Confirming signals are hard to resist: Blessing and curse of information under confirmation bias

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Abstract

Ample empirical evidence documents that human beings pay more attention to confirming than to contradicting evidence. This paper takes a closer look at the effects of the confirmation bias on financial markets in an overlapping generations model with a continuous signal distribution. We find that several results of more simplified models do not hold true any more. Confirmation bias leads to initial underreaction. In a framework with endogenous market entry and exit, this initial underreaction, however, triggers long-term overreaction. Other findings include a connection between trading volume and volatility, momentum effect, a time variation in market participation and novel varying responses on trading experience, as well as various announcement day and market depth effects. Moreover, we provide testable hypotheses such as that managers inform about bad news in a more diffuse signal compared to good news and that the frequency of information release matters.

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1 Introduction

Confirmation bias is a dominant cognitive bias that is empirically well-documented. Cognitive biases were first introduced by Tversky and Kahneman (1974) and the confirmation bias results from agents' willingness to reduce discomforting experience through cognitive dissonance. Said differently, agents that suffer from confirmation bias have the tendency to search for or interpret information in a way that confirms their prior beliefs and may even completely discredit contradictory information (e.g. Azzopardi (2012)). Despite the empirical interest it raised, our current understanding of confirmation bias is mainly focused on very young market setups and few period models in which exogenous priors play a key role.

We contribute to the literature by investigating confirmation bias in financial markets and specifically in a mature market setup, in which signals are continuous. Moreover, we consider endogenous market entry and exit in an overlapping generations model. The avoidance of the simplifying assumptions prevalent in current literature changes the effects of confirmation bias significantly and the variety of testable predictions. One example among others of such a change is the effect, that confirmation bias leads to initial underreaction and positive serially correlated returns, compared to overreaction in traditional literature.

We consider two aspects of confirmation bias, which can be found in the empirical literature. First, agents choose the public signal that is in consonance with their prior belief (e.g. Nickerson (1998) and Azzopardi (2012)). Second, agents tend to underestimate the mean-reversion in processes, so the expected change in future beliefs (Beshears et al. (2013)). In this paper, we investigate both aspects of confirmation bias, which interplay as a new source of mispricing and consecutive over-correction.

Starting with a representative agent model, we find that confirmation bias leads to initial underreaction to new information. Agents are conservative and focus on confirming information. Thus, they concentrate on signals that mitigate the effects of changes in the fundamental value. Interestingly, in an overlapping generations model, especially with endogenous market entry and exit, this initial underreaction triggers subsequent overreaction. Due to the resulting mispricing in the case of changes in the fundamental value, new, so far unbiased, investors have an incentive to enter. Becoming confirmation seeking when entering the market, the new signal becomes overrepresented in the entering agents expectations. As older agents exit the market due to an underestimation of the signal's relevance, we observe an underreaction, which finally results in an overreaction. A consequence is momentum in stock returns especially around earnings announcement days. This post earnings announcement drift, as first documented by Ball and Brown (1968), shows that firms reporting unexpectedly high earnings subsequently outperform firms reporting unexpectedly low earnings.

Second, the investor population changes over time and especially around announcement days. The market entry due to mispricing incentivises new investors to enter. Additionally, due to fading mispricing, older investors suffer from worse performance and as a result leave the market. Thus, after announcement days the investor population becomes younger, less experienced and less biased.

Around announcement days, we observe an enhanced belief dispersion, which is in line with the findings of Morse et al. (1991), among others. It results from underreaction and the resulting mispricing triggering, new disagreeing individuals to enter the market. As a result, beliefs among trading individuals differ more after changes in the fundamental. This phenomenon is intensified by the confirmation bias. Indeed, the model demonstrates that confirmation bias increases belief dispersion around announcement days and also on a general level. Such an effect is well-known in the context of confirmation bias as the attitude polarization or belief polarization (Lord et al. (1979), Barberis and Thaler (2003) and Taber and Lodge (2006)). The arrival of new information may widen, instead of narrow, the disagreement between the biased agents, as each of them looks at news in a way to reinforce their own beliefs.

Several predictions of the model are in line with market stylized facts. The model exhibits volatility clustering (Mandelbrot (1963)), momentum (Jegadeesh and Titman (1993)), positive correlation between trading volume and volatility (Lamoureux and Lastrapes (1990)) as well as the tendency of biased agents to trade more frequently and to obtain lower investment performance (Park et al. (2013)).

Finally, we pay attention to the impact of confirmation bias on the optimal information disclosure policy of managers. We find that in the case of bad news, managers have an incentive to disclose the news in a more diffuse signal than in the case of good news. It results from the fact, that in the case of a diffuse signal, confirmation biased investors have an opportunity to focus on those signals that reduce the impact of the information on firm value. In contrast, in the case of a precise signal confirmation biased investors have no possibility for an alternative interpretation. Additionally, the frequency of information disclosure play a role in the dispersion of agents' beliefs.

So far, only few papers investigate the confirmation bias in the context of financial markets. Pouget et al. (2014) builds on Rabin and Schrag (1999) and proposes a dynamic asset pricing model in which a part of the traders is subject to the bias. They provide rationale for excess volatility, excess volume and momentum in financial markets. However, several of their major results, such as overreaction and bubbles, are primarily driven by the simplifying assumption that both the signals as well as the final states of the world are Bernoulli distributed. Our paper takes a closer look at these results in the context of continuous signal distributions. In addition, Park et al. (2013) conduct an empirical investigation on the bias. Interestingly, they study how stock message boards influence investors trading decisions and investment performance within a field experiment in South Korea. They show that investors use message boards to seek information that confirms their prior beliefs. Moreover, the confirmation bias tends to make investors overconfident, which affects their investment performance. Confirmation bias is thus likely to contribute to overconfidence (Park et al. (2013) and Barber and Odean (2001)). Hence, the paper is as well associated to several studies on the effect of overconfidence on financial markets; specifically to Kyle and Wang (1997), Odean (1998), Benos (1998) and Garcia et al. (2007).

The paper is related to overlapping generations model, such as e.g. Kubler and Schmedders (2015), Gârleanu and Panageas (2014), and Albagli (2015). Compared to these papers our focus is not on the heterogeneity resulting from investment horizon or wealth effects, but on the differences in their priors when entering the market. Another paper which includes the effects of differences in experience on investment decisions is Schraeder (2015). However, her paper focuses on the effects of availability bias and all signals after market entrance are weighted equally.

Hence, to the best of our knowledge, this paper is the first to include confirmation bias in an overlapping generations model together with endogenous market entry and exit. Again, this allows us to better capture difference between generations and additionally investigate several other empirical model predictions.

The remainder of the paper is structured as follows. Section 2 provides a detailed description of the model considering only one generation of agents. We focus on the impact of the confirmation bias on individual expectations and solve the equilibrium model. Section 3 extends the model to a multi-generation setup. We consider a fixed lifespan and then we endogenize both individual market entry as well as market exit decision. Section 4 matches our model prediction with empirical evidence. Finally, Section 5 concludes.

2 One-generation model

2.1 Confirmation bias and signal processing

We consider an economy with one risky and one riskless asset. The riskless asset is in perfectly elastic supply and generates a constant return r_f. The risky asset pays dividends. As the confirmation bias is related to the effect of new information on belief formation, we distinguish between non-announcement days on which dividends are normally distributed around the mean, and announcement days (high-volatility days), in which the dividend mean is subject to change. The dividend process is defined by the following set of equations

$$d_t = x_t + \sigma_d \epsilon_{d,t}, \tag{1}$$

$$x_t = x_{t-1} + \mathbf{1}_{t \in I} \cdot \left[\kappa(\mu - x_{t-1}) + \sigma_x \epsilon_{x,t} \right]. \tag{2}$$

κ captures the mean reversion in the fundamental dividend and σ_x captures the change in the mean dividend throughout event times I, which are announcement days.

At each point in time t, agents receive a number $i = 1, .., n_s$ of independent signals about the mean of the dividend stream, which follows a normal distribution

$$s_{i,t} \sim N\left(x_t, \sigma_d^2\right).$$

The average of these n_s signals equals to the dividend, so that in principal it would be sufficient to focus on the mean signal as the only source of information

$$d_t = \frac{1}{n_s}\sum_{i=1}^{n_s} s_{i,t} = \bar{s}_t.$$

Rational agents filter this process using the Kalman filter procedure described in Appendix A.1 and we denote by $\hat{x}_t^a$ the *a posteriori* estimate of the fundamental part of the dividend x_t made by an agent a. On non-announcement days, agent a's prior expectation is

$$\mathbb{E}_{t-1}^a [x_t] = \hat{x}_{t-1}^a,$$

and on announcement days, agents a's prior expectation equals

$$\mathbb{E}_{t-1}^a [x_t] = (1-\kappa)\hat{x}_{t-1}^a + \kappa\mu.$$

In our model agents are not aware of their bias and assume to be rational and act accordingly. Thus, when updating their model, agents believe to estimate the joint system of Equations (1) and (2). They try to optimally filter the information out of their observed signals, but commit the mistake of confirmation bias. According to recent empirical and theoretical evidence, agents subject to confirmation bias selectively focus on information, which is in line with their prior belief and are unable to filter mean-reversion. Analytically, the signal most fitting to the previous perceptions equals[1]

$$s_t^{m,a} = \text{argmin}_i \left| s_{i,t} - \hat{x}_{t-1}^a \right|.$$

Investors do not resist to reducing their emotional cost of changing their percep-

[1] In this specification agents do not like to change their mind, independently of whether it is an unexpected shock or mean-reversion (compare Beshears et al. (2013)). An alternative specification, only taking unexpected shocks is $s_t^{m,a} = \text{argmin}_i \left| s_{i,t} - E_{t-1}^a [x_t] \right|$.

tion when updating in the Kalman procedure. Agents subject to confirmation bias replace the rational measurement by a weighted average of the rational estimate and the dividend measurement, which is closest to their initial estimate

$$d_t^{c,a} = \frac{\bar{s}_t + c\ s_t^{m,a}}{1+c},$$

where c is the confirmation bias parameter. When $c = 0$, agents correctly update with the dividend process d_t. The higher is c, the more an agent is said to be subject to confirmation bias, and the higher is the weight she puts on the signal which is closest to his prior belief.

2.2 Utility and prices with one generation

In this economy, each agent maximizes exponential utility of the next period wealth given his perception of the price process

$$\max \mathbb{E}_t^a \left[U(w_{t+1}^a) \right] = \max \mathbb{E}_t^a \left[1 - \exp(-\gamma w_{t+1}^a) \right], \tag{3}$$

where γ is the absolute risk aversion coefficient and subject to the budget constraint

$$w_{t+1}^a = w_t^a \cdot (1 + r_f) + y_t^a \left(p_{t+1} + d_{t+1} - (1 + r_f) p_t \right). \tag{4}$$

To determine the price process, we adopt the so-called *guess and verify methodology.* Under the assumption that p_{t+1} is normally distributed, w_{t+1}^a follows a normal distribution. The first order optimality condition results in

$$y_t^a = \frac{\mathbb{E}_t^a \left[p_{t+1} + d_{t+1} \right] - (1 + r_f) p_t}{\gamma Var_t^a (p_{t+1} + d_{t+1})}. \tag{5}$$

Using the market clearing condition ($y_t^a = 1$), we obtain the current market price as a function of next-time period price

$$p_t^j = \frac{\mathbb{E}_t^a \left[p_{t+1}^{j+1} + d_{t+1}\right] - \gamma Var_t^a(p_{t+1}^{j+1} + d_{t+1})}{1 + r_f}, \quad j \in \mathbb{Z}/n\mathbb{Z}, \tag{6}$$

where n indicates the frequency at which high-volatility days occur in this economy. In the following, we assume $n = 10$.[2] If $j = \bar{n} = \bar{0}$, there is a high-volatility day, and thus j days passed since the last high-volatility day. In mature markets, *a priori* and *a posteriori* estimates error covariance of the Kalman filter only depend on the position within the cycle of announcement days j, but no longer on t, as the initial uncertainty has vanished.

Following, the *guess and verify* technique, we conjecture that for all $j \in \mathbb{Z}/n\mathbb{Z}$, the price process has the following linear structural form

$$p_t = a^j + b^j \hat{x}_t^a.$$

The parameters a and b, depend on where time is located relatively to the business cycle j; as there is more risk around announcement days and the sensitivity to new information increases on and directly after announcement days. Please, refer to Appendix A.2 for an analytical derivation of the exact values of a^j and b^j.

This model allows us to investigate the effects of confirmation bias on prices, portfolio returns and volatility, as well as the impacts of the number of signals in Section 4. A dominant feature of this model is underreaction to new information. However, in the current version the model does not include differences in beliefs due to different starting points of the confirmation bias. Depending on when agents enter the market, they start their process with different beliefs. Thus, in the next section, we extend the model to account for overlapping generations and endogenous

[2] We also generalize the procedure for any n.

entry and exit.

3 Overlapping generations model

In this extended version of the model, agents enter at different points in time, leading to different priors. We assume that every agent enters the market with rational expectations. The market entrance determines the point in time in which agents start searching for stable perceptions and replace $\bar{s}_t$ by $d_t^{c,a}$. This model has several implications for well-known market characteristics. Newly entering agents disagree most with respect to the interpretation of recent signals especially around high volatility days. Moreover, those agents entering in extreme situations do not change their interpretation significantly after market entrance and, therefore, continue holding extreme perceptions. In this sense the initial underreaction leads to terminal overreaction especially in the case of endogenous market entry and exit.

3.1 Fixed lifespan

We consider agents living for m periods and we label generations according to their point of entry. For example, at time t generation $a = t - m$ up to generation $a = t$ are actively trading in the market as illustrated in Figure 1.

In a first step, we consider myopic agent, who just maximize utility out of next period wealth. Hence, the individual optimization problem follows the Equations (3), (4) and (5). In a second step, we use the solution technique as described in Schraeder (2015), in which agents believe to have drawn the right conclusions out of their observations. Thus, they consider deviations from their perceived rational price

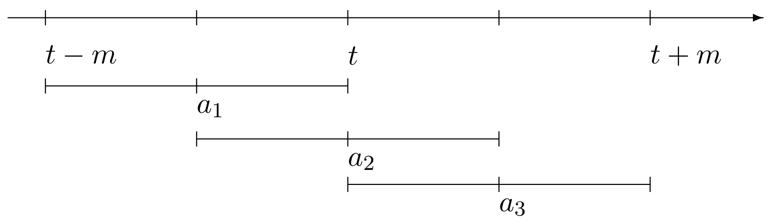


Figure 1: **Overlapping generation.** At each point in time one generation enter the market and lives for m periods. As a result, there are always m different generations actively trading in the market.

(which equals Equation (6)), as noise trading. With the market clearing condition

$$1 = \frac{1}{m}\sum_{i=1}^{m} y_t^{t-i}, \tag{7}$$

and the individual's optimal portfolio holdings for $j = n-1$

$$y_t^a = \frac{a^{j+1} + (b^{j+1}+1)((1-\kappa)\hat{x}_t^a + \mu\kappa) - (1+r_f)p_t^j}{\gamma\left[(1+b^{j+1}K^{j+1})^2 \cdot (P_{t+1}^- + \sigma_d^2)\right]},$$

and for $j \neq n-1$

$$y_t^a = \frac{a^{j+1} + (b^{j+1}+1)\hat{x}_t^a - (1+r_f)p_t^j}{\gamma\left[(1+b^{j+1}K^{j+1})^2 \cdot (P_{t+1}^- + \sigma_d^2)\right]}.$$

We obtain the market realizing price process for $j \neq n-1$,

$$p_t^j = \frac{a^{j+1} + (1+b^{j+1})\frac{1}{m}\sum_{i=1}^{m}\hat{x}_t^{t-i} - \gamma[(1+b^{j+1}K^{j+1})^2(P_{t+1}^- + \sigma_d^2)]}{(1+r_f)},$$

and for $j = n-1$,

$$p_t^j = \frac{a^{j+1} + (1+b^{j+1})\left(\kappa\mu + (1-\kappa)\frac{1}{m}\sum_{i=1}^{m}\hat{x}_t^{t-i}\right) - \gamma[(1+b^{j+1}K^{j+1})^2(P_{t+1}^- + \sigma_d^2)]}{(1+r_f)}.$$

P_t^- is the *a priori* estimation error covariance of the state estimate. K_t is the

Kalman gain that minimises the *a posteriori* estimation error covariance P_t of the state estimate.[3] Parameters a and b are defined as in the previous section.

The realizing price process aggregates all the different perceptions about the fundamental process x. For each generation, the market conditions around their market entrance determines their prior, which significantly influences which information is considered to be in line with their perception. There is a mixed effect of recent information on the realizing price. For those generations, who entered after the information is revealed the information is overly influential, as they ignore mean reversion, and, therefore, overestimate the longtime importance of their information. In contrast agents that have already been in the market around that time, were influenced by confirmation bias when evaluating the different signals transferring the information. Thus, they undervalue the importance of contrary information. Which effect dominates depends on the extend to which a signal is confirming the prevailing average beliefs. Another influencing factor is the relative strength of the cohorts, who consider the signal to be confirmatory. So far, all the generations were equally present in the market. However, the presence of different generations should depend on the profit they can make in the market by entering. This is what we are doing in the following section.

3.2 Endogenous market entry and exit

In this section, we extend the overlapping generations model to an environment in which agents face market entry costs and, therefore, market entry becomes an endogenous decision[4]. The benefits of market entry, resulting from mispricing, in-

[3]Please, refer to Appendix A.1 for analytical details on the Kalman procedure.

[4]As an extension, we work on assuming that agents have to pay fixed costs f to enter the market. Thus, agents are willing to enter the market only, if the market realizing price differs from their prior to such a degree that the expected future profits are higher than the cost of market entrance f.

creases in the deviation of the average belief in the market (reflected in prices) from the rational expectation, held by entering investors. Entering investors have formed rational prior beliefs about the fundamental value. We assume that they are not (or at least less) affected by cognitive dissonance bias as long as they are not yet involved in the market. Figure 2 illustrates the intraperiod timeframe of the endogenous market entry and exit.

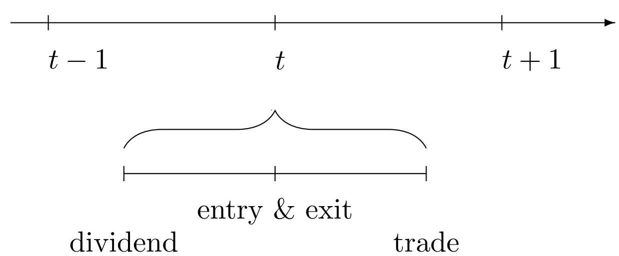


Figure 2: **Intraperiod timeframe of market events.** In one period, first dividends are paid out. Thus, time t dividends are paid to investors who bought the stock at time $t-1$. Then, agents enter end exit the market, before investors start trading. Thus, the periods of market entrance is the first period of trading. Market exit happens one period after the last active trade.

Therefore, the percentage of the currently inactive investors belonging to generation $t-i$ entering at time t equals

$$\eta_t^{t-i} = 1 - \exp\left(-\upsilon \left| \hat{x}_t^{t-i} - \frac{1}{m} \sum_{j=1}^{m} \hat{x}_t^{t-j} \right| \right).$$

If the average market perception, excluding the agent one, (second term in the exponential) deviates from the individual's perception (first term in the exponential), agents tend to enter the market. In contrast, if one generation's perceptions converge towards market perception, market participation converges to zero. Accumulating market entry over time, after one time period, the percentage of agents of one generation having entered the market equals at time t

$$\text{entry}_t^{t-i} = \text{entry}_{t-1}^{t-i} + (1 - \text{entry}_{t-1}^{t-i}) \cdot \eta_t^{t-i}.$$

We also include the possibility of market exit in our model. The probability that agents exit the market decreases with the utility they generate out of their investments. Hence, the number of currently active investors in cohort a at time t decreases by

$$\xi_t^{t-i} = 1 - h \cdot U(w_t^{t-i}),$$

with $U(w)$ being as defined in Equation (3) and h a parameter. Thus, the percentage of agents, who after an initial market entry have left the market because of the preferences, equals

$$\text{exit}_t^{t-i} = \text{exit}_{t-1}^{t-i} + (\text{entry}_{t-1}^{t-i} - \text{exit}_{t-1}^{t-i}) \cdot \xi_t^{t-i}$$

As a result, the percentage of one generation belonging to the active trading population equals

$$\pi_t^{t-i} = \text{entry}_t^{t-i} - \text{exit}_t^{t-i}$$

The population of young agents is high in times of higher disagreement and deviation from the fundamental, which is the case if there have been recent shocks. This is also the time in which older agents leave the market due to poor prior performance. For a more detailed description of the participation dynamics can be found in Section 4.3.

The time-varying active trading population has an impact on the market clearing mechanism, because only the active trading population directly influences prices through trade. Thus, the market clearing condition adapts to

$$1 = \frac{\sum_{i=1}^{m} \pi_t^{t-i} \cdot y_t^{t-i}}{\sum_{i=1}^{m} \pi_t^{t-i}}. \tag{8}$$

.

This modification affects realizing market clearing prices. Now, not only changes in portfolio holdings itself, but also market entry and exit impact prices.

For prices one day, prior to non-announcement days ($j \neq n-1$), the pricing equation equals

$$p_t^j = \frac{a^{j+1} + (1+b^{j+1}) \left(\frac{1}{\Pi_t} \sum_{i=1}^{m} \pi_t^{t-i} \hat{x}_t^{t-i} \right) - \gamma[(1+b^{j+1}K^{j+1})^2 (P_{t+1}^{-} + \sigma_d^2)]}{(1+r_f)}.$$

For days prior to announcement days ($j = n-1$), the pricing equation equals

$$p_t^j = \frac{a^{j+1} + (1+b^{j+1}) \left(\kappa\mu + (1-\kappa) \frac{1}{\Pi_t} \sum_{i=1}^{m} \pi_t^{t-i} \hat{x}_t^{t-i} \right)}{(1+r_f)} - \frac{\gamma[(1+b^{j+1}K^{j+1})^2 (P_{t+1}^{-} + \sigma_d^2)]}{(1+r_f)}.$$

In these equations, $\Pi_t = \sum_{i=1}^{m} \pi_t^{t-i}$ equals the total market participation. Likewise, P_t^- is the *a priori* estimation error covariance of the state estimate and K_t is the Kalman gain. Parameters a and b are defined as previously.

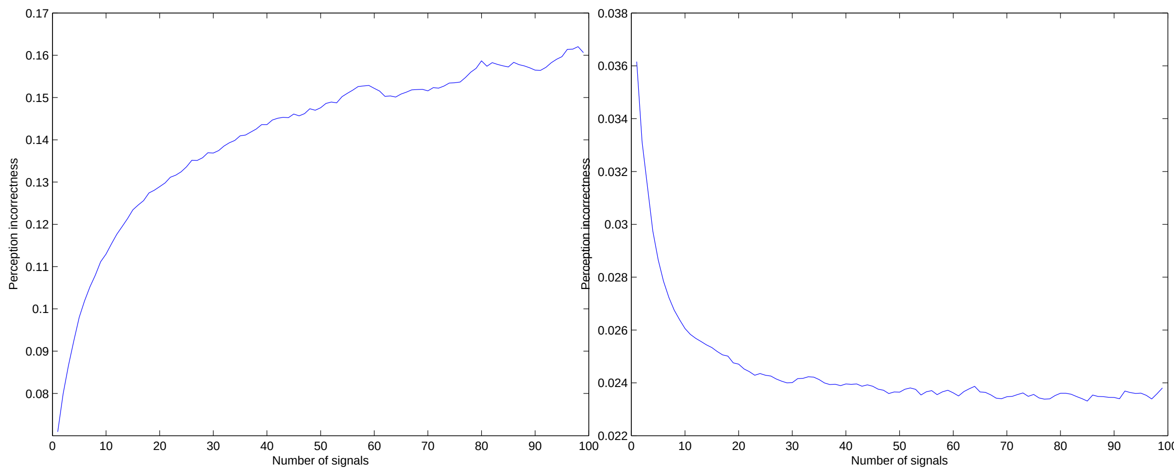
4 Effects of confirmation bias

In this section, we give market predictions that follow from the different model specifications presented above.

4.1 Additional signals and bias strength

We investigate the effect of additional signals in an economy without overlapping generations and endogenous market entry and exit on the estimation of the fundamental value. In the presence of confirmation bias, there are two conflicting aspects of having more signals on price correctness. On the one hand, more signals increase the accuracy of the mean signal, and therefore the correctness of the estimation. On the other hand, having more signals increases the possibility for the agent to choose the best fitting signal from a larger pool. This in turn decreases the need to deal with potential changes in the underlying and decreases the correctness of the estimate. Figure 3 displays the average effect of one additional signal on the perception incorrectness in the fundamental value of the dividend process, computed as $k_t = |x_t - \hat{x}_t^{c,a}|$, both in the case of rationality $c = 0$ and in the case of highly confirmation bias $c = 1000$ and where $\hat{x}_t^{c,a}$ is the *a posteriori* estimate of the fundamental part of the dividend for a agent subject to confirmation bias.

While the perception incorrectness in the case of rationality is distributed around the true value, in the case of confirmation bias, the deviation goes in the opposite direction. Having a large number of signals allows biased invertors to overweight their preferred one. This leads to the interesting testable hypothesis, that managers may disclose a broad range of information especially around negative announcement. This conveys biased investors to underreact to bad news, as they focus on the better news, and favour managers for a time.



(a) (b)

Figure 3: **Perception correctness as a function of the number of signals.** This figure shows the perception correctness of agents as a function of the number of signals. Subfigure (a) illustrates the case of $c = 1000$, which is the case of highly confirmation biased agents. As a comparison, Subfigure (b) shows the rational case of $c = 0$. Perception incorrectness decreases in the number of signals for rational agents. For highly confirmation biased agents in contrast the effect of signal selection is dominant. In this case signal incorrectness increases in the number of signals.

In Table I, we report the serial correlation in returns for different strength of the confirmation bias and the number of signals. Positive serial correlation is stronger as agents are more biased and as the number of signals increases. It verifies the fact that when agents are not fully rational and many signals are available to them, agents are even more seeking for information confirming their prior beliefs, leading to the existence of momentum. A testable hypothesis is that the more disperse is the information disclosed by managers, the stronger is the momentum in returns. Additionally, this supports evidence given in Pouget et al. (2014).

Table I: **Serial correlation in returns.** The table reports serial correlation in returns for different numbers of signals up to four lags and different confirmation bias parameter. The remaining model parameters are $n = 4$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$.

	$n_s = 3$			$n_s = 30$		
	$c = 0$	$c = 1$	$c = 1000$	$c = 0$	$c = 1$	$c = 1000$
lag 1	0.091	0.116	0.055	0.214	0.310	0.2835
lag 2	0.018	0.057	0.043	0.073	0.156	0.206
lag 3	-0.004	0.063	0.057	0.052	0.164	0.260

Table II illustrates the effect of confirmation bias on volatility clustering. The higher the number of signals, and the stronger the effect of confirmation bias on belief formation, the stronger is also the effect of volatility clustering. Around announcement days investors with a higher degree of confirmation bias tend to search for a confirming signal and underreact. Thus, the process of adapting to the new situation takes longer and clustering is stronger. The reluctance for immediate adaption increases in the desire to do so (confirmation bias parameter) as well as the opportunity (number of potential signals).

Table II: **Serial correlation in squared returns.** The table reports serial correlation in volatility for different numbers of signals up to four lags and different confirmation bias parameters. The remaining model parameters are $n = 4$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$.

	$n_s = 3$			$n_s = 30$		
	$c = 0$	$c = 1$	$c = 1000$	$c = 0$	$c = 1$	$c = 1000$
lag 1	0.004	0.013	0.040	0.048	0.147	0.174
lag 2	-0.035	-0.029	0.001	-0.035	0.023	0.113
lag 3	-0.019	0.000	0.017	-0.031	0.050	0.160

We are interested in the effect of confirmation bias on performance. Table III indicate that agents that are subject to the confirmation bias earn statistically significant lower returns, on average.

Table III: **Descriptive statistics.** This table reports returns means and standard deviations (std. dev.) depending on the value of the confirmation bias parameter. Results are in percentage. Two-sample t-tests of equal means are performed and we do reject the null of equal means for returns with $c = 0$ and $c = 1$, as well as for $c = 0$ and $c = 1000$, but we do not reject the null hypothesis for returns with $c = 1$ and $c = 1000$ at level 5%. The parameters used in the simulation are $n = 20$, $n_s = 30$, $m = 15$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$.

	$c = 0$	$c = 1$	$c = 1000$
Mean	5.64	5.54	5.51
Std. dev	2.03	2.01	1.98

4.2 Underreaction and overreaction

Figure 4 compares a sample price path under confirmation bias with the rational one. It shows that in a model with a continuous state distribution the confirmation bias results in an underreaction to new information. This differs from the results derived in Rabin and Schrag (1999) and Pouget et al. (2014), as their finding rely on the assumption of a binary distribution with two extreme signals and outcomes. In a continuous state distribution agents choose to pay attention especially to those realizations, which are closest to their original prior. In the case of an extreme event, agents pay most attention to signals close to their priors beliefs, which is one of the least extreme signals. Hence, they underreact as illustrated in Figure 4.

In the multiple generations model with endogenous market entry and market exit, another effect interferes with the initial underreaction. After an extreme signal, new generations enter the market without being affected by previous priors. They do not underreact, but initially form rational believes about the underlying fundamental value. Due to the underreaction prevailing in the market they observe deviations of the market price from the fundamental value. This potential for future profits results in an incentive for market entry. As a first consequence, this market

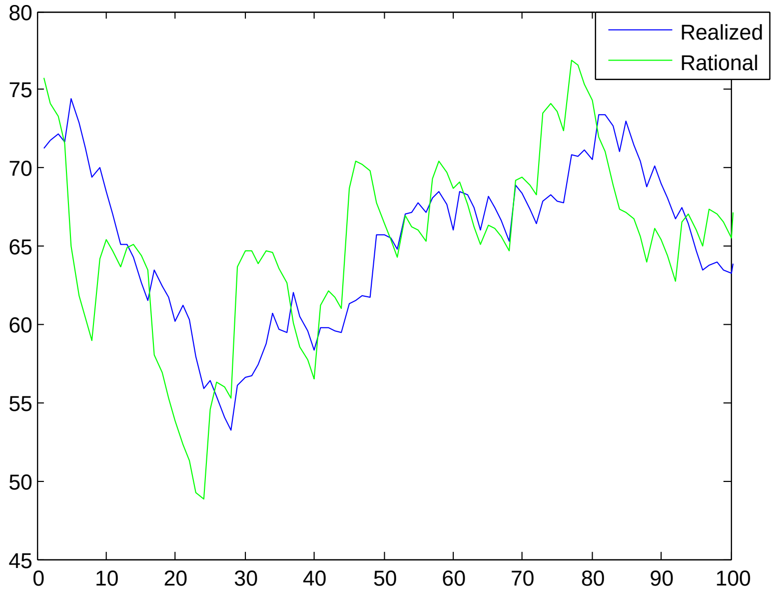


Figure 4: **Price process.** This figure plots the price process for rational and biased agents. In the case of confirmation bias the price process underreacts to new information and lacks behind. The price process is computed using $c = 1000$ for the confirmation biased price process, $n = 4$, $n_s = 30$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$.

entry mitigates the impact of biased agents on prices. Second, the newly entering agents start with a prior, which is strongly influenced by the extreme signal. Underreacting to future information, a strong cohort of newly entering agents leads to the information remaining in the market for an extended period of time. As such the initial underreaction triggers consequent overreaction. Figure 5 shows the average deviation of an average active investor's belief from the fundamental value, in the case of a negative shock to the fundamental. The results are calculated in a model with endogenous market entry and exit. We first observe the underreaction, namely a deviation in the opposite direction of the shock, and then the overreaction, meaning a deviation in the same direction of the shock. Moreover, along with the latter, empirical research reports underreaction to news as earning announcements, but overreaction to bad and good news, Barberis et al. (1998).

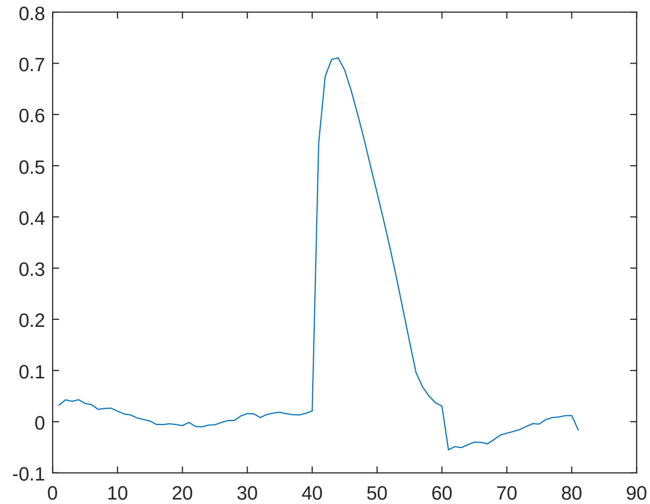


Figure 5: **Agent deviation.** This figure shows the average deviation of an average active investor's belief from the fundamental value, in the case of a negative shock to the fundamental. The results are calculated in a model with endogenous market entry and exit. We first observe the underreaction, namely a deviation in the opposite direction of the shock, and then the overreaction, meaning a deviation in the same direction of the shock. The average price reaction is calculated from $N = 1'000$ price paths and parameter values equal to $c = 1000$, $n = 20$, $n_s = 30$, $m = 15$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$.

4.3 Age generation and market participation

In the endogenous market entry and exit model, the participation in the different population groups is not shared equally. Around an announcement day, the incumbent agents underreact to new information due to confirmation bias. This leads to an under-performance in the following time and and a higher incentive to leave the market. Again, market exit depends on agent preference. Therefore, the percentage of older agents decreases after a negative shock to the fundamental and after high volatility days. In contrast, the younger generations adapt to the new information and are not influenced by previous priors. Prices deviate from the fundamental value and create an incentive for young, unbiased agents to enter. In addition, the general average age of the agents active in the market when there is some confirmation bias, is inferior to the average age of rational agents only, regardless of the announcement days. Meaning that this bias can acutely incentivise biased agents to leave the market earlier than if they were fully rational. This gives indication on the survival of biased investors in the market.

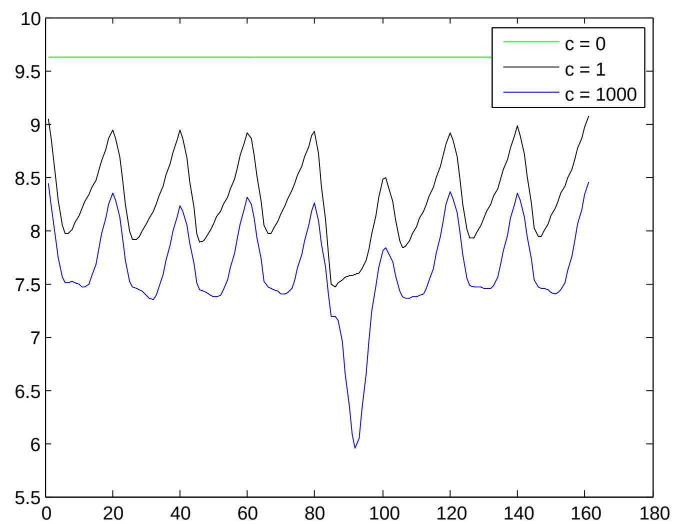


Figure 6: **Average Age.** This figure plots the average age throughout an announcement-cycle in the case of a negative shock to the fundamental. After announcement days the average age decreases as old agents leave the market due to bad performance and new agents enter, as they disagree with the average (biased) perception reflected in prices. We choose parameters as $n = 20$, $n_s = 30$, $m = 15$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$ and run 1'000 simulation paths.

4.4 Trading volume and its relation to volatility

In our model, especially in the endogenous entry and exit model, trading volume consists of two components. The first component results from agents changing their portfolio holdings, while remaining in the market. The second component results from agents buying and selling their shares, when entering or leaving the market.

The first component of trading volume PV is defined as the change in an individual's portfolio holdings, relative to his previous periods holdings

$$PV_t = \sum_{i=1}^{m} \pi_t^{t-i} \left[y_t^{t-i} - y_{t-1}^{t-i-1} \right].$$

The second component of trading volume EH is defined as the change in holdings of agents who enter or leave the market

$$EH_t = \sum_{i=1}^{m} \left[y_t^{t-i} \cdot \delta_t^{\text{entry}} + y_t^{t-i} \cdot \delta_t^{\text{exit}} \right],$$

with δ^{entry} and δ^{exit} being defined as the number of agents entering or respectively leaving the market.

Total trading volume then is defined as $TV_t = PV_t + EH_t$.

Table IV: **Different sources of trading volume.** This table reports the different sources of trading volume. It splits the total trading volume TV into the trading volume due to changes in the portfolio holdings of active market participants PV and the trading volume generated by market entry and market exit EH. Both sources of trading volume increase in the confirmation bias coefficient. The parameters used in the simulation are $n = 20$, $n_s = 30$, $m = 15$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$.

	$c = 0$	$c = 1$	$c = 1000$
PV	1.66	7.42	22.56
EH	2.29	3.08	5.52
TV	3.95	10.50	28.08

A higher degree of confirmation bias leads to a higher trading volume, both due to changes in the holdings of active investors, as well as due to market entrance and exit. A higher degree of confirmation bias changes the disagreement in the investor population and results in trade due to portfolio restructuring. The higher confirmation bias coefficients also lead to a higher entry or exit trading volume. This can be attributed to the effect, that not only the disagreement among active investors increases with confirmation bias, but also the disagreement of still passive agents with the aggregate market belief. This, in turn, triggers market entry. Moreover, the differences in belief also lead to a wider distribution in portfolio returns and a higher frequency of market exit due to negative performance shocks. This finding is empirically supported among others by Park et al. (2013), who show that biased investors have the tendency to trade more frequently.

As especially extreme events trigger both disagreement as well as changes in the market prices, we observe a positive correlation between trading volume and volatility. Table V shows that the correlation coefficient is increasing in the confirmation bias parameter. This is consistent with numerous empirical papers on trading volume and volatility interaction, such as Jones et al. (1994).

Table V: **Correlation between trading volume and volatility.** The table reports pairwise linear correlation coefficient between trading volume and squared returns. We observe the correlation coefficient to be positive. It is increasing in the confirmation bias parameter. Parameters equal $n = 20$, $n_s = 30$, $m = 20$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$. ** indicates that we reject the null hypothesis of zero correlation between rational and realized returns at the 5% significance level.

	$c = 0$	$c = 1$	$c = 1000$
ρ	0.030	0.324**	0.466**

4.5 Market depth

In our model, we measure market depth as the inverse derivative of prices with respect to demand

$$\lambda = \left(\sum_{i=1}^{m} \frac{\partial p}{\partial y_t^{t-i}} \right)^{-1} .$$

To estimate this price impact numerically, we use a simple centered finite difference approximation. In our estimation, we focus on the overlapping generation models in the case of a constant market population and in the case of endogenous market participation. Table VI shows the results.

Table VI: **Absolute average value of market depth approximation.** The table reports the average value of market depth approximated in the absence ($\hat{\lambda}_{\text{constant}}$) and in presence of endogenous market entry and exit ($\hat{\lambda}_{\text{endogenous}}$). Results are in absolute terms. Market depth is increasing with confirmation bias in the multiple overlapping generations model, while it is decreasing when one allows for endogenous market entry and exit. Two-sample t-tests are performed and we do reject the null of equal means for $c = 0$ and $c = 1$, as well as for $c = 0$ and $c = 1000$, but we do not reject the null hypothesis for $c = 1$ and $c = 1000$ at level 5%. Parameters equal $n = 20$, $n_s = 30$, $m = 10$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$ and run 1'000 simulation paths.

	$c = 0$	$c = 1$	$c = 1000$
$\hat{\lambda}_{\text{constant}}$	10^{-7}	0.5491	0.8413
$\hat{\lambda}_{\text{endogenous}}$	10^{12}	0.6185	0.5204

These two market's models capture two opposite effects of confirmation bias on liquidity: market participation and the willingness to trade. Which effect dominates, depends on the considered model. If market participation is constant, market depth is increasing in the confirmation bias, in absolute terms. However, in the case of endogenous market entry and exit market, depth decreases in confirmation bias (i.e. the price impact of demand is larger, in absolute terms). We demonstrate that the overall market participation is decreasing in the confirmation bias level (see Figure 6), which leads to a decrease in market depth. This effect is not caught in the case of constant market participation.

In the context of overconfidence, the papers by Odean (1998), Benos (1998) and Garcia et al. (2007)[5] examine, amongst others, the impact of overconfidence on market depth. Odean (1998) and Benos (1998) show that the presence of overconfi-

[5]Overconfidence is often described as an overestimation by traders of the precision of their private or public signals. Again, in this paper, confirmation bias leads agents to choose signals that are closer to their prior beliefs and to neglect mean-reversion.

dent traders increase market depth. In contrast, more recently Garcia et al. (2007) demonstrate that this does not hold any more when information is endogenously acquired. Market depth is decreasing in the overconfidence level, as rational agents leave the market. This is in line with our findings in the case of endogenous market entry and exit. Furthermore, as in all pre-mentioned papers regarding the impact of overconfidence, in our model a higher degree of confirmation bias leads to higher volume.

4.6 Announcement day effects

Announcement days differ from low-volatility days in several ways. First, Figure 7 demonstrates that around announcement days the effect of confirmation bias on prices is most pronounced. Prices deviate from their fundamental value especially around these times. As time passes by and agents have to face the fact that the signals constantly deviate from their perception and biased agents exit due to under-performance, the effect of confirmation bias fades.

Figure 8 shows an increase in the dispersion of beliefs around announcement days and at a general level. This phenomenon is enhanced by the confirmation bias and depicts the so-called attitude polarization fact. Such an effect is known to stem from confirmation bias. Indeed, it is too difficult for these agents to change their beliefs. Thus, new information tends to be distorted to fit prior beliefs and this in turn widens disagreements even more. We observe that, the more frequent the announcements arrive, the more disperse beliefs are. Disagreements among agents may augment as new announcements arrive often. This leads to an empirical and testable prediction on the effect of the frequency of information disclosed by managers on beliefs dispersion of analysts.

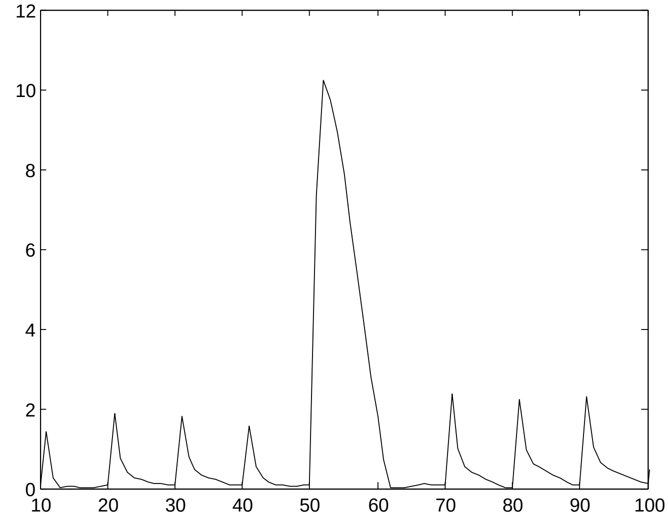


Figure 7: **Absolute price process difference.** This figure plots the difference in absolute value between realized price and rational price processes. Announcement days happen every n. Moreover, a negative shock of two standard-deviation happens at the same time of an announcement. The variance results from $N = 1'000$ price paths and parameter values equal to $c = 1000$, $n = 10$, $n_s = 30$, $m = 15$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$.

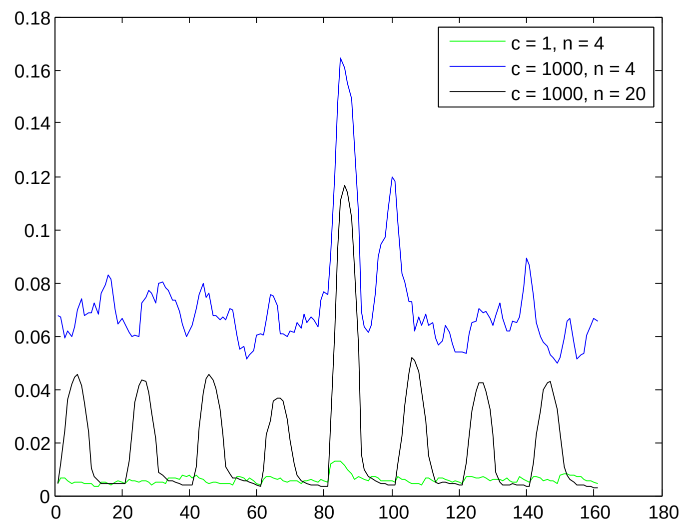


Figure 8: **Belief dispersion.** This figure plots the variance of beliefs of the fundamental value. Announcement days happen every n. Moreover, a negative shock of two standard-deviation happens at the same time of an announcement. The variance results from $N = 1'000$ price paths and parameter values equal to $m = 15$, $n_s = 30$ $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$.

Finally, in line with the Section 4.5, we look at the effect of announcement days on market depth. Table VII reports the absolute value of the average market depth estimation for days prior to announcement days. Chordia et al. (2001) demonstrate that liquidity increases prior to gross domestic product and unemployment announcements, but it falls back to its normal level on announcement days. Such results are attributed in Chordia et al. (2001) to the dispersion in beliefs of market agents prior the announcement dates and the entry in the market of more informed traders. Figures in Table VII support these findings, as we observe a decreasing trend as announcement days approach.

Table VII: **Relative to announcement days, average market depth.** The table reports absolute value of the average market depth, in the case of constant market participation. Parameters equal $n = 20$, $n_s = 30$, $m = 10$, $\mu = 4$, $\sigma_d = 0.5$, $\sigma_x = 0.5$, $\kappa = 0.1$, $r_f = 0.05$ and $\gamma = 0.075$ and run 1'000 simulation paths.

	$c = 0$	$c = 1$	$c = 1000$
$n - 2$	10^{-5}	0.3373	1.1456
$n - 1$	10^{-5}	1.479	0.0974
n	10^{-7}	0.0098	0.0059

5 Conclusion

This paper transfers early, simplifying models of confirmation bias into a more complex overlapping generations model, with endogenous market entry and exit. This leads to different results, and a changing intuition in many cases. For example, we find that confirmation bias is consistent with underreaction, compared to overreaction in previous models. In a framework with endogenous market entry and exit, the initial underreaction triggers a consequent over-correction and as such momentum in prices.

Other model implications are a higher trading frequency of biased agents. We find a positive relation between trading volume and volatility, a time variation in the investor's trading experience, as well as announcement days and liquidity effects of the confirmation bias on financial markets. Moreover, the effect that an initial misreaction finally results in an over-correction in the opposite direction, may not be only linked to confirmation bias and can be subject to future research also in other fields.

Finally, we provide a testable insight into managers' incentives regarding information policy towards investors. In the case of bad news, a higher dispersion in information results in a reduction in market reaction. If transferring good news, however, a strong market reaction is desirable and as a result less dispersed signals should be observed. Furthermore, the frequency, at which managers release information, matters as well.

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A Appendix

A.1 Kalman filtering

The optimal filtering technique, that permits to estimate the state x from Equations (1) and (2), differs between announcement days and non-announcement days. In the following, we first define as $\hat{x}_t^-$ to be the *a priori* estimate at step t, given the process prior to t. $\hat{x}_t$ is the *a posteriori* state estimate at step t, given measurement d_t. Then, we discuss the filter time update equations.

The *a priori* and *a posteriori* estimate error covariance are respectively

$$P_t^- = \mathbb{E}\left((x_t - \hat{x}_t^-)(x_t - \hat{x}_t^-)^T\right), \text{ and}$$

$$P_t = \mathbb{E}\left((x_t - \hat{x}_t)(x_t - \hat{x}_t)^T\right).$$

For non-announcement days we obtain

$$\hat{x}_t^- = \hat{x}_{t-1},$$

$$P_t^- = P_{t-1}.$$

For announcement days the filter time update equations equal

$$\hat{x}_t^- = (1-\kappa)\hat{x}_{t-1} + \kappa\mu,$$

$$P_t^- = (1-\kappa)^2 P_{t-1} + \sigma_x^2.$$

The filter measurement update equations are equivalent for both announcement and non-announcement days

$$K_t = P_t^-(P_t^- + \sigma_d^2)^{-1},$$

$$\hat{x}_t = \hat{x}_t^- + K_t \cdot (d_t - \hat{x}_t^-), \text{ and}$$

$$P_t = (1 - K_t)P_t^-.$$

K_t is the Kalman gain that minimises the *a posteriori* estimation error covariance P_t.

A.2 Derivation of pricing equation parameters

Under the assumption that we are dealing with mature markets, $K_t = K^j$ with $j = t \bmod n$. As $j \in \mathbb{Z}/n\mathbb{Z}$, we also obtain that $j + 1 = j - n + 1$ and the

$$\mathbb{E}_t^a[d_{t+1}] = E_t^a[x_{t+1}] = \hat{x}_t^a \qquad \text{for } j \neq n-1$$

$$\mathbb{E}_t^a[d_{t+1}] = E_t^a[x_{t+1}] = (1-\kappa)\hat{x}_t^a + \kappa\mu \quad \text{for } j = n-1$$

$$\mathbb{E}_t^a[p_{t+1}] = a^{j+1} + b^{j+1}\hat{x}_t^a \qquad \text{for } j \neq n-1$$

$$\mathbb{E}_t^a[p_{t+1}] = a^{j+1} + b^{j+1}\left[(1-\kappa)\hat{x}_t^a + \kappa\mu\right] \quad \text{for } j = n-1$$

$$\mathrm{Var}_t^a(d_{t+1}) = P_{t+1}^- + \sigma_d^2$$

$$\mathrm{Var}_t^a(p_{t+1} + d_{t+1}) = (1 + b^{j+1}K^{j+1})^2 \cdot \left(P_{t+1}^- + \sigma_d^2\right)$$

Inserting into the pricing equation, we obtain that

for $j \neq n-1$

$$p_t^j(1 + r_f) = a^{j+1} + (b^{j+1} + 1)\hat{x}_t^a - \gamma(1 + b^{j+1}K^{j+1})^2 \cdot \left(P_{t+1}^- + \sigma_d^2\right)$$

and for $j = n-1$

$$p_t^j(1+r_f) = a^{j+1} + (b^{j+1}+1)\left((1-\kappa)\hat{x}_t^a + \kappa\mu\right) - \gamma(1+b^{j+1}K^{j+1})^2 \cdot \left(P_{t+1}^- + \sigma_d^2\right).$$

Separating the term with and without $\hat{x}_t^a$

for $j \neq n-1$

$$b^j(1+r_f) = b^{j+1} + 1$$

for $j = n-1$

$$b^j(1+r_f) = (b^{j+1}+1)(1-\kappa)$$

for $j \neq n-1$

$$a^j(1+r_f) = a^{j+1} - \gamma(1+b^{j+1}K^{j+1})^2 \cdot \left(P_{t+1}^- + \sigma_d^2\right)$$

for $j = n-1$

$$a^j(1+r_f) = a^{j+1} + (b^{j+1}+1)\kappa\mu - \gamma(1+b^{j+1}K^{j+1})^2 \cdot \left(P_{t+1}^- + \sigma_d^2\right).$$

With the iterative procedure and for $n = 4$, we have

$$b^1 = \frac{(b^2+1)}{1+r_f} = \frac{b^3}{(1+r_f)^2} + \frac{1}{(1+r_f)^2} + \frac{1}{(1+r_f)} = \frac{b^4(1-\kappa)}{(1+r_f)^3} + \frac{1-\kappa}{1+r_f^3} + \frac{1}{(1+r_f)^2} + \frac{1}{(1+r_f)}$$

$$b^1 = \frac{(b^1+1)(1-\kappa)}{(1+r_f)^4} + \frac{1-\kappa}{(1+r_f)^3} + \frac{1}{(1+r_f)^2} + \frac{1}{(1+r_f)}$$

$$b^1 = \frac{1}{(1+r_f)^4 - (1-\kappa)}\left[(1-\kappa) + (1-\kappa)(1+r_f) + (1+r_f)^2 + (1+r_f)^3\right].$$

$$b^2 = \frac{b^3+1}{1+r_f} = \frac{(b^4+1)(1-\kappa)}{(1+r_f)^2} + \frac{1}{(1+r_f)} = \frac{b^1(1-\kappa)}{(1+r_f)^3} + \frac{(1-\kappa)}{(1+r_f)^3} + \frac{(1-\kappa)}{(1+r_f)^2} + \frac{1}{(1+r_f)},$$

$$b^3 = \frac{(b^4+1)(1-\kappa)}{(1+r_f)} = \frac{b^1(1-\kappa)}{(1+r_f)^2} + \frac{1-\kappa}{(1+r_f)^2} + \frac{1-\kappa}{(1+r_f)},$$

$$b^4 = b^0 = \frac{b^1+1}{(1+r_f)}.$$

where b^1 is defined above.

We have as well

$$a^1(1+r_f) = a^2 - \gamma(1+b^2K^2)^2 \cdot \left(P_2^- + \sigma_d^2\right),$$

$$a^2(1+r_f) = a^3 - \gamma(1+b^3K^3)^2 \cdot \left(P_3^- + \sigma_d^2\right),$$

$$a^3(1+r_f) = a^4 + (b^4+1)\kappa\mu - \gamma(1+b^4K^4)^2 \cdot \left(P_4^- + \sigma_d^2\right),$$

$$a^4(1+r_f) = a^0(1+r_f) = a^1 - \gamma(1+b^1K^1)^2 \cdot \left(P_1^- + \sigma_d^2\right).$$

With the iterative procedure, we get

$$\begin{aligned}
a^1 =& \frac{1}{(1+r_f)}\Big(a^2 - \gamma(1+b^2K^2)^2 \cdot (P_2^- + \sigma_d^2)\Big)\\
=& \frac{1}{(1+r_f)^2}a^3 - \frac{\gamma}{(1+r_f)^2}(1+b^3K^3)^2 \cdot (P_3^- + \sigma_d^2) - \frac{\gamma}{+r_f}(1+b^2K^2)^2 \cdot (P_2^- + \sigma_d^2)\\
=& \frac{1}{(1+r_f)^3}a^4 + \frac{1}{(1+r_f)^3}(b^4+1)\kappa\mu - \frac{\gamma}{(1+r_f)^3}(1+b^4K^4) \cdot (P_2^- + \sigma_d^2) -\\
& \frac{\gamma}{(1+r_f)^2}(1+b^3K^3) \cdot (P_3^- + \sigma_d^2) - \frac{\gamma}{1+r_f}(1+b^2K^2) \cdot (P_2^- + \sigma_d^2)\\
=& \frac{a^1}{(1+r_f)^4} + \frac{(b^4+1)(\kappa\mu}{(1+r_f)^3} - \gamma\Big[\frac{(1+b^1K^1)^2}{(1+r_f)^4} \cdot (P_1^- + \sigma_d^2) +\\
& \frac{(1+b^4K^4)^2}{(1+r_f)^3} \cdot (P_4^- + \sigma_d^2) + \frac{(1+b^3K^3)^2}{(1+r_f)^2} \cdot (P_3^- + \sigma_d^2) + \frac{(1+b^2K^2)^2}{(1+r_f)} \cdot (P_2^- + \sigma_d^2)\Big]\\
=& \frac{1}{(1+r_f)^4 - 1}\Big[-\gamma\Big((1+b^1K^1)^2 \cdot (P_1^- + \sigma_d^2) + (1+b^2K^2)^2 \cdot (P_2^- + \sigma_d^2)(1+r_f)^3\\
& + (1+b^3K^3)^2 \cdot (P_3^- + \sigma_d^2)(1+r_f)^2 + (1+b^4K^4)^2 \cdot (P_4^- + \sigma_d^2)(1+r_f)\Big) + (b^4+1)\kappa\mu(1+r_f)\Big].
\end{aligned}$$

Then, all are expressed in function of a_1, such that

$$\begin{aligned}
a^2 =& \frac{a^3}{(1+r_f)} - \gamma(1+b^3K^3)^2 \cdot (P_3^- + \sigma_d^2) \\
=& \frac{a^4}{(1+r_f)^2} + \frac{(b^4+1)\kappa\mu}{(1+r_f)^2} - \gamma\Big[\frac{(1+b^1K^1)^2 \cdot (P_1^- + \sigma_d^2)}{(1+r_f)^3} + \\
& \frac{(1+b^4K^4)^2 \cdot (P_4^- + \sigma_d^2)}{(1+r_f)^2}\Big] \\
=& \frac{a^1}{(1+r_f)^3} + \frac{(b^4+1)\kappa\mu}{(1+r_f)^2} - \gamma\Big[\frac{(1+b^1K^1)^2 \cdot (P_1^- + \sigma_d^2)}{(1+r_f)^3} + \frac{(1+b^4K^4)^2 \cdot (P_4^- + \sigma_d^2)}{(1+r_f)^2} + \\
& \frac{(1+b^3K^3)^2 \cdot (P_3^- + \sigma_d^2)}{(1+r_f)}\Big],
\end{aligned}$$

$$\begin{aligned}
a^3 =& \frac{a^4}{(1+r_f)} + \frac{(b^4+1)\kappa\mu}{1+r_f} - \frac{\gamma}{1+r_f}(1+b^4K^4)^2 \cdot (P_4^- + \sigma_d^2) \\
=& \frac{a^1}{(1+r_f)^2} + \frac{(b^4+1)\kappa\mu}{(1+r_f)} - \gamma\Big[\frac{(1+b^1K^1)^2 \cdot (P_1^- + \sigma_d^2)}{(1+r_f)^2} + \frac{(1+b^4K^4)^2 \cdot (P_4^- + \sigma_d^2)}{(1+r_f)}\Big],
\end{aligned}$$

$$a^4 = \frac{a^1}{(1+r_f)} - \frac{\gamma}{(1+r_f)}(1+b^1K^1)^2 \cdot (P_1^- + \sigma_d^2).$$

The procedure is generalized for any n in the code.