t)(q^2-q)} \tfrac{1}{2}k^2(T-t)(2q-1) \end{aligned} \tag{C.9}$$

For $q = 1$ we get $V(t,r) = r\left(\ln r + \frac{1}{2}k^2(T-t)\right) + 1 - r$. Hence, we can alternatively characterise the class of time-consistent indistinguishable models with $|\lambda(t,\omega)| \le 2.48/\sqrt{T}$, which implies a maximum attainable Burg entropy of $\frac{1}{2}(2.48)^2 = 3.08$.

C.3 J-Divergence

Jeffreys (1946) J-divergence is $D(\mathbb{P}\|\mathbb{A})+D(\mathbb{A}\|\mathbb{P})$ which equals $\mathbb{E}^{\mathbb{P}}[-\ln R(T)]+\mathbb{E}^{\mathbb{P}}[R(T)\ln R(T)]$. We can formulate this as a stochastic optimisation problem of the form

$$\begin{aligned} \max_{|\lambda(t,\omega)|\le k} \; & \mathbb{E}[-\ln R(T)] + \mathbb{E}[R(T)\ln R(T)] \\ \text{s.t. } & dR = \lambda(t,\omega) R dW \\ & R > 0 \end{aligned} \tag{C.10}$$

Again the associated value function is convex in R at terminal time T, $V(T,R) = (R-1)\ln R$. The lognormal candidate leads to $V(t,r) = (r-1)\ln r + \frac{1}{2}k^2(T-t)(r+1)$, and hence the maximum J-divergence is obtained for $V(0,1) = k^2 T = (2.48)^2 = 6.15$.

C.4 χ^2-Distance

Actually for all $\frac{\partial^2\phi(R)}{\partial R^2} \ge 0$ the associated value function is convex for $R > 0$ and hence $|\lambda(t,\omega)| = 2.48/\sqrt{T}$ leads to the maximum attainable distance. The $\mathbb{P}$-expectation can be obtained from plugging in

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[R(T)\phi\left(\frac{1}{R(T)}\right)\right] &= \mathbb{E}^{\mathbb{P}}\left[R(T)^2 - 2R(T) + 1\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[R(T)^2\right] - 2\mathbb{E}^{\mathbb{P}}[R(T)] + 1 \end{aligned} \tag{C.11}$$

For the associated optimisation problem the HJB representation is similar as the ones before. If we set $V(t,r) := \mathbb{E}\left[R(T)^2 - 2R(T) + 1 \,\middle|\, R(t) = r\right]$, then the terminal condition $V(T,R) = R^2 - 2R + 1$ is convex. For $|\lambda(t,\omega)| = k$ the value function is

$$\begin{aligned} V(t,r) &= r^2 e^{k^2(T-t)} - 2r + 1 \\ V(0,1) &= e^{2.48^2} - 1 = 467.90 \end{aligned} \tag{C.12}$$

C.5 Modified χ^2-Distance

The objective function of the modified χ^2-distance under $\mathbb{P}$ is

$$\mathbb{E}^{\mathbb{P}}\left[R(T)\phi\left(\frac{1}{R(T)}\right)\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{R(T)} - 2 + R(T)\right] \tag{C.13}$$

The value function is $V(t,r) = \mathbb{E}^{\mathbb{P}}\left[\frac{1}{R(T)} - 2 + R(T) \,\middle|\, R(t) = r\right]$ then the terminal condition is convex in R. For the lognormal Radon-Nikodym derivative the modified χ^2-divergence bounded by

$$\begin{aligned} V(t,r) &= \frac{1}{r}e^{k^2(T-t)} - 2 + r \\ V(0,1) &= e^{k^2 T} - 1 = 467.90 \end{aligned} \tag{C.14}$$

represents the set of indistinguishable models for a probability of 5% for a Type I error occurring and all models yield a power of at most 80%. The derivation is obtained by applying Ito's Lemma

$$d\left(\frac{1}{R(t)}\right) = \frac{1}{R(t)}\left(k^2 dt - kW^{\mathbb{P}}(t)\right) \tag{C.15}$$

C.6 Hellinger Distance

The Hellinger distance equals

$$\mathbb{E}^{\mathbb{P}}\left[R(T)\phi\left(\frac{1}{R(T)}\right)\right] = \mathbb{E}^{\mathbb{P}}\left[1-2\sqrt{R(T)}+R(T)\right] \tag{C.16}$$

The associated value function is $V(t,r)=\mathbb{E}^{\mathbb{P}}\left[1-2\sqrt{R(T)}+R(T) \mid R(t)=r\right]$ Since

$$\begin{aligned} \sqrt{R(T)} &= \sqrt{R(t)}e^{-\frac{1}{4}k^2(T-t)+\frac{1}{2}k(W^{\mathbb{P}}(T)-W^{\mathbb{P}}(t))} \\ d\sqrt{R(T)} &= -\tfrac{1}{8}k^2\sqrt{R(t)}dt+\tfrac{1}{2}k\sqrt{R(t)}dW^{\mathbb{P}}(t) \end{aligned} \tag{C.17}$$

The optimal control solves for $|\lambda(t,\omega)|=k$

$$\begin{aligned} V(t,r) &= 1-2\sqrt{r}e^{-\frac{1}{8}k^2T}+r \\ V(0,1 &= 1.07 \end{aligned} \tag{C.18}$$

C.7 Variation Distance

The variation distance can be decomposed into a call and put option and is defined as follows

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[R(T)\phi\left(\frac{1}{R(T)}\right)\right] &= \mathbb{E}^{\mathbb{P}}\left[\left|\frac{1}{R(T)}-1\right|R(T)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\max\left(\frac{1}{R(T)}-1,0\right)R(T)\right]+ \\ &\quad\ \mathbb{E}^{\mathbb{P}}\left[\max\left(1-\frac{1}{R(T)},0\right)R(T)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\max(1-R(T),0)\right]+\mathbb{E}^{\mathbb{P}}\left[\max(R(T)-1,0)\right] \end{aligned} \tag{C.19}$$

The objective is convex by convexity of the max-operator. The associated value function is $V(t,r)=\mathbb{E}^{\mathbb{P}}\left[\left|\frac{1}{R(T)}-1\right|R(T) \;\middle|\; R(t)=r\right]$. Therefore we propose $dR(t)=kR(t)dW^{\mathbb{P}}(t)$ for the value function

$$\begin{aligned} V(t,r) &= N(d_1)r-N(d_2)+N(-d_2)-N(-d_1)r \\ d_1 &= \frac{1}{k\sqrt{T-t}}\left(\ln r+\tfrac{1}{2}k^2(T-t)\right) \\ d_2 &= d_1-k\sqrt{T-t} \end{aligned} \tag{C.20}$$

The second derivative is positive thus the optimal value is obtained for

$$\begin{aligned} V(0,1) &= N(d_1)-N(d_2)+N(-d_2)-N(-d_1) \\ d_1 &= \tfrac{1}{2}k\sqrt{T} \\ d_2 &= -d_1 \\ N(-d_1) &= 1-N(d_1) \\ V(0,1) &= 4N(d_1)-2 \end{aligned} \tag{C.21}$$

For $d1=\frac{1}{2}k\sqrt{T}=1.24$, the bound on the variation distance is $V(0,1)=1.57$.

C.8 χ-Divergence of Order $\theta > 1$

For general θ strictly above 1 the χ-divergence can only be solved numerically, though for integers analytically. The divergence functions is defined by

$$\mathbb{E}^{\mathbb{P}}\left[R(T)\phi\left(\frac{1}{R(T)}\right)\right] = \mathbb{E}^{\mathbb{P}}\left[\left|\frac{1}{R(T)}-1\right|^{\theta}R(T)\right] \tag{C.22}$$

For different values of $\theta = \{1.5, 2, 2.5, 3\}$ and $k^2T = 2.48^2$, we integrate

$$\mathbb{E}^{\mathbb{P}}\left[\left|\frac{1}{R(T)}-1\right|^{\theta}R(T)\right] = \int_{-\infty}^{\infty}\left|e^{\frac{1}{2}k^2T-k\sqrt{T}z}-1\right|^{\theta}e^{-\frac{1}{2}T+k\sqrt{T}z}n(z)dz$$

If $\theta = 2$ this coincides with modified χ^2-Divergence.

C.9 Cressie-Read

The Cressie-Read divergence is

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[R(T)\phi\left(\frac{1}{R(T)}\right)\right] &= \mathbb{E}^{\mathbb{P}}\left[\frac{1-\theta+\theta\frac{1}{R(T)}-\left(\frac{1}{R(T)}\right)^{\theta}}{\theta(1-\theta)}R(T)\right] \\ &= \frac{1}{\theta(1-\theta)}\mathbb{E}^{\mathbb{P}}\left[R(T)-\theta R(T)+\theta-R(T)^{1-\theta}\right]\end{aligned} \tag{C.23}$$

This is convex in r therefore if $R(T)$ is lognormal distributed with volatility k,

$$V(t,r) = \frac{1}{\theta(1-\theta)}\left(r-\theta\cdot r+\theta-r^{1-\theta}e^{-\frac{1}{2}k^2\theta(1-\theta)(T-t)}\right) \tag{C.24}$$

At time 0 this equals

$$\begin{aligned}V(0,1) &= \frac{1}{\theta(1-\theta)}\left(1-e^{-\frac{1}{2}k^2\theta(1-\theta)T}\right) \\ &= \frac{1}{\theta(1-\theta)}\left(1-e^{-3.08\theta(1-\theta)}\right) \\ &= \frac{1}{\theta(1-\theta)}\left(1-0.05e^{\theta(1-\theta)}\right)\end{aligned} \tag{C.25}$$

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