The Lost Capital Asset Pricing Model

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Abstract

A flat Securities Market Line is not evidence against the CAPM. Under the Roll (1977) critique, the CAPM is a "lost city of Atlantis," empirically invisible. In a noisy rational-expectations economy, there exists an information gap between the average investor who holds the market and the empiricist who does not observe the market portfolio. The CAPM holds for the investor, but appears flat to the empiricist. This distortion is empirically substantial and explains, for instance, why "Betting Against Beta" works; BAB really bets on true beta. Macroeconomic announcements reduce the distortion—for a fleeting moment the empiricist catches a glimpse of the CAPM.

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1 Introduction

The Capital Asset Pricing Model, one of the main pillars of modern finance theory, fails in empirical tests. A common explanation for this empirical failure is the Roll (1977) critique: testing the CAPM is unfeasible because the composition of the market portfolio is not observable. While various attempts have been undertaken to alleviate this criticism[1], the CAPM relation has yet to be found in the data, largely shaping the view that it does not hold. However, recent empirical findings challenge this view. Savor and Wilson (2014) document a strong CAPM relation that holds on days during which macroeconomic news is released and vanishes immediately right after. This result is puzzling as it suggests that the CAPM behaves like a hidden "Atlantis" that unveils on particular occasions.[2]

This paper is a theoretical attempt to understand why the CAPM fails most of the time. For the purposes of our argument, we assume that a CAPM correctly prices all assets and show that, under fairly general conditions, an empiricist will incorrectly reject it.[3] The main idea is that the empiricist—an *outside* observer—has a coarser information set than that of investors who trade *inside* the marketplace (Roll, 1978; Dybvig and Ross, 1985; Hansen and Richard, 1987). Because investors must hold the portfolio of aggregate wealth in equilibrium, the composition of this portfolio is a central determinant of equilibrium prices, in addition to investors' information. For the empiricist, observing the market portfolio and equilibrium prices is all she needs for asset pricing tests. But, when this portfolio is unobservable (Roll, 1977), the information gap between investors and the empiricist impairs asset pricing tests. Testing the CAPM illustrates a textbook example of this idea—using a proxy for the market portfolio is a necessity for carrying out the test. In this paper, we explore and quantify the equilibrium consequences of this observation.

Suppose an econometrician observes the time series of realized excess returns for a large number of assets. Using the law of total covariance, the unconditional variance-covariance matrix of excess returns the econometrician computes based on historical data, $\mathbb{V}[R]$, can

[1] Stambaugh (1982) investigates how sensitive the tests are to the choice of market proxy. Shanken (1987) develops a framework to incorporate explicitly the imperfect correlation between the true market portfolio and commonly-used market proxies. Theoretically, the cross sectional SML relation is very sensitive to the choice of the index (Roll and Ross, 1994). Jagannathan and Wang (1996) include non-traded wealth such as human capital and show that the results can change dramatically (this follows earlier work by Mayers (1972), who extended the CAPM relation to include nonmarketable assets).

[2] Seminars and conferences in which this paper was presented have been prolific with alternative metaphors. Hanno Lustig compares the CAPM to a "19th century debutante," Ian Martin to a "dormant dragon," Steve Heston to an "eclipse," David Hirshleifer to "the cicadas," and Zhenyu Wang calls it the "Holy CAPM."

[3] There are compelling theoretical reasons not to believe the CAPM is the correct canonical asset pricing model but these are not relevant to our argument. While our argument focuses on the Sharpe-Lintner CAPM, similar principles clearly apply to other equilibrium models such as the Merton ICAPM.

be decomposed into the observed variation "explained" by the information of the average investor, $\mathbb{V}\left[\mathbb{E}[R \mid \mathscr{F}]\right]$, and a remaining "unexplained" component, $\mathbb{E}\left[\mathbb{V}[R \mid \mathscr{F}]\right]$:

$$\mathbb{V}[R] = \mathbb{V}\left[\mathbb{E}[R \mid \mathscr{F}]\right] + \mathbb{E}\left[\mathbb{V}[R \mid \mathscr{F}]\right]. \tag{1}$$

This decomposition formalizes the notion of *informational distance* between the econometrician and the average investor.

The true betas in the economy—investor's betas—are solely based on the unexplained component of the variance. Econometrician's betas, instead, are based on the total variance, $\mathbb{V}[R]$. Thus, true and measured betas differ in proportion to the informational distance in Eq. (1). Importantly, in contrast to the traditional argument that variation in beta estimates invalidates an unconditional CAPM relation (Jagannathan and Wang, 1996; Lewellen and Nagel, 2006), in our static model a true unconditional CAPM relation always holds. If the econometrician observed the market portfolio, the informational distance in Eq. (1) would not affect the empiricist's ability to recover this unconditional relation. But, in fact, the econometrician does not observe the market portfolio (Roll, 1977) and investors' informational advantage does matter; it distorts the econometrician's perception of unconditional betas. Our objective is to place economic restrictions on this distortion. We show that in equilibrium the econometrician perceives a "flat" CAPM relation, which becomes steeper when public information is released.[4]

We build our argument on a rational-expectations model of informed trading in which a continuum of mean-variance investors trade multiple assets based on private and public information. A necessity for private information to be relevant in this framework is that the market portfolio be unobservable, thus sowing the seeds of the Roll (1977) critique right into the building blocks of the model. The original Roll critique states that not all assets are observable or tradable. In our model we assume that all assets are observable, but their supply is noisy (Grossman and Stiglitz, 1980). This assumption differs from the Roll critique, as originally intended. However, the model is equivalent to one with non-tradable, unobservable assets that are correlated with tradable assets (e.g., Wang, 1994). In that respect, noise in supply captures the spirit of the Roll critique.

Suppose now that an outside econometrician estimates a Securities Market Line (SML) in this equilibrium model. The econometrician has to take a stand *ex-ante* on the composition

[4] Other possible explanations for a flat CAPM exist. They include leverage constraints (Black, 1972; Frazzini and Pedersen, 2014), inflation (Cohen, Polk, and Vuolteenaho, 2005), disagreement (Hong and Sraer, 2016), preference for volatile, skewed returns (Kumar, 2009; Bali, Cakici, and Whitelaw, 2011), market sentiment (Antoniou, Doukas, and Subrahmanyam, 2015), stochastic volatility (Campbell, Giglio, Polk, and Turley, 2012), and benchmarking of institutional investors (Baker, Bradley, and Wurgler, 2011; Buffa, Vayanos, and Woolley, 2014). However, none directly result from the informational distance in Eq. (1).

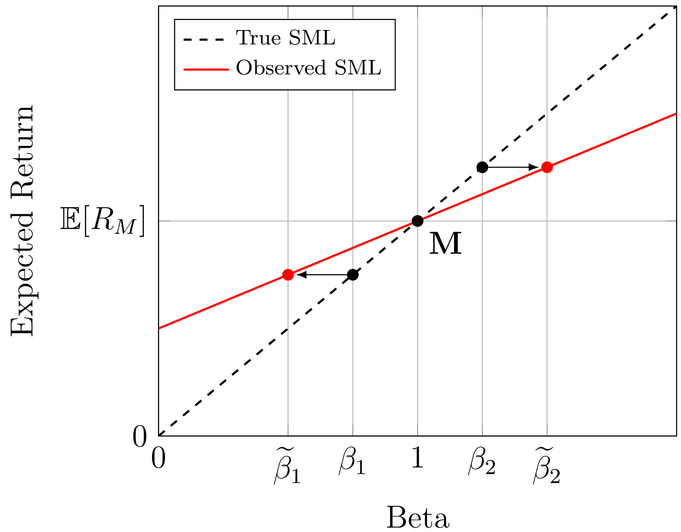

Figure 1: CAPM distortion. This figure illustrates the main result of the paper. The perceived SML is flatter than the actual SML in equilibrium. The black dashed line and the red solid line show the true and perceived SML. **M** represents the market portfolio.

of the market portfolio, which is unobservable (Roll, 1977). That CAPM tests fail when the market portfolio proxy is not mean-variance efficient is certainly true (Roll and Ross, 1994) and is not our question. Our question is whether the econometrician rejects the CAPM with a market proxy that is theoretically correct. Let the econometrician choose a proxy for the market portfolio that is unconditionally mean-variance efficient from the perspective of the "average investor," a fictitious agent who holds the market portfolio and whose beliefs define the market consensus. Paradoxically, even if this proxy is mean-variance efficient for the average investor, the econometrician perceives it as mean-variance *in*efficient.

The main argument is that the econometrician, who holds unconditional beliefs and thus faces more uncertainty than the average investor, perceives the comovement among assets differently. While the average investor sees the true SML—a line that crosses the origin with the market risk premium as its slope—the econometrician's SML is distinctly flatter. Figure 1 illustrates this distortion, plotting the true SML (the dashed line) and the perceived SML (the solid line). The informational distance in Eq. (1) amplifies the dispersion in econometrician's betas ($\widetilde{\beta}_1$ and $\widetilde{\beta}_2$) relative to true betas (β_1 and β_2). However, all betas—correct or incorrect—must average to one. The market beta thus becomes the "center of gravity" around which econometrician's betas "inflate" away from a value of one. Assets with a beta higher than one appear riskier than they really are, whereas assets with a beta lower than one appear safer than they really are. Since the empiricist and the investor agree on what unconditional returns are, econometrician's SML rotates clockwise around the market portfolio (denoted by M), which flattens its slope and creates a positive intercept.

The magnitude of this distortion in beta estimates is proportional to the informational distance between investors and the econometrician. Thus, the Roll critique resurrects the Hansen and Richard (1987) critique—it makes the informational distance between investors and the econometrician matter. This distortion arises because the econometrician ignores asset-pricing relevant information. Individual investors possess private information that prices do not fully reveal. Furthermore, the empiricist will never be able to state confidently that she has controlled for all publicly-available information that could have been relevant for investors in determining their trading strategies.

The law of total covariance in Eq. (1) serves as a tool for empirical work. When augmented with the endogenous relation between perceived and actual betas implied by our model, it generates novel empirical predictions. Specifically, there is an affine relation between econometrician's beta and a new notion of beta based on the time-series of expected excess returns on individual securities and on the market. However, the main matter we investigate in this paper is precisely that expected returns are unobservable. Measuring expected returns is difficult because expected returns are determined by information that investors possess but that empiricists do not; factors that are commonly used in the literature (e.g., Fama French) likely capture some of this information, but arguably will always leave a large fraction unexplained. Recently, Martin (2017) and Martin and Wagner (2017) overcome this difficulty by extracting this information all at once from option data. Option prices tell us what the market thinks about future returns, thus providing a proxy for what the "average investor's expectations" are. This is the methodology we adopt to conduct empirical tests.

We test the affine relation that our theory predicts and find strong support for a distortion in beta estimates. Back-of-the-envelope calculations show that the distortion can be large, leading to substantial flattening of the SML. That this distortion has plausibly large empirical magnitudes calls for a reinterpretation of existing findings in the literature. Under our theory, betting against *measured* beta (Frazzini and Pedersen, 2014) is really betting on *true* beta. Additionally, in the eyes of the econometrician the distortion in beta estimates creates the illusion of an idiosyncratic volatility puzzle—stocks with high idiosyncratic risk have implausibly low returns (Ang, Hodrick, Xing, and Zhang, 2006, 2009).

Extending the model to allow assets' payoffs to depend on multiple factors complicates the econometrician's problem of estimating a securities market line. Not only does the CAPM look flat, but the relation between average excess returns and betas is no longer a straight line. Allowing assets to be in heterogeneous supplies has a similar effect, and may further cause the econometrician to perceive a downward-sloping securities market line, although the actual SML is always upward-sloping. This situation occurs when assets that have a high market beta simultaneously have a low market supply.

We further allow the empiricist to estimate the CAPM relation conditioning on days when investors observe public announcements, as in Savor and Wilson (2014). The announcement involves macroeconomic uncertainty that raises the risk premium prior to its release; it also provides macroeconomic information about payoffs, which reduces the informational distance between investors and the empiricist. These two effects jointly lead the CAPM relation to look distinctly stronger on announcement days.

Section 2 presents our main result in a static model, provides intuition about the distortion in beta estimates, along with back-of-the-envelope calculations of the distortion, and reinterprets anomalies in light of our theoretical result. Section 3 studies extensions of the static model. Section 4 examines the distortion of the SML in a dynamic model and describes the effect of a periodic public announcement. Section 5 concludes. All proofs are relegated to the Appendix.

2 Perception of the SML under the Roll critique

We revisit an old asset-pricing question, "why is the Securities Market Line (SML) flat?". Suppose a true CAPM relation holds under a representative investor's information, which the empiricist does not observe (the "Hansen and Richard (1987) critique").[5] In this context, our focus is on how the Roll critique (Roll, 1977) distorts the empiricist's perception of the SML. We first clarify how these two critiques interact in a simple economic setup. We then explore their joint implications for tests of the CAPM in equilibrium—when the representative investor holds the market portfolio. The empiricist views low-beta assets as less risky and high-beta assets as riskier than they actually are. Thus, under the Roll critique the SML looks flatter than it really is. We argue that this distortion is empirically substantial.

2.1 Background: testing the CAPM under the Roll critique

In the context of our framework, we interpret the Roll critique as the empiricist's inability to observe the market portfolio. Of course, the empiricist likely ignores other information relevant to investors. Absent the Roll critique, however, this information is irrelevant for the purpose of testing the CAPM. What makes this information matter under the Roll critique is that the empiricist cannot simply "ignore the market portfolio"—using a proxy for the market portfolio is a necessity for testing the CAPM. The empiricist must take a stand

[5] The "Hansen-Richard critique" designation belongs to Cochrane (2009), by analogy to the "Roll critique." Along the same lines, Dybvig and Ross (1985) show that conditionally mean-variance returns from a managed portfolio might appear unconditionally mean-variance inefficient under a coarser information set.

on what the market portfolio is *ex-ante*. The information gap between investors and the empiricist then ensures that the empiricist rejects the CAPM.

Suppose the empiricist chooses a proxy for the market portfolio that investors find mean-variance optimal to hold unconditionally. Paradoxically, chances are the empiricist perceives this proxy as mean-variance suboptimal. To illustrate how this mechanism operates, we go through a typical exercise of deriving the CAPM. Consider a one-period economy populated by a representative investor who derives monotone increasing utility, $U(W)$, from consuming her wealth W in period one. For simplicity, suppose the investor has zero initial wealth—what she invests in the N stocks available in supply M and paying excess returns R she must borrow from a risk-free bond with gross return normalized to 1. In equilibrium the supply of stocks must equal investor's demand and thus M defines the market portfolio.

Investor's first-order condition for optimal portfolio choice leads to the standard asset-pricing equation (referred to as "Euler equation" thereafter):

$$\mathbb{E}[U_W(M'R)R|\mathscr{F}] = 0, \tag{2}$$

where $\mathscr{F} \equiv \{M, X\}$ denotes the investor's information set; it contains all information that is relevant for asset pricing. This information must include the market portfolio $M \in \mathscr{F}$, as well as additional asset-pricing relevant information, which we denote by X.

The conditional asset-pricing model in Eq. (2) need not imply an unconditional CAPM relation (Jagannathan and Wang, 1996). In this paper, however, we are interested in a model than can be easily conditioned down to an unconditional CAPM relation, thereby ruling out distortions that are of small empirical magnitude (Lewellen and Nagel, 2006). Specifically, we assume that asset-pricing information X and the vectors of market weights M and excess returns R are jointly Gaussian. Under this assumption Stein's lemma applies and conditional covariances are nonrandom. We further restrict preferences, $U(\cdot)$, to be exponential so that the market price of variance risk is a known constant, γ, the investor's coefficient of absolute risk aversion. This CARA-normal structure is one particular framework in which the unconditional CAPM holds (Cochrane, 2005, Chapter 9); it is also analytically convenient for incorporating disperse information, which we will do shortly.

The CAPM is derived in a few standard steps. First, rearrange Eq. (2) using covariance decomposition and apply Stein's lemma:

$$\mathbb{E}[R|\mathscr{F}] = -\frac{\mathbb{E}[U_{WW}(M'R)|\mathscr{F}]}{\mathbb{E}[U_W(M'R)|\mathscr{F}]}\,\mathbb{V}[R|\mathscr{F}]M = \gamma\,\mathbb{V}[R|\mathscr{F}]M. \tag{3}$$

Then use that this relation must also hold for the market portfolio and obtain a CAPM relation under investor's information set $\mathscr{F}$:

$$\mathbb{E}[R|\mathscr{F}] = \frac{\mathrm{Cov}[R, M'R|\mathscr{F}]}{\mathbb{V}[M'R|\mathscr{F}]}\,\mathbb{E}[M'R|\mathscr{F}], \tag{4}$$

a line that crosses the origin in the beta-return space with the market premium as its slope. As anticipated, this conditional relation implies an unconditional CAPM relation, which follows after taking unconditional expectation of the equilibrium risk-return tradeoff in (3):

$$\mathbb{E}[R] = \frac{\mathrm{Cov}[R, R_M|\mathscr{F}]}{\mathbb{V}[R_M|\mathscr{F}]}\,\mathbb{E}[R_M] \equiv \beta[R, R_M|\mathscr{F}]\,\mathbb{E}[R_M], \tag{5}$$

where $R_M \equiv \overline{M}'R$ denotes excess returns on the unconditional market portfolio, $\overline{M} \equiv \mathbb{E}[M]$.

Absent the Roll critique, omitting asset-pricing relevant information $X \in \mathscr{F}$ (Hansen and Richard, 1987) is irrelevant for testing the CAPM. Suppose the empiricist observes the market portfolio M, but does not know X. She can condition the Euler equation (2) down to M, perform the steps above and find the CAPM relation knowing M exclusively:

$$\mathbb{E}[R|M] = \frac{\mathrm{Cov}[R, M'R|M]}{\mathbb{V}[M'R|M]}\,\mathbb{E}[M'R|M]. \tag{6}$$

The CAPM relations in Eqs. (4) and (6) are actually identical in a CARA-normal framework. In particular, the law of total covariance implies:

$$\mathbb{V}[R|M] = \mathbb{E}[\mathbb{V}[R|\mathscr{F}]|M] + \underbrace{\mathbb{V}[\mathbb{E}[R|\mathscr{F}]|M]}_{\equiv 0 \text{ from Eq. (3)}} \equiv \mathbb{V}[R|\mathscr{F}] \tag{7}$$

and thus the investor's beta and that of the empiricist coincide. Similarly, their expectations of market returns are identical (see Appendix A.1):

$$\mathbb{E}[M'R|M] = \mathbb{E}[M'R|\mathscr{F}]. \tag{8}$$

Hence, knowing the market portfolio is all we need to derive the CAPM relation in Eq. (4).

But, in fact, the empiricist does not observe the composition of the market (Roll, 1977). What makes the Roll critique ineluctable is that the empiricist must take a stand on what the market portfolio is. Hence, the question emerges which proxy for M she should use. In equilibrium, it must remain the case that markets clear unconditionally and thus the investor finds it mean-variance optimal to hold the average market portfolio, $\overline{M}$. Assuming the empiricist can only compute unconditional moments, her best proxy is the average market portfolio, $\overline{M}$. Allowing the empiricist to use this proxy ensures that she tests the CAPM under the best possible conditions. Note further that letting the proxy be arbitrary would open up a myriad of possible SML representations.[6] Fortunately, in an equilibrium context the proxy need not be arbitrary—it results directly from the market-clearing condition.

Although the investor finds it mean-variance optimal to hold the proxy $\overline{M}$ unconditionally, the empiricist perceives the same portfolio as mean-variance suboptimal. The Euler equation (2) states that holding the proxy $\overline{M}$ is mean-variance optimal after taking unconditional expectations, which is the meaning of Eq. (5). However, the empiricist must substitute her proxy in the Euler equation before taking unconditional expectations. She is thus bound to conclude that holding the proxy is not mean-variance optimal *ex-ante*:

$$\mathbb{E}[U_W(\overline{M}'R)R] = \mathbb{E}[U_{WW}(\overline{M}'R)](\underbrace{\mathbb{V}[R] - \mathbb{V}[R|\mathscr{F}]}_{\text{informational distance}})\overline{M} \neq 0, \tag{9}$$

a conclusion that follows directly from the law of total covariance (see Appendix A.1).

The Roll critique resurrects the Hansen and Richard (1987) critique—it makes the informational distance between the investor and the empiricist matter. Specifically, Eq. (9) shows that the empiricist's proxy violates optimality in proportion to this distance. Hence, under the Roll critique a test of the CAPM based on its associated Euler equation (Hansen and Singleton, 1982) does not give the empiricist the ability to ignore relevant information. The Roll critique makes the CAPM axiomatically untestable.

Although Eq. (9) shows that the informational distance distorts the test, it does not say how. To place restrictions on this distortion we need an equilibrium model of returns. This is the goal of the next subsection.

2.2 An equilibrium model of excess returns

We build a model of how investors form expectations, imposing an equilibrium structure on excess returns. A particularity of this model is that its building block is the Roll critique.

Consider the one-period economy of Section 2.1 in which the market consists of one risk-free asset with gross return normalized to 1 and N risky stocks indexed by $n = 1, ..., N$. Suppose now the risky stocks have payoffs D realized at the liquidation date (time 1). These

[6] Alternative market proxies that are *not* unconditionally mean-variance efficient have been explored by Roll and Ross (1994). See also Shanken (1987). In a related article, Roll (1978) shows that detecting "superior" performance based on the SML is strongly influenced by the choice of market index. In our context, assuming a market proxy that differs from $\overline{M}$ would presumably generate further bias.

payoffs are unobservable at the trading date (time 0) and have a common factor structure:

$$D \equiv \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_N \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix} F + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \end{bmatrix} \equiv \Phi F + \epsilon. \tag{10}$$

The common factor F and each stock-specific component ϵ_j of the payoff are independently normally distributed with means zero and precisions τ_F and τ_ϵ.

The economy is populated with a continuum of investors indexed by $i \in [0, 1]$, who choose their portfolio at time 0 and derive utility from terminal wealth with constant absolute risk aversion coefficient γ. Investors know the structure of realized payoffs in Eq. (10), but do not observe the common factor. Each investor i forms expectations about F based on both a private signal V_i and a public signal G:

$$V_i = F + \upsilon_i \tag{11}$$

$$G = F + \upsilon. \tag{12}$$

Signal noises υ and $\upsilon_i \perp \upsilon$, $\forall i$ are unbiased and independently normally distributed with precisions τ_G and τ_υ, respectively.

To allow prices to play an informational role in equilibrium, we make the customary assumption that the supply of stocks, $M \equiv [M_1 \ \dots \ M_N]'$, is noisy.[7] This assumption makes the market portfolio unobservable both to investors and to the econometrician, thus making the Roll (1977) critique a building block of the model. On average each stock has equal weight $1/N$ in the market portfolio and each weight is normally and independently distributed across stocks with precision τ_M.

Note how this economy relies on several simplifying assumptions. For instance, we have assumed that payoffs in Eq. (10) are driven by a single factor, as opposed to multiple factors. Stocks only differ according to their loading Φ on this common factor—they have equal weight (or size) in the market portfolio on average; their sizes and their idiosyncratic noises have equal precision of τ_M and τ_ϵ. Similarly, because the model is static, public and private signals have identical timing. These simplifying assumptions serve our immediate purpose of generating the main results in a simple model. We subsequently examine in Section 3 how relaxing these assumptions affects the results.

[7] Alternatively, the literature sometimes assumes that some assets are privately traded (Wang, 1994). The two approaches are equivalent.

We solve for a linear equilibrium of the economy in which prices satisfy

$$P = \alpha F + gG + \xi M, \tag{13}$$

where α and g are $N-$dimensional vectors and ξ is a $N \times N$ matrix, all of which are determined in equilibrium by imposing market clearing. Because returns are not normally distributed in this framework, a convention in the literature is to work with price changes instead (e.g., Dybvig and Ross, 1985). We follow this convention and refer to price changes, $R \equiv D - P$, as "excess returns".

Each investor i forms expectations about excess returns based on the information she observes:

$$\mathscr{F}^i = \{V_i, G, P\}. \tag{14}$$

Because private signals V_i all have identical precision, and the signal G and prices P are public, each investor i forecasts the common factor F with identical precision:

$$\tau \equiv \mathbb{V}[F \mid \mathscr{F}^i]^{-1} = \tau_F + \tau_\upsilon + \tau_G + \tau_P \Phi' \Phi. \tag{15}$$

The last term in Eq. (15) is the sum of squared noise-signal ratios over all prices, where τ_P is a scalar that measures price informativeness (see Appendix A.2). Prices have explicit solutions that we provide in the proposition below.

Proposition 1. ***(Equilibrium)*** *There exists a unique linear equilibrium in which prices take the linear form in Eq.* (13) *and are explicitly given by*

$$P = \frac{\tau - \tau_F - \tau_G}{\tau} \Phi F + \frac{\tau_G}{\tau} \Phi G - \left(\frac{\gamma + \sqrt{\tau_M \tau_P}}{\tau} \Phi \Phi' + \frac{\gamma}{\tau_\epsilon} \mathbf{I}_N \right) M, \tag{16}$$

where $\mathbf{I}_N$ *is the identity matrix of dimension* N*. The precision* τ *is defined in* (15) *and the scalar* τ_P *is the unique positive root of the cubic equation:*

$$\tau_P \left[\tau_F + \tau_\upsilon + \tau_G + (\tau_P + \tau_\epsilon) \Phi' \Phi \right]^2 \gamma^2 = \tau_M \tau_\epsilon^2 \tau_\upsilon^2. \tag{17}$$

In this economy investors have different perceptions of the mean-variance frontier, because each investor i observes a different information set $\mathscr{F}^i$. Hence, not only is it impossible for them to hold the market portfolio M, but it would not even be mean-variance optimal to do so from their perspective. Holding the market portfolio is both mean-variance efficient and feasible only for the *average investor*, a fictitious investor who defines the consensus (average) beliefs and clears the market. We denote the beliefs of this average investor by $(\overline{\mathbb{E}}[\cdot], \overline{\mathbb{V}}[\cdot])$ and describe them below.

Since the precision on the common factor in Eq. (15) is identical for all investors, they hold the same posterior variance of excess returns. Thus, the posterior variance of excess returns from the perspective of the average investor is:

$$\overline{\mathbb{V}}[R] \equiv \mathbb{V}[R \mid \mathscr{F}^i] = \frac{1}{\tau}\Phi\Phi' + \frac{1}{\tau_\epsilon}\mathbf{I}_N, \quad \forall i \in [0,1]. \tag{18}$$

The consensus beliefs are further defined by averaging over investors' expectations:

$$\overline{\mathbb{E}}[R] \equiv \int_i \mathbb{E}[R \mid \mathscr{F}^i]\mathrm{d}i. \tag{19}$$

Because the average investor clears the market, the market-clearing condition relates endogenously the consensus beliefs and posterior variance of excess returns:

$$\overline{\mathbb{E}}[R] = \gamma\overline{\mathbb{V}}[R]M, \tag{20}$$

which leads to the solution for equilibrium prices in Proposition 1. In other words, since the average investor clears the market she must find it mean-variance optimal to hold the market portfolio. Furthermore, Eq. (20) defines the true capital market line, which represents the particular form Eq. (3) takes in this model. By the law of iterated expectations, it follows as a corollary that a CAPM relation holds unconditionally for the average investor.[8]

Corollary 1.1. ***(CAPM)*** *In equilibrium in this model, an unconditional CAPM relation holds and admits the usual beta-return representation:*

$$\mathbb{E}[R] = \frac{\frac{1}{N}\overline{\mathbb{V}}[R]\mathbf{1}}{\overline{\mathbb{V}}[R_M]}\mathbb{E}[R_M] = \beta\,\mathbb{E}[R_M], \tag{21}$$

where β is a N-dimensional vector of betas, $\mathbf{1}$ is a N-dimensional vector of ones, $\mathbb{E}[R_M]$ is the unconditional expected dollar excess return on the market portfolio, and $\overline{\mathbb{V}}[R_M]$ is the posterior variance of excess returns on the market portfolio:

$$\mathbb{E}[R_M] = \frac{1}{N}\mathbf{1}'\mathbb{E}[R] \tag{22}$$

$$\overline{\mathbb{V}}[R_M] = \frac{1}{N^2}\,\mathbf{1}'\,\overline{\mathbb{V}}[R]\,\mathbf{1}\,. \tag{23}$$

[8]In this economy, the CAPM holds both unconditionally and conditionally (under the information set of the average investor). However, our focus is on the unconditional CAPM relation.

Corollary 1.1 presents the unconditional CAPM in this economy. Compared to Eq. (5), Eq. (21) further assumes that the unconditional market portfolio is equally weighted, $\overline{M} \equiv N^{-1}\mathbf{1}$. In this economy computing actual betas, β, simply requires knowing the posterior variance of excess returns measured by the average investor.

2.3 Empiricist's view: the Roll critique in equilibrium

Given the equilibrium representation of the CAPM in Corollary 1.1, we now investigate how the informational distance in Eq. (9) affects the empiricist's own representation of the CAPM. For simplicity, we assume the empiricist can only compute unconditional moments from realized excess returns. We relax this assumption in Section 3.1, allowing the empiricist to control for all publicly available information when testing the CAPM.

A key starting point is the law of total covariance in Eq. (1), which allows us to formalize the informational distance between the average investor and the empiricist:

$$\mathbb{V}[R] - \overline{\mathbb{V}}[R] = \mathbb{V}\left[\mathbb{E}[R \mid \mathscr{F}^i]\right] = \mathbb{V}\left[\overline{\mathbb{E}}[R] + \Phi\frac{\tau_v}{\tau}v_i\right] = \mathbb{V}\left[\overline{\mathbb{E}}[R]\right] + \frac{\tau_v}{\tau^2}\Phi\Phi'. \tag{24}$$

The first equality states that the empiricist perceives additional variation in realized returns relative to investors of the model—she observes none of the information contained in $\mathscr{F}^i$. Because this information is heterogenous across investors, their expectations differ from consensus beliefs by the idiosyncratic noise in their private signal (the meaning of the second equality); accordingly, the last equality decomposes the additional variation the empiricist perceives into variation in the consensus beliefs and variation in investors' dispersed information. That dispersed information causes the empiricist to perceive additional variation in returns has economic consequences that we discuss in Section 3.1.

Without adding economic content to Eq. (24), however, this statistical relation does not say how it distorts the beta the empiricist estimates. To augment this statistical decomposition with an economic argument, we substitute the equilibrium relation in Eq. (20), which produces an endogenous link between the variation in excess returns, as measured by the econometrician, $\mathbb{V}[R]$, and the average investor, $\overline{\mathbb{V}}[R]$, respectively.

Lemma 1. *In equilibrium the informational distance between the matrices* $\mathbb{V}[R]$ *and* $\overline{\mathbb{V}}[R]$ *satisfies:*

$$\mathbb{V}[R] - \overline{\mathbb{V}}[R] = \left(\kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right)\overline{\mathbb{V}}[R] - \kappa\mathbf{I}_N, \tag{25}$$

where κ is a strictly positive coefficient:

$$\kappa \equiv \frac{\gamma^2}{\tau_M \tau_\epsilon}\left(\frac{1}{\tau_\epsilon} + \frac{\Phi'\Phi}{\tau}\right) + \frac{\tau_\upsilon}{\tau \tau_\epsilon} > 0. \tag{26}$$

Absent the Roll critique the informational distance in Eq. (25) vanishes. In equilibrium, variation in consensus beliefs arise because they move with the market portfolio M (see Eq. 20). Thus, eliminating the Roll critique (i.e., $\tau_M \to \infty$) removes variation in consensus beliefs. Similarly, eliminating variation in the market portfolio allows investors to gain perfect knowledge of the common factor (i.e., $\tau \to \infty$), which makes variation in investors' private information irrelevant. Hence, not only does the Roll critique create the informational gap in Eq. (25), it also makes it matter for the CAPM estimation.

Remarkably, the informational distance in Eq. (25) is determined by a unique, positive coefficient κ in equilibrium. The first term in Eq. (25) implies that the empiricist's covariance is an inflated version of investors' covariance, while the second term reduces the variances the empiricist measures on individual stocks. Thus, on balance, the distortion this informational distance implies on the empiricist's perception of the CAPM is still unclear.

To obtain a definitive answer, we compute the vector of betas that the empiricist estimates from realized returns:

$$\widetilde{\beta} = \frac{\frac{1}{N}\,\mathbb{V}[R]\,\mathbf{1}}{\mathbb{V}[R_M]}. \tag{27}$$

As in Section 2.1, we allow the empiricist to use the best proxy available for the market portfolio, its average $\overline{M}$. This proxy *is* the market portfolio that the average investor finds mean-variance optimal to hold unconditionally (see Corollary 1.1). We then use Lemma 1 to determine how the empiricist perceives the CAPM relation, the main result of the paper.

Theorem 1. ***(CAPM tests based on realized returns)*** *In the eyes of the empiricist the expected excess return on each asset $n \in \{1, 2, ...N\}$ and on the market satisfy the relation:*

$$\mathbb{E}[R_n] = \underbrace{\delta(1+\delta)^{-1}(1-\widetilde{\beta}_n)\,\mathbb{E}[R_M]}_{\textit{perceived mispricing (alpha)}} + \widetilde{\beta}_n\,\mathbb{E}[R_M]. \tag{28}$$

In equilibrium the empiricist's vector of betas, $\widetilde{\beta}$, and the average investor's vector of betas, β, both net of their average (their average is the beta on the market portfolio and thus equals one) satisfy the proportionality relation:

$$\widetilde{\beta} - \mathbf{1} = (1+\delta)(\beta - \mathbf{1}), \tag{29}$$

where the strictly positive coefficient δ measures the magnitude of the distortion in Eq. (28)*:*

$$\delta \equiv \frac{\kappa/N}{\mathbb{V}[R_M]} = \frac{1}{N\,\mathbb{V}[R_M]}\left[\frac{\gamma^2}{\tau_M\tau_\epsilon}\left(\frac{1}{\tau_\epsilon}+\frac{\Phi'\Phi}{\tau}\right)+\frac{\tau_v}{\tau\tau_\epsilon}\right] > 0. \tag{30}$$

The empiricist perceives mispricing (a non-zero alpha) for all stocks, except those that have a beta of one. Eq. (28) shows that low-beta assets ($\widetilde{\beta}_n < 1$) earn a positive unconditional alpha, whereas high-beta assets ($\widetilde{\beta}_n > 1$) earn a negative unconditional alpha. To see how perceived mispricing distorts the empiricist's view of the SML, write (28) as:

$$\mathbb{E}[R_n] = \frac{\delta}{1+\delta}\,\mathbb{E}[R_M] + \widetilde{\beta}\frac{\mathbb{E}[R_M]}{1+\delta}, \tag{31}$$

which resembles the zero-beta CAPM (Black, 1972). The first term implies that the empiricist perceives a SML with a positive intercept. The other term states that the perceived SML is flatter than the true SML of Corollary 1.1. In that respect, Eqs. (28) and (31) describe what is the biggest failure of the CAPM to many (e.g., Black, Jensen, and Scholes, 1972, and the literature that followed): the high returns enjoyed by many apparently low-beta assets and the high intercept of the SML.

Figure 1 illustrates the SML distortion implied by Eq. (31). The econometrician's SML rotates clockwise around the market portfolio, which flattens its slope and creates a positive intercept. How much flatter the SML is, and how large its intercept is, depends on the magnitude of the coefficient δ in Eq. (30). This coefficient is proportional to κ, which determines the size of the informational distance in Eq. (25). Hence, the larger this informational distance is, the flatter an SML the empiricist perceives. Likewise, absent the Roll critique the informational distance vanishes and so does the distortion in the SML. Hence, investors' information is irrelevant for testing the CAPM (Hansen and Singleton, 1982) as long as the market portfolio is known; it only matters for the test (Hansen and Richard, 1987) under the Roll critique.

The main result is that, in equilibrium, true betas are shrunk towards one relative to empiricist's betas. The "degree of shrinkage" is determined by a unique coefficient, δ, which adjusts the empiricist's betas towards true betas. Interestingly, Eq. (29) is identical to the Bayesian estimator proposed by Vasicek (1973), an estimator that is popular in the financial industry ("ADJ BETA" on Bloomberg terminals).[9] We emphasize, however, that the result of Theorem 1 is not due to sampling error. Nor is this result a standard attenuation bias, which commonly plagues the second pass cross-sectional regression in the Fama and MacBeth

[9] This linear adjustment has been first proposed by Blume (1971) (due to mean reversion of betas over time) and then by Vasicek (1973) (due to measurement error). See Bodie, Kane, and Marcus (2007), Berk and DeMarzo (2007) among others. Levi and Welch (2017) give best-practice advice for beta-shrinkage.

(1973) method.[10] Rather, in equilibrium shrinkage in betas arises as a joint consequence of the Roll (1977) critique and the Hansen and Richard (1987) critique.

A question that remains is, what is the empirical magnitude of the SML distortion? In other words, what is an empirically plausible value for the coefficient δ in Theorem 1? This matter is the subject of the next section.

2.4 Empirical magnitude of the distortion in beta estimates

This section provides a back-of-the-envelope calculation of the distortion in beta estimates. Consider the unconditional beta of any individual security n, as computed by the econometrician from realized excess returns on the asset and on the market. We decompose this beta using the law of total covariance in Eq. (24):

$$\widetilde{\beta}_n = \frac{\text{Cov}[R_n, R_M]}{\mathbb{V}[R_M]} \approx \frac{\overline{\text{Cov}}[R_n, R_M] + \text{Cov}[\overline{\mathbb{E}}[R_n], \overline{\mathbb{E}}[R_M]]}{\mathbb{V}[R_M]}. \tag{32}$$

The law of total covariance further implies that variation in investors' private information causes the empiricist to perceive additional variation in realized returns (the last term in Eq. 24). However, empirically disentangling this source of variation from variation in consensus beliefs requires observing data on individual investors' information and this data is unavailable. Since our purpose is simply to approximate the empirical magnitudes of the distortion in betas, we abstract from this source of variation in realized returns. As a result, the second equality above holds as an approximation. We also assume that the covariance $\overline{\text{Cov}}[R_n, R_M]$ is nonrandom (i.e. that betas of securities are not moving over time).[11]

We can further decompose the approximation above as follows:

$$\widetilde{\beta}_n \approx \underbrace{\frac{\overline{\text{Cov}}[R_n, R_M]}{\overline{\mathbb{V}}[R_M]}}_{=\beta_n} \underbrace{\frac{\overline{\mathbb{V}}[R_M]}{\mathbb{V}[R_M]}}_{\equiv 1-\mathcal{R}^2} + \underbrace{\frac{\text{Cov}[\overline{\mathbb{E}}[R_n], \overline{\mathbb{E}}[R_M]]}{\mathbb{V}[\overline{\mathbb{E}}[R_M]]}}_{\equiv \beta_{E,n}} \underbrace{\frac{\mathbb{V}[\overline{\mathbb{E}}[R_M]]}{\mathbb{V}[R_M]}}_{\equiv \mathcal{R}^2}. \tag{33}$$

The empiricist's beta is a weighted average of two terms. The first term is the "true beta" of security n as measured by the average investor. The second term, $\beta_{E,n}$, has a "beta-like" structure; it is a term that we could compute by means of simple (OLS) regression if only we observed time series of expected excess returns on stock n and the market. Both terms

[10] In Vasicek (1973), the degree of adjustment depends on the sample size and converges to zero as the sample size increases. Similarly, Shanken (1992) shows that the attenuation bias becomes negligible as the length of the sample period grows indefinitely (see also Jagannathan and Wang, 1998; Shanken and Zhou, 2007; Kan, Robotti, and Shanken, 2013). In our case, the adjustment is necessary even in infinite samples.

[11] This is equivalent to abstracting away from effects that can arise from fitting an unconditional model on a conditional one (Jagannathan and Wang, 1996; Lewellen and Nagel, 2006).

are weighted by $\mathcal{R}^2$, which represents the coefficient of determination from regressing excess returns of the market portfolio on the information set of the average investor (who holds the market portfolio).

Although the relation in Eq. (33) is merely a statistical decomposition, when combined with our theory it becomes a testable prediction. Namely, replacing the result of Theorem 1 in Eq. (33) produces an affine relation between the beta of the econometrician and $\beta_{E,n}$:

$$\widetilde{\beta}_n \approx \frac{\delta(1-\mathcal{R}^2)}{\mathcal{R}^2+\delta} + \left[1 - \frac{\delta(1-\mathcal{R}^2)}{\mathcal{R}^2+\delta}\right]\beta_{E,n}. \tag{34}$$

This relation forms the basis of our empirical tests. If we could measure β_E on individual stocks, then a cross-sectional regression would allow us to test whether a distortion exists at all: if $\delta > 0$, the intercept is positive and the slope is lower than one.[12]

At this stage, the empirical challenge is to obtain time series of expected returns on the market and, most importantly, on individual securities. From these time series we can then estimate $\beta_{E,n}$ on each stock and run the distortion test associated with Eq. (34). However, the main issue we investigate in this paper is precisely that investors' expectations are unobservable. The common approach to deal with this problem is to compute expected returns from factor models (e.g., Fama-French factors). A limitation of this approach is that investors likely possess information that typical asset-pricing factors do not capture—we will never be able to state confidently that we have taken into account all information that could have been relevant for investors. Recently, Martin (2017) and Martin and Wagner (2017) propose extracting this information all at once from option data. Option prices tell us what the market thinks about future returns and thus what the "average investor's expectations" are. This is the strategy we adopt here and that we explain next.[13]

Martin (2017) derives a lower bound on the equity premium using index option prices. Martin and Wagner (2017) extend this approach to compute expected returns on individual stocks, using index and stock option prices. Expected excess returns are derived on a daily basis at the different maturities of traded options: 30, 91, 182 and 365 days. These papers actually provide "bounds" on expected excess returns, as opposed to expected excess returns

[12] A positive intercept can also arise if δ is negative and larger in absolute value than $\mathcal{R}^2$. We discuss this (unlikely) possibility below.

[13] In related calculations, Buss and Vilkov (2012) use forward-looking information extracted from option prices to estimate *implied* market betas. Using these forward-looking betas, they find a monotonically increasing risk-return relation, with a slope close to the historical equity premium. In the context of our model, what Buss and Vilkov (2012) do is to compute the true betas based on the posterior variance of excess returns of the average investor, as in Corollary 1.1. In light of our theory, we interpret the findings of Buss and Vilkov (2012) as an alternative way of testing our result that the CAPM should look stronger if the econometrician uses the correct covariance matrix of excess returns.

Maturity	365 days		182 days		91 days		30 days	
	Estimate	Tests	Estimate	Tests	Estimate	Tests	Estimate	Tests
a	0.74^{***} (0.073)	10.21	0.67^{***} (0.056)	11.89	0.70^{***} (0.057)	12.28	0.70^{***} (0.059)	11.86
b	0.32^{***} (0.058)	11.65	0.40^{***} (0.044)	13.64	0.38^{***} (0.054)	11.48	0.35^{***} (0.049)	13.27
R^2	0.148		0.252		0.331		0.395	

Table 1: Evidence of distortion in beta estimates. Results of the regression specification (35), in which the true betas of the econometrician, $\widetilde{\beta}_n$, are regressed cross-sectionally onto the expected betas, $\beta_{E,n}$, at different horizons according to option maturities. Bootstrapped standard errors are provided in parentheses. The columns labeled "Tests" show t-stats for the separate null hypotheses $a = 0$ and $b = 1$.

themselves. However, all we need for our tests are covariances between expected returns (second moments), as opposed to levels (first moments). As in Martin and Wagner (2017), we compute expected excess returns for S&P 500 firms at the individual stock level. We obtain daily equity index prices and return data from CRSP and daily equity index options on the S&P 500 from OptionMetrics. We replicate the approach from Martin and Wagner (2017) and compute three measures of risk-neutral variance, which are then substituted into a parameter-free formula for expected returns on individual stocks. We use the resulting series of expected excess returns to compute $\beta_{E,n}$ defined in Eq. (33) on individual stocks.

Based on Eq. (34), we then regress econometrician's betas $\widetilde{\beta}$ onto "expected betas" β_E:

$$\widetilde{\beta}_n = a + b\beta_{E,n} + e_n. \tag{35}$$

Table 1 shows the estimation results. As expected, the intercept is strongly statistically significant at all maturities. In particular, Table 1 shows intercepts ranging from 0.67 to 0.74, with t-stats exceeding ten in all cases (standard errors are bootstrapped and provided in parentheses.) The slope coefficients are all lower than one, with t-stats highly statistically significant (these t-stats correspond to the null hypothesis that $b = 1$). These results strongly suggest a positive distortion in beta estimates.[14]

Importantly, the regression results of Table 1 allow us to perform a "back-of-the-envelope" calculation of δ. Specifically, Table 1 shows that the intercept at the 365-days horizon belongs

[14] If δ is negative and larger in absolute value than $\mathcal{R}^2$, the intercept of the regression (35) becomes positive again. But this case is unlikely. When $\delta < -\mathcal{R}^2$, it must be that $a > \underline{a}$, where $\underline{a} = \lim_{\delta \to -\infty} a = 1 - \mathcal{R}^2$. Taking it to the extreme and assuming that δ is huge and negative, with $\mathcal{R}^2 \in [0, 0.1]$, we should obtain $a > 0.9$. This hypothesis is rejected with the numbers from Table 1.

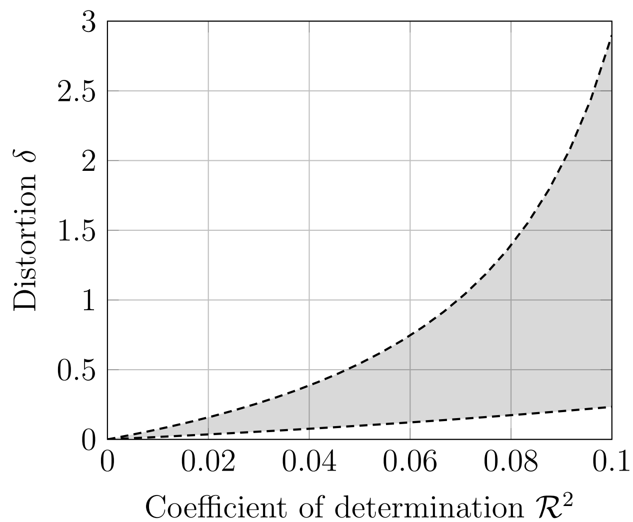

Figure 2: Empirically plausible range for the distortion δ. The shaded area shows the 90 percent confidence region for δ based on Eq. (36) and a 90 percent confidence range for the intercept a: $a \in [0.63, 0.87]$. The distortion is plotted as a function of the coefficient of determination from regressing excess returns of the market portfolio on the information set of the average investor, $\mathcal{R}^2 \equiv \mathbb{V}[\mathbb{E}^a[R_M]]/\,\mathbb{V}[R_M]$.

to the 90% confidence interval $a \in [0.63, 0.87]$. Furthermore, from Eq. (34) we obtain

$$\delta = \frac{a\mathcal{R}^2}{1 - a - \mathcal{R}^2}, \tag{36}$$

which provides a 90% confidence interval for δ. Determining this confidence interval further requires estimating the coefficient of determination from regressing market excess returns on the average investor's information, $\mathcal{R}^2 \equiv \mathbb{V}[\overline{\mathbb{E}}[R_M]]/\,\mathbb{V}[R_M]$. This coefficient is typically around 10%, depending on the horizon of option prices (Martin, 2017, Table 1). By comparison, in Cochrane (2011, Table 1), it is approximatively 11%, using return-forecasting regressions; accordingly, Figure 2 considers a range from zero to ten percent for $\mathcal{R}^2$ on the horizontal axis and depicts the resulting 90% confidence interval for δ in the shaded area.

The plot shows that the distortion can be significant, ranging from 0.3 to 3 if $\mathcal{R}^2 = 10\%$. By comparison, the Vasicek (1973) shrinkage proposed in finance textbooks (Bodie et al., 2007; Berk and DeMarzo, 2007) and adopted by practitioners is $\delta = 0.5$ (our 365-day point estimate and $\mathcal{R}^2 = 10\%$ jointly imply $\delta = 0.46$). The 90% confidence interval shows that the distorition can in fact be much larger (Levi and Welch (2017) is the only reference we know that advocates for a larger shrinkage). We emphasize that our calculations leave aside relevant sources of variation that can further magnify the distortion. For instance, we have ignored variation arising from differential information across agents (the last term in Eq. 24). Similarly, we have neglected the fact that betas may vary systematically with the market

Panel A: Ten beta-sorted portfolios					
	(a)	(b)	(c)	(d)	(e)
Portfolio	Avg. excess returns	Sample betas	Adj. betas ($\delta = 0.5$)	Adj. betas ($\delta = 3$)	Adj. betas ($\delta = 4.5$)
Low	0.54	0.61	0.74	0.90	0.93
2	0.51	0.73	0.82	0.93	0.95
3	0.58	0.83	0.89	0.96	0.97
4	0.66	0.97	0.98	0.99	0.99
5	0.54	1.01	1.01	1.00	1.00
6	0.63	1.08	1.05	1.02	1.01
7	0.51	1.15	1.10	1.04	1.03
8	0.65	1.27	1.18	1.07	1.05
9	0.63	1.39	1.26	1.10	1.07
High	0.61	1.61	1.40	1.15	1.11
Panel B: Securities Market Line					
Intercept		0.49	0.44	0.20	0.06
		(0.09)	(0.09)	(0.23)	(0.32)
Slope		0.09	0.14	0.38	0.52
		(0.06)	(0.09)	(0.23)	(0.32)

Table 2: Resurrecting the CAPM. Columns (a) and (b) of Panel A report average monthly excess returns for for ten beta-sorted portfolios, using monthly returns from July 1963 to July 2017, the market return and risk-free rate from Professor French's website. Columns (c) to (d) adjust betas according to Eq. (29), for three different values of the distortion δ. Panel B reports the intercept and the slope of the fitted Securities Market Line in each case. Standard errors of regression estimates are provided in brackets.

risk premium or market volatility (Cederburg and O'Doherty, 2016). In that respect, we believe our assessment of the distortion is conservative. The large numbers in Figure 2 thus suggest that the distortion in beta estimates implied by the Roll critique is substantial.

Can these magnitudes resurrect the CAPM as a valid asset pricing model? Using monthly returns for ten beta-sorted portfolios from July 1963 to July 2017, the market return and the risk-free rate (from Professor French's website), Panel A of Table 2 reports estimates of average excess returns and betas in columns (a) and (b). Eq. (29) then implies adjusted or "true" betas for different assumed values of the distortion parameter δ (a distortion of zero implies that measured and true beta are the same, for example). These adjusted betas are reported in columns (c) to (e) for δ equal to 0.5, 3 and 4.5 respectively. Each set of such betas, together with estimated average returns, leads to a different intercept and slope estimate of the Securities Market Line implied by the unconditional CAPM. These slope and intercept estimates, together with associated standard errors beneath, are reported in Panel B of Table 2 for measured beta and for the different assumed values of the distortion.

For $\delta = 0.5$, the performance of the CAPM is only marginally better than it is for measured beta, with a positive intercept and a low slope of around 0.1 (compared to the market premium of 0.52). At $\delta = 3$, the CAPM can no longer be rejected as the intercept is not significantly positive and the slope is not significantly below 0.52. At $\delta = 4.5$ our simple model fits the corrected estimates perfectly.

2.5 Interpreting anomalies in light of the Roll critique

In our view, two well-known anomalies naturally result from the Roll critique. One is the underperformance of high-beta stocks (Friend and Blume, 1970; Black et al., 1972)—exploited by "betting against beta" (Frazzini and Pedersen, 2014). The other is the underperformance of stocks that have high idiosyncratic volatility (Ang et al., 2006, 2009).

2.5.1 Betting against *measured* beta really means betting on *true* beta

Theorem 1 shows that the econometrician obtains an unconditional alpha for all the assets, but those with a beta of one. Specifically, the alpha of any asset or portfolio thereof satisfies:

$$\widetilde{\alpha}_n = \frac{\delta}{1+\delta}(1 - \widetilde{\beta}_n)\,\mathbb{E}[R_M] = \delta(1 - \beta_n)\,\mathbb{E}[R_M], \tag{37}$$

where the second equality follows from Theorem 1. Because the econometrician inflates betas above one and deflates those below, low-beta stocks ($\widetilde{\beta}_n < 1$) earn a positive unconditional alpha, whereas high-beta stocks ($\widetilde{\beta}_n > 1$) earn a negative unconditional alpha.

It follows that a long-short portfolio that goes long low-beta stocks and short high-beta stocks earns a positive unconditional alpha. This result is consistent with prior findings that a "betting against beta" (BAB) strategy generates positive abnormal returns (Friend and Blume, 1970; Black et al., 1972; Frazzini and Pedersen, 2014). However, its interpretation differs. Eq. (37) shows the unconditional alpha of asset n depends on its true beta. In fact, the empiricist overestimates high betas and underestimates low betas. As a result, low-beta portfolios seem less risky than they really are and high-beta portfolios seem riskier than they really are; this leads to a reinterpretation of abnormal returns earned by the BAB strategy—betting against *measured* beta really is betting on *true* beta.

This reinterpretation finds empirical support in recent work by Cederburg and O'Doherty (2016), who show that perceived abnormal performance on a BAB strategy disappears after properly incorporating conditioning information. This result suggests, as Theorem 1 predicts, that the informational distance between the empiricist and market participants gives the illusion of positive abnormal returns earned by BAB.

2.5.2 The idiosyncratic volatility puzzle

Idiosyncratic variance is the variation in returns that the market does not explain. Because the empiricist misperceives beta, her perception of idiosyncratic variance:

$$\widetilde{\sigma}^2_{n,id} \equiv \mathbb{V}[R_n] - \widetilde{\beta}^2_n \, \mathbb{V}[R_M], \tag{38}$$

on an individual stock n is mechanically distorted (her measure of beta appears on the right-hand side of Eq. 38). Applying the result of Lemma 1 to $\mathbb{V}[R_n]$ and $\mathbb{V}[R_M]$, we rewrite Eq. (38) to make its dependence on the asset's true beta explicit. For simplicity, we let the number of assets in the economy grow large, $N \to \infty$, and further assume that all assets have the same *true* idiosyncratic variance, which we denote by $\overline{\mathbb{V}}[R_n] \equiv \sigma^2_{id}$. Eq. (38) becomes:

$$\tilde{\sigma}^2_{n,id} = \underbrace{\left(1 + \kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M \tau_\epsilon}\right)\sigma^2_{id} - \kappa}_{\text{individual variance } \mathbb{V}[R_n]} - \left(1 + \kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M \tau_\epsilon}\right) \underbrace{(\tilde{\beta}^2_n - \beta^2_n)}_{\text{measurement error}} . \tag{39}$$

Idiosyncratic volatility and expected returns are negatively related (the idiosyncratic volatility puzzle). The first term in Eq. (39) is simply the variance the empiricist measures on an individual stock, which is affine in the true idiosyncratic variance (the result of Lemma 1). The second term is the mismeasurement in idiosyncratic variance inherited from beta mismeasurement. Based on the relation between measured and true betas in Theorem 1, it follows that high-beta stocks appear to have lower idiosyncratic volatility than low-beta stocks. As an illustration, let us define "idiosyncratic volatility distortion" as:

$$\textit{Idiosyncratic volatility distortion} \equiv \frac{\widetilde{\sigma}_{n,id}}{\sigma_{id}}, \tag{40}$$

the ratio of perceived to actual idiosyncratic volatility. We plot this distortion as a function of the asset's true beta in Figure 3. If the view of the empiricist and that of investors coincided, this distortion would be identically one for all assets (the dashed line). The informational distance instead causes the empiricist to underestimate the idiosyncratic volatility of high-beta stocks and vice-versa (the solid line).[15]

In the framework of Theorem 1, betting against beta and the idiosyncratic volatility puzzle are two anomalies that originate from the same effect—the econometrician mis-measures unconditional betas because the composition of the market portfolio is unobservable. In

[15]The distortion is non-linear, which generates a peculiar effect when β_j gets close to zero. In this case, the beta of the econometrician $\tilde{\beta}_j$ can become negative and larger in absolute value than the true beta. Empirically, however, about 95% of stocks have a *measured* beta higher than zero (Cederburg and O'Doherty, 2016, Figure 1), which suggest that this effect is likely to be small.

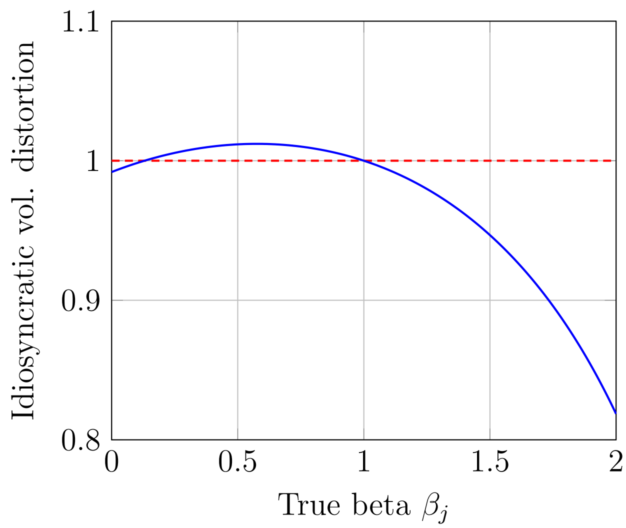

Figure 3: The idiosyncratic volatility puzzle. The figure depicts the idiosyncratic volatility distortion (defined in Eq. (40) as the idiosyncratic volatility perceived by the econometrician divided by the true idiosyncratic volatility) in two situations. The red dashed line is obtained when the view of the econometrician and the view of market participants coincide (i.e., the econometrician finds the CAPM). The blue solid line depicts the distortion when $\tilde{\beta}_j \neq \beta_j$. In this latter case, the econometrician (falsely) uncovers the idiosyncratic volatility puzzle.

other words, both phenomena are merely collateral effects of the Roll critique.

3 Extensions

We extend the model analyzed in Section 2 along several dimensions. We first allow the econometrician to control for all publicly available information when testing the CAPM. If investors possess private information that prices do not fully reveal, conditioning on public information may not prove helpful and may even be detrimental to the test. We also study how heterogeneity in asset supplies and the presence of multiple factors driving asset payoffs affect CAPM tests. Although none of these features compromise the validity of the *true* CAPM relation, they complicate the econometrician's task. Both features break the linearity of the perceived CAPM relation and heterogeneity in supplies may further lead the econometrician to uncover a downward-sloping SML.

3.1 Conditioning on public information

The main result of the paper (Theorem 1) assumes the empiricist's information is limited to realized returns. Hence, augmenting the empiricist's dataset with all relevant public information would seem a natural remedy for the distortion of Theorem 1. However, controlling for public information may not help the empiricist improve the CAPM test. For instance, suppose we allow the econometrician to condition on the public signal G when computing betas of securities. Equilibrium excess returns in Proposition 1 then imply that

$$\mathrm{Cov}[R, G] = \mathbf{0}\,. \tag{41}$$

It follows that controlling for the public signal cannot possibly improve the empiricist's estimate of the covariance matrix of returns.

The public signal does not improve the empiricist's estimate because prices already incorporate all public information available in the economy. Thus, the econometrician should control jointly for prices and the signal G, both of which are publicly observable. Although the econometrician does not know the equilibrium structure of prices in Eq. (13), she knows that prices necessarily reveal information about the market portfolio. She thus uses all prices in the economy as controls in regressions, together with the public signal G.[16] Under this augmented information set, we show in Appendix A.5 that the relation in Eq. (29) still holds, but with a different coefficient of distortion, $\widehat{\delta}$:

$$\widehat{\delta} = \frac{1}{N\,\mathbb{V}[R_M|\{P, G\}]}\frac{\tau_v}{\tau_\epsilon(\tau - \tau_v)} \geq 0, \tag{42}$$

where $\mathbb{V}[R_M|\{P, G\}]$ denotes the variance of excess returns on the market portfolio conditional on observing all publicly available information.

In this economy without private information ($\tau_v = 0$), the CAPM is untestable *in practice*. If only the empiricist could condition on all publicly available information, the distortion of Theorem 1 would vanish and the empiricist would recover the true CAPM relation. The presence of dispersed private information makes the CAPM untestable *in principle* (Roll, 1977). It will always remain the case that the empiricist observes a flatter SML, even after controlling for all publicly available information.

Surprisingly, comparing Eqs. (30) and (42), it remains unclear whether the econometrician is always better off conditioning on all publicly available information. If the precision of private information, τ_v, is small enough, the distortion $\widehat{\delta}$ is smaller than δ. However, if private information is sufficiently precise, conditioning on publicly available information amplifies the distortion, $\widehat{\delta} > \delta$ (see Appendix A.5). Intuitively, in our model the signal-noise ratio of equilibrium prices may decrease as the precision of private information increases. This phe-

[16] Although we have shown above that conditioning on the public signal G alone does not help, it must be included in the information set when conditioning on prices. If not, then it creates an omitted variable bias (because both prices and realized returns depend on the public signal).

nomenon occurs when the market portfolio is noisy and private information is simultaneously precise (see Appendix A.5). We conclude that conditioning on public information only offers an unsatisfactory solution. Not only would it be a daunting task for the econometrician to take into account all relevant public information, but it may not even be optimal to do so.

Nevertheless, many of the asset pricing anomalies in the literature involve conditioning on publicly available information, such as book-to-market ratios, past returns, etc. Our analysis may well prove relevant to these anomalies, as suggested by Eq. (28). This equation further licenses a look at how measured alpha depends on measured beta.

3.2 Multiple factors driving asset payoffs

In this section we investigate how the presence of multiple factors may affect our results. The main implication is that, although the CAPM holds for the average investor, expected excess returns and econometrician's betas do not plot on a straight line—the econometrician now faces a "broken" SML.

Denote a vector of $K \leq N$ independent factors by $\mathbf{F} \equiv [F_1 \quad F_2 \quad \cdots \quad F_K]'$ and let this vector be normally distributed with mean zero and covariance matrix $\tau_F^{-1}\mathbf{I}_K$. Furthermore, let the vector of realized dividends have the following structure:

$$D = \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,K} \\ \phi_{2,1} & \phi_{2,2} & \cdots & \phi_{2,K} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{N,1} & \phi_{N,2} & \cdots & \phi_{N,K} \end{bmatrix} \mathbf{F} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \end{bmatrix} \equiv \Phi \mathbf{F} + \epsilon. \tag{43}$$

The only difference relative to the setup from Section 2.2 is that payoffs are now driven by multiple factors. For expositional convenience, we present the solution of this generalized setup in Appendix A.6. One complication associated with this setup is that investors' precision regarding the factors driving the payoff structure in Eq. (43),

$$\tau \equiv \mathbb{V}[\mathbf{F}|\mathscr{F}^i]^{-1}, \tag{44}$$

is a $K \times K$ matrix. Nonetheless, in the presence of multiple factors, it must remain the case that the average investor plots the SML on a straight line that crosses the origin with the market risk premium as its slope.

As in Section 2.2, we use the law of total covariance to obtain a relation between the

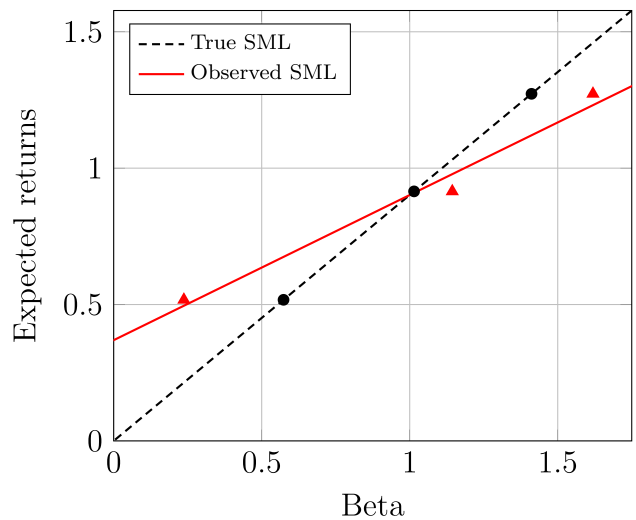

Figure 4: Securities Market Line under Multiple Factors. This figure plots the SML in a three-asset economy with two factors. The matrix of loadings on the two factors is given in Eq. (46). All remaining parameters are set to unity. The dashed black line represents the true SML, while the solid red line is the observed SML. The solid line depicts the linear fit on the three red triangles.

empiricist's covariance matrix, $\mathbb{V}[R]$, and that of the average investor, $\overline{\mathbb{V}}[R]$:

$$\mathbb{V}[R] = \overline{\mathbb{V}}[R] + \frac{\gamma^2}{\tau_M^2}\overline{\mathbb{V}}[R]\overline{\mathbb{V}}[R] + \tau_v \Phi \tau^{-1}\tau^{-1}\Phi'. \tag{45}$$

From this relation we then derive the econometrician's view of the SML. Notice, however, that the result of Lemma 1 does not obtain in this context. Because asset payoffs are driven by multiple factors, the relation between the econometrician's view and that of investors cannot be characterized by a unique number, as opposed to the relation in Lemma 1.

To illustrate how the presence of multiple factors affects the distortion in econometrician's betas, we use a numerical example. We consider an economy with three assets and two factors. The matrix of loadings on the two factors is:

$$\Phi = \begin{bmatrix} 1.5 & 1.2 & -2 \\ 1.1 & 2 & 0.8 \end{bmatrix}'. \tag{46}$$

We plot in Figure 4 the true and observed SML associated with this economy. In this example, the presence of multiple factors breaks the linearity of the perceived SML. Hence, not only does the observed SML rotate clockwise around the market portfolio, but the econometrician also observes a "broken" SML.

3.3 Heterogeneity in asset supplies

The previous section shows that the presence of multiple factors driving asset payoffs causes the proportionality result of Theorem 1 to fail. Assuming that assets are in unequal supplies has a similar effect. Heterogeneous asset supplies have yet another aggravating effect: the econometrician may perceive a downward-sloping Securities Market Line, although the actual Securities Market Line is *always* upward-sloping. In the model this situation typically occurs when assets that have a high measured beta simultaneously have a low market supply (and vice-versa). Intuitively, there is a conflicting relation between, on the one hand, how much variation in an asset's returns the market can explain—the asset's beta—and, on the other hand, the importance of this asset in the market portfolio—its relative supply.

We start by showing that heterogeneity in asset supplies violates the result of Theorem 1. In Section 2.3, we showed that the variance of the econometrician satisfies Lemma 1, a result that still holds under unequal supplies. Following the steps of Appendix A.4 for an arbitrary vector $\overline{M}$ of unconditional supply (whose elements are positive and sum up to one), the average investor's betas and the econometrician's betas satisfy the following relation:

$$\widetilde{\beta}_n - 1 = (1+\delta)(\beta_n - 1) - \frac{\kappa}{\mathbb{V}[R_M]}(\overline{M}_n - \|\overline{M}\|^2), \quad \text{for } n = 1, 2, ...N, \tag{47}$$

where κ is defined in Lemma 1 and the distortion coefficient δ satisfies:

$$\delta = \frac{\kappa \|\overline{M}\|^2}{\mathbb{V}[R_M]}. \tag{48}$$

Compared to the proportionality relation of Theorem 1, Eq. (47) includes an additional term, the sign and magnitude of which depend on the difference $\overline{M}_n - \|\overline{M}\|^2$. This difference represents the unconditional supply of asset n in excess of the weighted average of supplies across assets.[17] We recover the result of Section 2 when assets are in equal supply, $\|\overline{M}\|^2 = 1/N$, in which case this difference is zero and the last term in Eq. (47) thus drops out. To see how the last term in Eq. (47) affects the measured slope of the SML, multiply Eq. (47) with $\mathbb{E}[R_M]$ and rearrange to obtain the relation:

$$\mathbb{E}[R_n] - \mathbb{E}[R_M] = \frac{\mathbb{E}[R_M]}{1+\delta}\left[(\widetilde{\beta}_n - 1) + \delta\left(\frac{\overline{M}_n}{\|\overline{M}\|^2} - 1\right)\right], \tag{49}$$

which describes returns of an individual asset n in excess of market returns.

Heterogeneity in supplies gives the empiricist the illusion of an additional factor. For

[17] The squared norm, $\|\overline{M}\|^2$, can be interpreted as a weighted average in this context since $\sum_{n=1}^{N} \overline{M}_n = 1$.

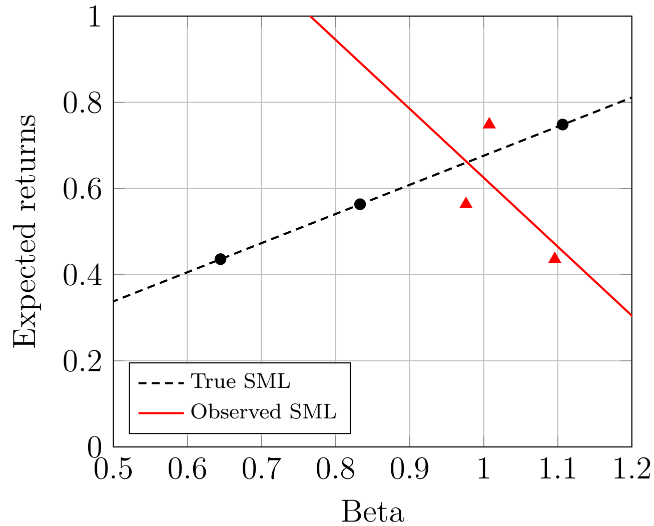

Figure 5: Securities Market Line under Heterogeneous Supplies. This figure illustrates the true (dashed black line) and observed (solid red line) SML when stocks are in heterogeneous supplies. This illustrative economy has three assets with factor loadings, $\phi_1 > \phi_2 > \phi_3 > 0$, and with supplies in the market portfolio, $0 < \overline{M}_1 < \overline{M}_2 < \overline{M}_3$.

instance, consider an asset that earns negative returns in excess of the market portfolio, i.e., the left-hand side in Eq. (49) is negative. Suppose the empiricist finds this asset is *riskier* than the market ($\widetilde{\beta}_n > 1$) and thus contemplates a downward-sloping SML. She rationalizes this outright rejection of the CAPM with the second term in Eq. (49), which misleadingly suggests the asset's relative supply in the market matters. Adding a "supply factor" to the CAPM relation, the empiricist concludes this high-beta asset earns negative returns relative to the market simply due to its small relative supply. Of course, this supply effect is merely an illusion, since the true CAPM never ceases to hold. Supply appears as a priced factor because beta is mismeasured—because relative supply in Eq. (49) scales with the distortion δ, there can be no supply effect without distortion in betas.

More broadly, Eq. (49) highlights a tension between an asset's measured beta and its relative importance in the market portfolio. When sufficiently strong, this tension leads the empiricist to perceive a downward-sloping SML, as illustrated in Figure 5, which depicts the true and observed SML in an economy with three assets. Not only do assets no longer plot on a straight line (solid line), but the empiricist also perceives a downward-sloping SML. Although a true CAPM relation holds, the empiricist rejects the CAPM at once.

4 CAPM on announcement days in a dynamic model

We consider a dynamic version of the model of Section 2.2, assuming a public announcement is made periodically (e.g., FOMC meetings, unemployment or inflation announcements). The purpose is to show the CAPM relation measured by the econometrician is distinctly steeper on announcement days relative to non-announcement days (Savor and Wilson, 2014).

4.1 A dynamic model with periodic public announcements

Consider a discrete-time economy that goes on forever (e.g. Spiegel, 1998; Watanabe, 2008; Andrei, 2013). As in Section 2.2, the economy is populated with a continuum of investors indexed by $i \in [0, 1]$, who have CARA utility with common risk aversion γ. Each investor i lives for two periods, entering period t with wealth W_t^i and consuming W_{t+1}^i next period. There are N risky assets (stocks) and an exogenous riskless bond with constant gross interest rate $R_f > 1$. At each period t the N stocks pay a vector D of dividends satisfying the dynamic equivalent to the common factor structure in Eq. (10):

$$D_t = \overline{D}\,\mathbf{1} + \Phi F_t + \epsilon_t^D, \tag{50}$$

where $\epsilon_t^D \sim \mathscr{N}(\mathbf{0}_N, \tau_\epsilon^{-1}\,\mathbf{I}_N)$ is an i.i.d. asset-specific innovation and F denotes a factor that commonly affects all dividends in the cross section of stocks. As in Section 2.2, the vector Φ of loadings is the only source of heterogeneity across assets.

We assume the factor F_t mean-reverts over time around zero according to

$$F_t = \kappa_F F_{t-1} + \epsilon_t^F, \quad \epsilon_t^F \sim \text{i.i.d. } \mathscr{N}(0, \tau_F^{-1}), \tag{51}$$

with $0 \leq \kappa_F \leq 1$. As in Section 2.2, the market portfolio—the per-capita supply of stocks—is random. We further assume that the market portfolio mean-reverts around its average, $\overline{M}$, a vector of dimension N with identical elements that sum up to one, $\overline{M} = \mathbf{1}/N$:

$$M_t = (1 - \kappa_M)\overline{M} + \kappa_M M_{t-1} + \epsilon_t^M, \quad \epsilon_t^M \sim \text{i.i.d. } \mathscr{N}(\mathbf{0}_N, \tau_M^{-1}\,\mathbf{I}_N). \tag{52}$$

Investors observe private information at each trading date. Formally, at any date s, each investor i receives a private signal v^i about the current factor innovation:

$$V_s^i = \epsilon_s^F + v_s^i, \quad v_s^i \sim \mathscr{N}(0, \tau_v^{-1}),\ v_s^i \perp v_s^k,\ \forall k \neq i. \tag{53}$$

Importantly, in addition to private signals, we introduce periodic public announcements.

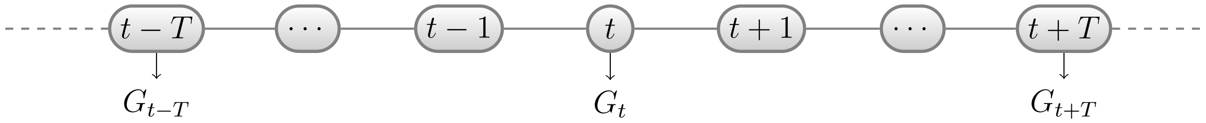

Figure 6: Timeline of periodic public announcements.

Investors observe every T periods a new public signal centered on the fundamental F, according to the sequence illustrated in Figure 6. As a convention, we denote by t the time at which a public announcement is made. Then, for any date $t-k$, with $k \in \{-T+1, ..., T\}$ public announcements take the form:

$$G_{t-k} = \begin{cases} F_{t-T} + v^G_{t-T}, & \forall k \in \{1, ..., T\} \\ F_t + v^G_t, & \forall k \in \{-T+1, ..., 0\}, \end{cases} \tag{54}$$

with normal, independent noise $v^G_{t-T},\ v^G_t \sim \mathscr{N}(0, \tau_G^{-1})$. We further assume that all investors commonly observe the fundamental at lag T and beyond (Townsend, 1983; Singleton, 1987).[18]

Based on this information, along with current and past dividends and prices, each investor i builds at time $t-k$ a forecast of cash flows next period, $P_{t-k+1}+D_{t-k+1}$ (see Appendix A.7). As in Section 2.2, we focus on equilibria in which the price P is a linear function of the state variables of the economy.[19] Since investors are myopic and have CARA preferences, at each lag k in the announcement cycle their asset demands take the following standard form:

$$x^i_{t-k} = \frac{1}{\gamma}\, \mathbb{V}^i_{t-k}\left[P_{t-k+1} + D_{t-k+1}\right]^{-1} \left(\mathbb{E}^i_{t-k}\left[P_{t-k+1} + D_{t-k+1}\right] - R_f P_{t-k}\right), \tag{55}$$

where $R_{t+1} \equiv P_{t+1} + D_{t+1} - R_f P_t$ represents the dollar excess returns on individual securities.

In Section 2.2 we defined the average investor as a fictitious agent who holds consensus beliefs. Keeping the same notation we denote by $(\overline{\mathbb{E}}_{t-k}[\cdot], \overline{\mathbb{V}}_{t-k}[\cdot])$ the dynamic equivalent to consensus beliefs at lag k in the cycle. In contrast to the static beliefs of Section 2.2, however, these beliefs move over the announcement cycle, hence the index $t-k$. Because

[18] We make this assumption for tractability—without it the information structure would introduce an infinite-regress inference problem whereby investors would need to infer unobservables shocks over infinitely many periods back in time. This assumption eliminates the infinite-regress problem: at any time s, the fundamental F_{s-T} becomes public information and thus investors only need to infer unobservable shocks up to lag $T-1$.

[19] With N assets, there are 2^N linear equilibria in this model. We focus on the low-volatility equilibrium, as this is the only stable equilibrium to which a finite horizon economy would converge. Bacchetta and Wincoop (2008), Banerjee (2010), Watanabe (2008), and Andrei (2013) discuss the multiplicity of equilibria in infinite horizon models.

the market portfolio M_{t-k} is mean-variance efficient for the average investor, we obtain a *conditional* pricing relation at each date of the announcement cycle:

$$\overline{\mathbb{E}}_{t-k}\left[R_{t-k+1}\right] = \gamma \overline{\mathbb{V}}_{t-k}\left[R_{t-k+1}\right] M_{t-k}. \tag{56}$$

An important difference with the static model of Section 2.2 is that the posterior variance of the average investor moves *deterministically* with each lag k. As a result, the unconditional risk premium now varies deterministically over the announcement cycle.

Finally, taking unconditional expectations of the pricing relation in Eq. (56) and performing a few manipulations yields an unconditional CAPM relation from the perspective of the average investor who holds the unconditional market portfolio, $\overline{M} = \mathbf{1}/N$:

$$\mathbb{E}[R_{t+1}] = \frac{\frac{1}{N}\widehat{\mathbb{V}}\left[R_{t+1}\right]\mathbf{1}}{\widehat{\mathbb{V}}\left[R_{M,t+1}\right]}\,\mathbb{E}[R_{M,t+1}]. \tag{57}$$

By analogy to the static model, $R_{M,t+1} \equiv \overline{M}' R_{t+1}$ is the realized return on the market portfolio. Furthermore, since the posterior variance $\overline{\mathbb{V}}_{t-k}[\cdot]$ varies at each lag k in the cycle, $\widehat{\mathbb{V}}[\cdot]$ represents the unconditional average over all lags in the announcement cycle.

4.2 The CAPM on announcement days

We now estimate a Securities Market Line in this dynamic equilibrium model. Following the steps of Section 2.3, we start with the law of total covariance:

$$\mathbb{V}[R_{t+1}] = \widehat{\mathbb{V}}_t[R_{t+1}] + \mathbb{V}\left[\overline{\mathbb{E}}_t[R_{t+1}]\right] + \sigma_v^2 \Pi\Pi', \tag{58}$$

where Π is a matrix of conformable dimension that multiplies the vector of private signals in each investor i's individual expectation, $\mathbb{E}_t^i[R_{t+1}]$. The market-clearing condition (Eq. 56) then produces a relation between expected excess returns and their conditional variance:

$$\mathbb{V}[R_{t+1}] = \widehat{\mathbb{V}}_t[R_{t+1}] + \gamma^2\,\mathbb{V}\left[\overline{\mathbb{V}}_t[R_{t+1}]M_t\right] + \sigma_v^2 \Pi\Pi'. \tag{59}$$

The vector of econometrician's betas is defined analogously to the static setup:

$$\widetilde{\beta} = \frac{\frac{1}{N}\,\mathbb{V}\left[R_{t+1}\right]\mathbf{1}}{\mathbb{V}\left[R_{M,t+1}\right]}. \tag{60}$$

Relative to the static framework of Section 2.2, the empiricist perceives additional variation in excess returns, because the conditional covariance matrix of the average investor,

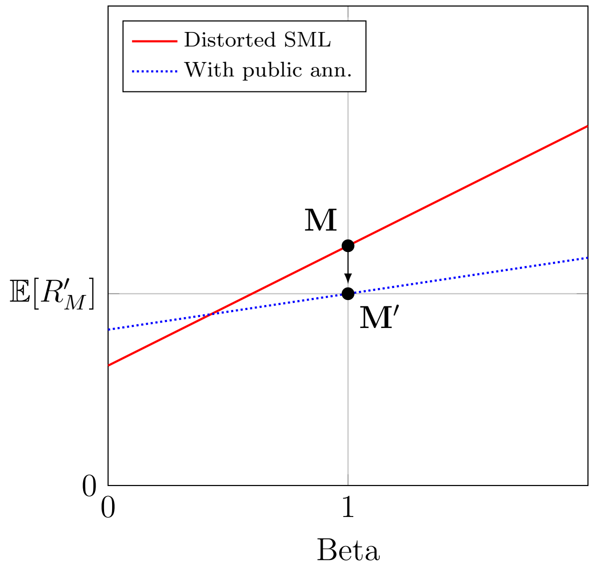

Figure 7: Distortion of the Securities Market Line and Public Announcements. This figure illustrates the observed SML in an economy with and without public announcements (dotted blue line and the solid red line, respectively). Periodic public announcements adds to the SML flattening due to fluctuations in the conditional variance of returns.

$\overline{\mathbb{V}}_t[R_{t+1}]$, varies over the announcement cycle. This dynamic effect captures the notion of "conditional CAPM" (Jagannathan and Wang, 1996). To illustrate how this additional variation affects the SML, we plot in Figure 7 the observed SML in a dynamic economy *with* and *without* public announcements. The implication of Theorem 1 remains valid in this dynamic setup—the observed SML in an economy without public announcements (the red line) rotates clockwise around the market portfolio coordinate, $\mathbf{M} = (1, \mathbb{E}[R_M])$.

Introducing periodic announcements causes the covariance matrix $\mathbb{V}_t[R_{t+1}]$ of excess returns to vary over the announcement cycle. This additional source of variation—the second term on the right-hand side of Eq. (59)—reinforces the distortion in the observed SML, as illustrated by the dotted line in Figure 7. Because announcements make investors relatively better informed, they require a lower premium for holding stocks *on average*, which reduces the *unconditional* risk premium in the economy. Therefore, the market portfolio moves down towards the point $\mathbf{M'} = (1, \mathbb{E}[R'_M])$ in the plot, adding to the SML flattening.

We further decompose the CAPM relation over a typical announcement cycle. As opposed to averaging the CAPM relation over the cycle as we do in Figure 7, we now let the empiricist measure the SML on announcement and non-announcement days separately. For illustrative purposes, we fix the number of stocks to $N = 4$ and assume a public announcement is made every $T = 4$ periods.[20] We then follow the methodology in Savor and Wilson (2014) and

[20]Other parameters are: $\gamma = 2$, $\kappa_F = \kappa_M = 0.99$, $\phi_1 = 1$, $\phi_2 = 1/2$, $\phi_3 = 1/3$, $\phi_4 = 1/4$, $\sigma_F = \sigma_M = 0.06$, $\sigma_D = 0.32$, $\sigma_G = 0.001$, $\overline{X} = 1$, $\overline{D} = 0$, and $R = 1.22$.

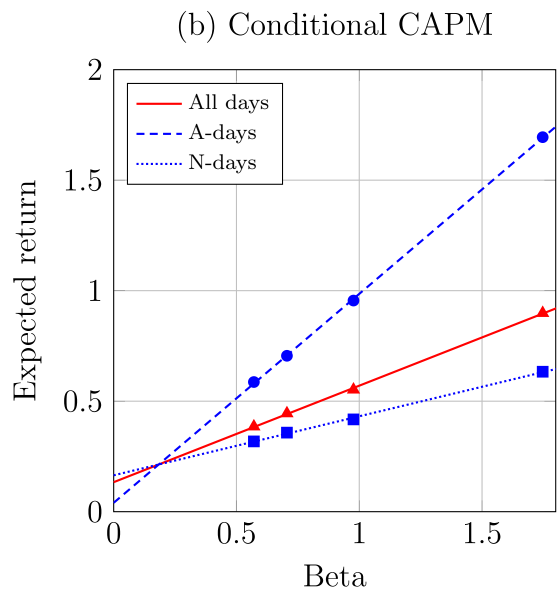

Figure 8: CAPM during announcement and non-announcement days. This figure shows the SML during all days (solid), announcement days (dashed), and non-announcement days (dotted). The CAPM looks stronger on announcement days (Savor and Wilson, 2014).

estimate betas over the whole sample.[21] Figure 8 shows the SML estimated on announcement days (dashed line) and non-announcement days (dotted line). For comparison we also report the SML averaged over all days (solid line), which is identical to the dotted line in Figure 7. The three lines pivot around the same point, which corresponds to a portfolio that does not load on the factor F and that macroeconomic information thus does not affect.

The observed SML steepens on announcement days, thus moving closer to the true SML (Savor and Wilson, 2014). In our model this phenomenon arises through two channels. On the morning of the announcement day, investors forecast how prices will move after the announcement is made. These prices depend on the public signal G_t, which investors do not observe yet. Because there is uncertainty about the announcement, uncertainty about future cash flows, $\overline{\mathbb{V}}_{t-1}[P_t + D_t]$, is higher prior to the announcement. Investors thus require a higher risk premium prior to the announcement (see Eq. (56)), making expected and realized return on the market higher on *A-days*. Holding betas of assets constant, this effect is stronger for high-beta assets (Proposition 1), leading to a steeper SML. Second, the informational distance between investors and the empiricist shrinks as public information is released. Therefore, the distortion in the observed SML is reduced on *A-days* and for a fleeting moment the observed SML moves closer to the true SML.

[21] Our results do not depend on holding the betas constant. In separate calculations, we estimate different betas on different types of days. Although there is variation in betas across types of days, the same steepening (flattening) obtains during *A-days* (*N-days*).

5 Conclusion

It has not escaped our notice that the distortion analyzed in this article carries over to factor models in asset pricing. Some variables may appear to the econometrician as priced factors simply because betas are mismeasured. Rather than being priced factors these variables are instruments for the measurement error in betas. Do there exist economic criteria that would allow the empiricist to distinguish variables that are economically meaningful from those that are not? This matter opens up fascinating directions for future research.

The basic premise of this paper is that investors actually hold the market portfolio, while the econometrician merely tests whether the market portfolio is mean-variance efficient. Holding the market portfolio presumably gives investors an informational edge over the econometrician. Observing investors' actions (for instance, their investment decisions) likely reveals some of this information. This approach based on "revealed preference" has caught on recently (Barber, Huang, and Odean, 2016; Berk and Van Binsbergen, 2016).

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A Appendix

A.1 Background

This appendix derives Equations (8) and (9) in the text. Eq. (8) follows from the law of iterated expectations:

$$\mathbb{E}[M'R|M] = \mathbb{E}[\mathbb{E}[M'R|\mathscr{F}]|M] = \mathbb{E}[M'\gamma\mathbb{V}[R|\mathscr{F}]M|M] = M'\gamma\mathbb{V}[R|\mathscr{F}]M = \mathbb{E}[M'R|\mathscr{F}]. \tag{61}$$

For Eq. (9), use the definition of covariance:

$$\begin{aligned}
\mathbb{E}[U_W(\overline{M}'R)R] &= \mathbb{E}[U_W(\overline{M}'R)]\,\mathbb{E}[R] + \mathrm{Cov}[U_W(\overline{M}'R), R] & (62)\\
&= \mathbb{E}[U_W(\overline{M}'R)]\,\mathbb{E}[R] + \mathbb{E}[U_{WW}(\overline{M}'R)]\,\mathbb{V}[R]\overline{M} & (63)\\
&= \left(-\frac{\mathbb{E}[U_{WW}(\overline{M}'R)]}{\gamma}\right)\gamma\,\mathbb{V}[R|\mathscr{F}]\overline{M} + \mathbb{E}[U_{WW}(\overline{M}'R)]\,\mathbb{V}[R]\overline{M} & (64)\\
&= \mathbb{E}[U_{WW}(\overline{M}'R)](\mathbb{V}[R] - \mathbb{V}[R|\mathscr{F}])\overline{M}. & (65)
\end{aligned}$$

The third equality follows from using the unconditional version of Eq. (3) and from the CARA utility assumption. The law of total covariance shows that the term $\mathbb{V}[R] - \mathbb{V}[R|\mathscr{F}]$ is non-zero:

$$\begin{aligned}
\mathbb{V}[R] &= \mathbb{V}[R|\mathscr{F}] + \mathbb{V}[\mathbb{E}[R|\mathscr{F}]] & (66)\\
&= \mathbb{V}[R|\mathscr{F}] + \mathbb{V}\left[\gamma\,\mathbb{V}[R|\mathscr{F}]M\right] & (67)\\
&= \mathbb{V}[R|\mathscr{F}] + \gamma^2\,\mathbb{V}[R|\mathscr{F}]^2\,\mathbb{V}[M], & (68)
\end{aligned}$$

and thus, as long as the econometrician does not observe the market portfolio M,

$$\mathbb{V}[R] - \mathbb{V}[R|\mathscr{F}] = \gamma^2\,\mathbb{V}[R|\mathscr{F}]^2\,\mathbb{V}[M] \neq 0. \tag{69}$$

A.2 Proof of Proposition 1

Start by conjecturing a linear price function of the form:

$$\underbrace{\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix}}_{\alpha} F + \underbrace{\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix}}_{g} G + \underbrace{\begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1N} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2N} \\ \vdots & \vdots & \ddots & \\ \xi_{N1} & \xi_{N2} & \cdots & \xi_{NN} \end{bmatrix}}_{\xi} \underbrace{\begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_N \end{bmatrix}}_{M}, \tag{70}$$

where the undetermined coefficients multiplying the random variables F, G, and M will be pinned down by the market clearing condition. Any investor i has three sources of information gathered in $\mathscr{F}^i$ in Eq. (14): (*i*) N public prices, (*ii*) one private signal V^i, (*iii*) and one public signal G. We isolate the informational part of prices by subtracting the (known) public signal:

$$P^a = P - gG = \alpha F + \xi M, \tag{71}$$

and stack all information of investor i, both private and public, into a single vector

$$S^i = \begin{bmatrix} P^a \\ V^i \\ G \end{bmatrix} = \begin{bmatrix} \alpha \\ 1 \\ 1 \end{bmatrix} F + \begin{bmatrix} \xi & 0_{N\times 1} & 0_{N\times 1} \\ 0_{1\times N} & 1 & 0 \\ 0_{1\times N} & 0 & 1 \end{bmatrix} \begin{bmatrix} M \\ v^i \\ v \end{bmatrix} \equiv HF + \Theta \begin{bmatrix} M \\ v^i \\ v \end{bmatrix}, \tag{72}$$

where the vector of noise in the signals, $[M\ v^i\ v]'$, is jointly Gaussian with covariance matrix:

$$\Sigma = \begin{bmatrix} \tau_M^{-1}\mathbf{I}_N & 0_{N\times 1} & 0_{N\times 1} \\ 0_{1\times N} & \tau_v^{-1} & 0 \\ 0_{1\times N} & 0 & \tau_G^{-1} \end{bmatrix}. \tag{73}$$

Applying standard projection techniques we define

$$r \equiv (\Theta\Sigma\Theta')^{-1} = \begin{bmatrix} \tau_M(\xi\xi')^{-1} & 0_{N\times 1} & 0_{N\times 1} \\ 0_{1\times N} & \tau_v & 0 \\ 0_{1\times N} & 0 & \tau_G \end{bmatrix}, \tag{74}$$

and obtain that an investor i's total precision on the common factor satisfies

$$\tau \equiv (\mathbb{V}[F|\mathscr{F}^i])^{-1} = \tau_F + H'rH = \tau_F + \tau_G + \tau_v + \tau_M\alpha'(\xi\xi')^{-1}\alpha. \tag{75}$$

The precision τ is the same across investors. Furthermore, an investor i's expectation of F satisfies

$$\mathbb{E}[F|\mathscr{F}^i] = \frac{1}{\tau}H'rS^i = \frac{1}{\tau}\left[\alpha'(\xi\xi')^{-1}\tau_M \quad \tau_v \quad \tau_G\right] S^i. \tag{76}$$

Using the definition of the total precision (75), it follows that average market expectation regarding dividends is

$$\overline{\mathbb{E}}[D] = \Phi\frac{1}{\tau}\left[(\tau - \tau_F - \tau_G)F + \tau_G G + \tau_M\alpha'(\xi\xi')^{-1}\xi M\right] = \mathbb{E}[D|\mathscr{F}^i] - \Phi\frac{\tau_v}{\tau}v^i, \tag{77}$$

and average market uncertainty regarding dividends is

$$\overline{\mathbb{V}}[D] = \frac{1}{\tau}\Phi\Phi' + \frac{1}{\tau_\epsilon}\mathbf{I}_N. \tag{78}$$

Because agents hold mean-variance portfolios, the market-clearing condition implies:

$$P = \overline{\mathbb{E}}[D] - \gamma\overline{\mathbb{V}}[D]M \tag{79}$$

$$= \Phi\frac{\tau - \tau_F - \tau_G}{\tau}F + \Phi\frac{\tau_G}{\tau}G + \left[\Phi\frac{\tau_M}{\tau}(\xi^{-1}\alpha)' - \gamma\left(\frac{1}{\tau}\Phi\Phi' + \frac{1}{\tau_\epsilon}\mathbf{I}_N\right)\right]M, \tag{80}$$

where we have used the simplification

$$\tau_M\alpha'(\xi\xi')^{-1}\xi = \tau_M(\xi^{-1}\alpha)'. \tag{81}$$

The initial price conjecture then yields the following fixed point:

$$\alpha = \Phi \frac{\tau - \tau_F - \tau_G}{\tau} \tag{82}$$

$$g = \Phi \frac{\tau_G}{\tau} \tag{83}$$

$$\xi = \Phi \frac{\tau_M}{\tau} (\xi^{-1}\alpha)' - \gamma \left(\frac{1}{\tau} \Phi\Phi' + \frac{1}{\tau_\epsilon} \mathbf{I}_N \right). \tag{84}$$

Multiply both sides of the last equation by $\xi^{-1}\alpha$ (to the right):

$$\alpha = \Phi \frac{\tau_M}{\tau} (\xi^{-1}\alpha)' \xi^{-1}\alpha - \gamma \left(\frac{1}{\tau} \Phi\Phi' + \frac{1}{\tau_\epsilon} \mathbf{I}_N \right) \xi^{-1}\alpha, \tag{85}$$

and then recognize that $\tau_M(\xi^{-1}\alpha)'\xi^{-1}\alpha = \tau_M \alpha'(\xi\xi')^{-1}\alpha = \tau - \tau_F - \tau_G - \tau_v$ (from Eq. 75), which can be replaced above, together with the solution for α to obtain (after multiplication with τ):

$$\Phi \tau_v = -\gamma \left(\Phi\Phi' + \frac{\tau}{\tau_\epsilon} \mathbf{I}_N \right) \xi^{-1}\alpha, \tag{86}$$

which leads to an equation for $\xi^{-1}\alpha$:

$$\xi^{-1}\alpha = -\frac{\tau_v}{\gamma} \left(\Phi\Phi' + \frac{\tau}{\tau_\epsilon} \mathbf{I}_N \right)^{-1} \Phi. \tag{87}$$

Multiply both sides with Φ' (to the left):

$$\Phi'\xi^{-1}\alpha = -\frac{\tau_v}{\gamma} \Phi' \left(\Phi\Phi' + \frac{\tau}{\tau_\epsilon} \mathbf{I}_N \right)^{-1} \Phi = -\frac{\tau_v \tau_\epsilon \Phi'\Phi}{\gamma(\tau + \tau_\epsilon \Phi'\Phi)}, \tag{88}$$

where the second equality follows from the Woodbury matrix identity. Conjecture

$$\xi^{-1}\alpha \equiv -\frac{\sqrt{\tau_P}}{\sqrt{\tau_M}} \Phi, \tag{89}$$

where τ_P is an unknown positive scalar. Replacing Eq. (89) in Eq. (75) yields the total precision τ as a function of this scalar:

$$\tau = \tau_F + \tau_G + \tau_v + \tau_P \Phi'\Phi. \tag{90}$$

Furthermore, replacing the conjecture (89) in Eq. (88) yields

$$\frac{\sqrt{\tau_P}}{\sqrt{\tau_M}} = \frac{\tau_v \tau_\epsilon}{\gamma(\tau + \tau_\epsilon \Phi'\Phi)} \tag{91}$$

which leads to Eq. (17) in the text:

$$\tau_P \left[\tau_F + \tau_v + \tau_G + (\tau_P + \tau_\epsilon)\Phi'\Phi \right]^2 = \frac{\tau_M \tau_\epsilon^2 \tau_v^2}{\gamma^2}. \tag{92}$$

This is a cubic equation in τ_P. It can be shown that the discriminant of this cubic equation is strictly negative and thus it has a unique real root. Since it cannot have a negative root (the right hand side is strictly positive), it follows that τ_P is indeed a unique positive scalar, as conjectured. Finally, the conjecture (89) can now be replace in the fixed point solution (82) to obtain the undetermined coefficients ξ:

$$\xi = -\left(\frac{\gamma + \sqrt{\tau_M \tau_P}}{\tau}\Phi\Phi' + \frac{\gamma}{\tau_\epsilon}\mathbf{I}_N\right). \tag{93}$$

This completes the proof of Proposition 1.

A.3 Proof of Lemma 1

Write the excess returns as

$$D - P = \frac{\tau_F}{\tau}\Phi F - \frac{\tau_G}{\tau}\Phi \upsilon + \left(\frac{\gamma + \sqrt{\tau_M \tau_P}}{\tau}\Phi\Phi' + \frac{\gamma}{\tau_\epsilon}\mathbf{I}_N\right) M + \epsilon, \tag{94}$$

and thus their unconditional variance is

$$\mathbb{V}[D-P] = \underbrace{\left[\frac{\tau_F + \tau_G}{\tau^2} + \frac{\Phi'\Phi}{\tau_M}\left(\frac{\gamma + \sqrt{\tau_M \tau_P}}{\tau}\right)^2 + \frac{2\gamma(\gamma + \sqrt{\tau_M \tau_P})}{\tau_M \tau_\epsilon \tau}\right]}_{X}\Phi\Phi' + \left(\frac{1}{\tau_\epsilon} + \frac{\gamma^2}{\tau_M \tau_\epsilon^2}\right)\mathbf{I}_N\,. \tag{95}$$

Develop the term in square brackets:

$$X = \frac{\tau_F + \tau_G + \tau_P \Phi'\Phi}{\tau^2} + \frac{\gamma^2 \Phi'\Phi}{\tau_M \tau^2} + \frac{2\gamma^2}{\tau_M \tau_\epsilon \tau} + 2\frac{\gamma\sqrt{\tau_M \tau_P}(\tau + \tau_\epsilon \Phi'\Phi)}{\tau_\epsilon \tau_M \tau^2}. \tag{96}$$

We know from Eq. (92) that

$$\gamma\sqrt{\tau_M \tau_P}(\tau + \tau_\epsilon \Phi'\Phi) = \tau_M \tau_\epsilon \tau_\upsilon. \tag{97}$$

Replacing this in Eq. (96) yields

$$X = \left(1 + \frac{2\gamma^2}{\tau_M \tau_\epsilon} + \frac{\gamma^2 \Phi'\Phi + \tau_M \tau_\upsilon}{\tau_M \tau}\right)\frac{1}{\tau}, \tag{98}$$

and thus

$$\mathbb{V}[R] = \left(1 + \frac{2\gamma^2}{\tau_M \tau_\epsilon} + \frac{\gamma^2 \Phi'\Phi + \tau_M \tau_\upsilon}{\tau_M \tau}\right)\frac{1}{\tau}\Phi\Phi' + \left(1 + \frac{\gamma^2}{\tau_M \tau_\epsilon}\right)\frac{1}{\tau_\epsilon}\mathbf{I}_N\,. \tag{99}$$

Using (78), it follows that

$$\mathbb{V}[R] = \left(1 + \frac{2\gamma^2}{\tau_M \tau_\epsilon} + \frac{\gamma^2 \Phi'\Phi + \tau_M \tau_\upsilon}{\tau_M \tau}\right)\overline{\mathbb{V}}[R] - \kappa\,\mathbf{I}_N, \tag{100}$$

where κ is defined as in (26):

$$\kappa \equiv \frac{\gamma^2}{\tau_M \tau_\epsilon}\left(\frac{1}{\tau_\epsilon} + \frac{\Phi'\Phi}{\tau}\right) + \frac{\tau_\upsilon}{\tau \tau_\epsilon}. \tag{101}$$

Thus,

$$\mathbb{V}[R] - \overline{\mathbb{V}}[R] = \left(\kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right)\overline{\mathbb{V}}[R] - \kappa\mathbf{I}_N, \tag{102}$$

which proves Lemma 1.

A.4 Proof of Theorem 1

Start with the unconditional true CAPM (21):

$$\mathbb{E}[R] = \frac{\frac{1}{N}\overline{\mathbb{V}}[R]\mathbf{1}}{\overline{\mathbb{V}}[R_M]}\mathbb{E}[R_M], \tag{103}$$

and replace the following relationship, which results from Lemma 1:

$$\overline{\mathbb{V}}[R] = \frac{\mathbb{V}[R] + \kappa\,\mathbf{I}_N}{1 + \kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M\tau_\epsilon}}. \tag{104}$$

This yields

$$\mathbb{E}[R] = \frac{\mathbb{V}[R_M]}{\left(1 + \kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right)\overline{\mathbb{V}}[R_M]}\widetilde{\beta}\mathbb{E}[R_M] + \frac{\kappa\,\mathbf{1}}{N\left(1 + \kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right)\overline{\mathbb{V}}[R_M]}\mathbb{E}[R_M], \tag{105}$$

where $\widetilde{\beta}$ is defined in (27). Multiply both sides with the equally weighted market portfolio $\mathbf{1}/N$ and use the fact that the weighted average of econometrician's betas is one:

$$1 = \frac{\mathbb{V}[R_M]}{\left(1 + \kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right)\overline{\mathbb{V}}[R_M]} + \frac{\kappa}{N\left(1 + \kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right)\overline{\mathbb{V}}[R_M]}, \tag{106}$$

which can be written without loss of generality:

$$\frac{1}{1+\delta} + \frac{\delta}{1+\delta} = \frac{\mathbb{V}[R_M]}{\left(1 + \kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right)\overline{\mathbb{V}}[R_M]} + \frac{\kappa/N}{\left(1 + \kappa\tau_\epsilon + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right)\overline{\mathbb{V}}[R_M]}. \tag{107}$$

We obtain

$$\delta = \frac{\kappa/N}{\mathbb{V}[R_M]}, \tag{108}$$

and Eq. (105) can now be written for any individual stock n:

$$\mathbb{E}[R_n] = \frac{\delta}{1+\delta}\,\mathbb{E}[R_M] + \frac{1}{1+\delta}\widetilde{\beta}_n\,\mathbb{E}[R_M] = \frac{\delta}{1+\delta}(1-\widetilde{\beta}_n)\,\mathbb{E}[R_M] + \widetilde{\beta}_n\,\mathbb{E}[R_M], \tag{109}$$

which is Eq. (28) in the text. Also, the first equality above is Eq. (31) in the text. The remainder of Theorem 1 follows from recognizing that the econometrician and the average agent obtain the same average expected returns for any asset n:

$$\frac{1}{1+\delta}\,\mathbb{E}[R_M](\delta+\widetilde{\beta}_n)=\beta\,\mathbb{E}[R_M] \tag{110}$$

and thus this relationship completes the proof of Theorem 1:

$$\widetilde{\beta}_n-1=(1+\delta)(\beta-1),\quad \forall n\in\{1,2,...N\}. \tag{111}$$

A.5 Conditioning on all available public information

This appendix derives the distortion in beta estimates $\widehat{\delta}$ (Eq. 42) when the econometrician conditions on all available public information. The econometrician now computes

$$\mathbb{V}[R|\{P,G\}]=\mathbb{V}[D|\{P,G\}]=\frac{1}{\tau-\tau_v}\Phi\Phi'+\frac{1}{\tau_\epsilon}\mathbf{I}_N, \tag{112}$$

and thus we can write[22]

$$\mathbb{V}[R|\{P,G\}]=\frac{\tau}{\tau-\tau_v}\overline{\mathbb{V}}[R]-\frac{\tau_v}{\tau_\epsilon(\tau-\tau_v)}\,\mathbf{I}_N\,. \tag{114}$$

The econometrician obtains a new set of betas:

$$\widehat{\beta}=\frac{\frac{1}{N}\,\mathbb{V}[R|\{P,G\}]\,\mathbf{1}}{\mathbb{V}[R_M|\{P,G\}]} \tag{115}$$

$$=\underbrace{\frac{\tau}{\tau-\tau_v}\frac{\overline{\mathbb{V}}[R_M]}{\mathbb{V}[R_M|\{P,G\}]}}_{\equiv(1+\widehat{\delta})}\beta-\frac{\frac{\tau_v}{\tau_\epsilon(\tau-\tau_v)}\,\mathbf{1}}{N\,\mathbb{V}[R_M|\{P,G\}]}. \tag{116}$$

Take average on both sides by multiplying with $\mathbf{1}\,/N$:

$$1=(1+\widehat{\delta})-\frac{1}{N\,\mathbb{V}[R_M|\{P,G\}]}\frac{\tau_v}{\tau_\epsilon(\tau-\tau_v)}, \tag{117}$$

and thus we obtain Eq. (42) in the text. Replacing $\widehat{\delta}$ in (116) and subtracting $\mathbf{1}$ on both sides yields

$$\widehat{\beta}-\mathbf{1}=(1+\widehat{\delta})(\beta-\mathbf{1}). \tag{118}$$

The sign of $\widehat{\delta}-\delta$ depends on the parameter τ_v. For large enough values of τ_v, the distortion $\widehat{\delta}$ can become larger than the initial distortion obtained without conditioning δ. To see this, replace

[22] Notice that it is not necessary to assume here that the econometrician knows the price coefficients. This is because

$$\mathbb{V}[R|\{P,G\}]=\mathbb{V}[R]-\mathbb{V}[\mathbb{E}[R|\{P,G\}]]. \tag{113}$$

The econometrician can compute both terms on the right hand side: the first term is the total covariance matrix; the second term represents explained variation (i.e., the numerator of the coefficient of determination R^2) after regressing realized returns on prices P and the public signal G.

(104) in (114):

$$\mathbb{V}[R|\{P,G\}] = \frac{\tau}{\tau-\tau_v}\frac{\mathbb{V}[R]+\kappa\,\mathbf{I}_N}{1+\kappa\tau_\epsilon+\frac{\gamma^2}{\tau_M\tau_\epsilon}} - \frac{\tau_v}{\tau_\epsilon(\tau-\tau_v)}\,\mathbf{I}_N \tag{119}$$

$$= \frac{\tau}{\tau-\tau_v}\frac{1}{1+\kappa\tau_\epsilon+\frac{\gamma^2}{\tau_M\tau_\epsilon}}\,\mathbb{V}[R] - \frac{\tau}{\tau-\tau_v}\left(\frac{\tau_v}{\tau\tau_\epsilon} - \frac{\kappa}{1+\kappa\tau_\epsilon+\frac{\gamma^2}{\tau_M\tau_\epsilon}}\right)\mathbf{I}_N\,. \tag{120}$$

One can thus define a distortion between the betas that the econometrician obtains by conditioning on all available information, $\widehat{\beta}$, and the betas that the econometrician obtains when using only realized returns, $\widetilde{\beta}$. Denoting this distortion by $\widetilde{\delta}$, Eq. (120) implies

$$\widehat{\beta}-1 = (1+\widetilde{\delta})(\widetilde{\beta}-1), \tag{121}$$

with

$$\widetilde{\delta} = \frac{1}{N\,\mathbb{V}[R_M|\{P,G\}]}\frac{\tau}{\tau-\tau_v}\left(\frac{\tau_v}{\tau\tau_\epsilon} - \frac{\kappa}{1+\kappa\tau_\epsilon+\frac{\gamma^2}{\tau_M\tau_\epsilon}}\right). \tag{122}$$

It follows that

$$\mathrm{sign}(\widetilde{\delta}) = \mathrm{sign}\left(\frac{\tau_v}{\tau\tau_\epsilon} - \frac{\kappa}{1+\kappa\tau_\epsilon+\frac{\gamma^2}{\tau_M\tau_\epsilon}}\right). \tag{123}$$

Clearly, if $\tau_v = 0$, the distortion $\widetilde{\delta}$ is negative, meaning that the econometrician's measurement improves by using all publicly available information. However, as τ_v grows, the sign in Eq. (123) may become positive (it can be shown that $\widetilde{\delta}$ grows arbitrarily large with τ_v). To gain some insight into the condition under which Eq. (123) is positive, we rewrite the term on the right-hand side as

$$\mathrm{sign}(\widetilde{\delta}) \equiv \mathrm{sign}\left(\tau_\epsilon\frac{\tau_v-\tau}{\tau_v}\kappa + 1 + \frac{\gamma^2}{\tau_M\tau_\epsilon}\right). \tag{124}$$

Thus, the distortion coefficient $\widetilde{\delta}$ is positive if the coefficient κ parametrizing the informational distance between the empiricist and investors is sufficiently small:

$$\kappa < \frac{(\gamma^2+\tau_M\tau_\epsilon)\tau_v}{\tau_M\tau_\epsilon^2(\tau-\tau_v)}, \tag{125}$$

which can be written in terms of the precision of private information:

$$\tau_v > \frac{\gamma^2(\tau-\tau_v)(\tau+\tau_\epsilon\Phi'\Phi)}{\gamma^2\tau+\tau_M\tau_v\tau_\epsilon}. \tag{126}$$

More explicitly, this expression is equivalent to the private precision being sufficiently large:

$$\tau_v > \gamma\sqrt{\frac{(\tau_F+\tau_G+\tau_P\Phi'\Phi)(\tau_F+\tau_G+(\tau_P+\tau_\epsilon)\Phi'\Phi)}{\gamma^2+\tau_M\tau_\epsilon}}. \tag{127}$$

When interpreting this condition, we emphasize that the right-hand side depends implicitly on τ_v, as well as on other parameters through the noise-signal ratio τ_P. For instance, τ_P is increasing in

τ_M, since the more precise prices are, the higher the noise-signal ratio is. However, its dependence on τ_v is ambiguous. Applying the implicit function theorem to Eq. (17) shows that

$$\frac{\mathrm{d}}{\mathrm{d}\tau_v}\tau_P(\tau_v) < 0, \quad \text{if} \quad 0 < \tau_M \leq \frac{\gamma^2\tau_P(\tau_F+\tau_G+\tau_v)}{\tau_v\tau_\epsilon^2} \tag{128}$$

That is, an increase in the precision of private information decreases price informativeness in our model if the market portfolio is sufficiently noisy. Combining this observation with Eq. (127), we conclude that conditioning on public information is detrimental to the test if the market portfolio is sufficiently noisy and private information is sufficiently precise.

A.6 Multiple Factors

This appendix solves an extension of the model when assets' payoffs are driven by multiple factors (Section 3.2). Denote a vector of K independent factors by

$$\mathbf{F} = \begin{bmatrix} F_1 & F_2 & \cdots & F_K \end{bmatrix}' \sim \mathcal{N}\left(\mathbf{0}_{K\times 1}, \tau_F^{-1}\mathbf{I}_K\right) \tag{129}$$

and a vector of $N \geq K$ realized dividends by

$$D = \underbrace{\begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,K} \\ \phi_{2,1} & \phi_{2,2} & \cdots & \phi_{2,K} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{N,1} & \phi_{N,2} & \cdots & \phi_{N,K} \end{bmatrix}}_{\Phi\ (N\times K)} \mathbf{F} + \underbrace{\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_N \end{bmatrix}}_{\epsilon\ (N\times 1)} \equiv \Phi\mathbf{F} + \epsilon \tag{130}$$

with $\epsilon \sim \mathcal{N}\left(\mathbf{0}_{N\times 1}, \tau_\epsilon^{-1}\mathbf{I}_N\right)$. Each asset is in supply $M \sim \mathcal{N}\left(\mathbf{1}_N/N, \tau_M^{-1}\mathbf{I}_N\right)$. Each agent i observes a vector of private signals about the K factors

$$V^i = \mathbf{F} + v^i, \quad v^i \sim \mathcal{N}\left(\mathbf{0}_{K\times 1}, \tau_v^{-1}\mathbf{I}_K\right), \tag{131}$$

as well as a common public signal

$$G = \mathbf{F} + v, \quad v \sim \mathcal{N}\left(\mathbf{0}_{K\times 1}, \tau_G^{-1}\mathbf{I}_K\right). \tag{132}$$

In addition they observe prices, which we conjecture to satisfy

$$P = \underbrace{\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,K} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,K} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{N,1} & \alpha_{N,2} & \cdots & \alpha_{N,K} \end{bmatrix}}_{\alpha\ (N\times K)} \mathbf{F} + \underbrace{\begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,K} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,K} \\ \vdots & \vdots & \vdots & \vdots \\ g_{N,1} & g_{N,2} & \cdots & g_{N,K} \end{bmatrix}}_{g\ (N\times K)} G + \underbrace{\begin{bmatrix} \xi_{1,1} & \xi_{1,2} & \cdots & \xi_{1,N} \\ \xi_{2,1} & \xi_{2,2} & \cdots & \xi_{2,N} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{N,1} & \xi_{N,2} & \cdots & \xi_{N,N} \end{bmatrix}}_{\xi\ (N\times N)} M. \tag{133}$$

Since agents observe the public signal G, the effective price signal is

$$P^a \equiv P - gG = \alpha\mathbf{F} + \xi M. \tag{134}$$

Regrouping all signals in a vector we obtain

$$S^i = \begin{bmatrix} P^a \\ V^i \\ G \end{bmatrix} = \begin{bmatrix} \alpha \\ \mathbf{I}_K \\ \mathbf{I}_K \end{bmatrix} \mathbf{F} + \begin{bmatrix} \xi & \mathbf{0}_{N\times K} & \mathbf{0}_{N\times K} \\ \mathbf{0}_{K\times N} & \mathbf{I}_K & \mathbf{0}_{K\times K} \\ \mathbf{0}_{K\times N} & \mathbf{0}_{K\times K} & \mathbf{I}_K \end{bmatrix} \begin{bmatrix} M \\ v^i \\ v \end{bmatrix} = H\mathbf{F} + \Theta \begin{bmatrix} M \\ v^i \\ v \end{bmatrix}, \tag{135}$$

where the vector of noise in the signals, $[M \; v^i \; v]'$, is jointly Guassian with covariance matrix:

$$\Sigma = \begin{bmatrix} \tau_M^{-1}\mathbf{I}_N & \mathbf{0}_{N\times K} & \mathbf{0}_{N\times K} \\ \mathbf{0}_{K\times N} & \tau_v^{-1}\mathbf{I}_K & \mathbf{0}_{K\times K} \\ \mathbf{0}_{K\times N} & \mathbf{0}_{K\times K} & \tau_G^{-1}\mathbf{I}_K \end{bmatrix}, \tag{136}$$

of dimension $(N + 2K) \times (N + 2K)$. Using these matrices we now define a matrix of dimension $(N + 2K) \times (N + 2K)$:

$$r \equiv (\Theta\Sigma\Theta')^{-1} = \begin{bmatrix} \tau_M(\xi\xi')^{-1} & \mathbf{0}_{N\times K} & \mathbf{0}_{N\times K} \\ \mathbf{0}_{K\times N} & \tau_v\mathbf{I}_K & \mathbf{0}_{K\times K} \\ \mathbf{0}_{K\times N} & \mathbf{0}_{K\times K} & \tau_G\mathbf{I}_K \end{bmatrix}. \tag{137}$$

We obtain that an investor i's total precision of the common factors (which is the same across investors and thus is the average precision in the market) satisfies:

$$\tau \equiv \left(\mathbb{V}[\mathbf{F}|\mathscr{F}^i]\right)^{-1} = \tau_F\,\mathbf{I}_K + H'rH = (\tau_F + \tau_G + \tau_v)\,\mathbf{I}_K + \tau_M\alpha'(\xi\xi')^{-1}\alpha. \tag{138}$$

Furthermore, the projection theorem also yields

$$\mathbb{E}[\mathbf{F}|\mathscr{F}^i] = \tau^{-1}H'rS^i = \tau^{-1}\left[\tau_M\alpha'(\xi\xi')^{-1} \quad \tau_v\,\mathbf{I}_K \quad \tau_G\,\mathbf{I}_K\right]S^i \tag{139}$$

It follows that the average market expectation regarding dividends is

$$\overline{\mathbb{E}}[D] = \Phi\tau^{-1}\left[(\tau - \tau_F\,\mathbf{I}_K - \tau_G\,\mathbf{I}_K)\mathbf{F} + \tau_G G + \tau_M\alpha'(\xi\xi')^{-1}\xi M\right] = \mathbb{E}[D|\mathscr{F}^i] - \Phi\tau^{-1}\tau_v v^i, \tag{140}$$

and the posterior variance regarding dividends is

$$\overline{\mathbb{V}}[D] = \Phi\tau^{-1}\Phi' + \frac{1}{\tau_\epsilon}\,\mathbf{I}_N\,. \tag{141}$$

The market-clearing condition then requires that

$$P = \overline{\mathbb{E}}[D] - \gamma\overline{V}[D]M \tag{142}$$

$$= \Phi\tau^{-1}(\tau - \tau_F\,\mathbf{I}_K - \tau_G\,\mathbf{I}_K)\mathbf{F} + \Phi\tau^{-1}\tau_G G + \left[\Phi\tau^{-1}\tau_M(\xi^{-1}\alpha)' - \gamma\left(\Phi\tau^{-1}\Phi' + \frac{1}{\tau_\epsilon}\,\mathbf{I}_N\right)\right]M. \tag{143}$$

We obtain the following fixed point solution for the undetermined price coefficients:

$$\alpha = \Phi\tau^{-1}(\tau - \tau_F\,\mathbf{I}_K - \tau_G\,\mathbf{I}_K) \tag{144}$$

$$g = \Phi\tau^{-1}\tau_G \tag{145}$$

$$\xi = \Phi\tau^{-1}\tau_M(\xi^{-1}\alpha)' - \gamma\left(\Phi\tau^{-1}\Phi' + \frac{1}{\tau_\epsilon}\,\mathbf{I}_N\right). \tag{146}$$

Multiply both sides of the last equation by $\xi^{-1}\alpha$ (to the right):

$$\alpha = \Phi\tau^{-1}\tau_M(\xi^{-1}\alpha)'\xi^{-1}\alpha - \gamma\left(\Phi\tau^{-1}\Phi' + \frac{1}{\tau_\epsilon}\mathbf{I}_N\right)\xi^{-1}\alpha, \tag{147}$$

and then recognize that (From Eq. 138):

$$\tau_M(\xi^{-1}\alpha)'\xi^{-1}\alpha = \tau_M\alpha'(\xi\xi')^{-1}\alpha = \tau - (\tau_F + \tau_G + \tau_v)\,\mathbf{I}_K\,. \tag{148}$$

Replacing this, together with the solution for α, in Eq. (147) yields:

$$\Phi\tau^{-1}\tau_v = -\gamma\left(\Phi\tau^{-1}\Phi' + \frac{1}{\tau_\epsilon}\mathbf{I}_N\right)\xi^{-1}\alpha, \tag{149}$$

which leads to an equation for $\xi^{-1}\alpha$:

$$\xi^{-1}\alpha = -\frac{\tau_v}{\gamma}\left(\Phi\tau^{-1}\Phi' + \frac{1}{\tau_\epsilon}\mathbf{I}_N\right)^{-1}\Phi\tau^{-1}. \tag{150}$$

Multiply both sides by $\tau^{-1}\Phi'$ (to the left):

$$\tau^{-1}\Phi'\xi^{-1}\alpha = -\frac{\tau_v}{\gamma}\tau^{-1}\Phi'\left(\Phi\tau^{-1}\Phi' + \frac{1}{\tau_\epsilon}\mathbf{I}_N\right)^{-1}\Phi\tau^{-1} \tag{151}$$

$$= -\frac{\tau_v}{\gamma}\left[\tau^{-1} - (\tau + \tau_\epsilon\Phi'\Phi)^{-1}\right], \tag{152}$$

where the second equality follows from the Woodbury matrix identity. Conjecture

$$\xi^{-1}\alpha = -\frac{1}{\sqrt{\tau_M}}\Phi\tau_P, \tag{153}$$

where τ_P is a symmetric $K \times K$ matrix with $K(K+1)/2$ unknown coefficients. This can be replaced in the total precision (Eq. 138):

$$\tau = (\tau_F + \tau_G + \tau_v)\,\mathbf{I}_K + \tau_P\Phi'\Phi\tau_P. \tag{154}$$

Finally, replacing the conjecture (153) in Eq. (152) yields an equation to be solved by τ_P:

$$\frac{1}{\sqrt{\tau_M}}\tau^{-1}\Phi'\Phi\tau_P = \frac{\tau_v}{\gamma}\left[\tau^{-1} - (\tau + \tau_\epsilon\Phi'\Phi)^{-1}\right]. \tag{155}$$

This is a system of $K(K+1)/2$ equations with $K(K+1)/2$ unknowns, which can be solved numerically. Once τ_P is determined, it can be replaced in Eq. (154), which determines τ. Finally, the conjecture (153) is then replaced in the fixed point solution for the undetermined coefficients in the conjectured price to obtain

$$\xi = -\sqrt{\tau_M}\Phi\tau^{-1}\tau_P\Phi' - \gamma\left(\Phi\tau^{-1}\Phi' + \frac{1}{\tau_\epsilon}\mathbf{I}_N\right). \tag{156}$$

This completes the equilibrium solution for the case when there are multiple factors.

A.7 Dynamic model

This appendix solves the equilibrium in the dynamic model of Section 4. We first conjecture that prices at time $t-k$, with $k \in \{0,...,T-1\}$, take the following linear form:

$$\begin{aligned} P_{t-k} = \underbrace{\bar{\alpha}_k}_{N\times 1} \overline{D} + \underbrace{\alpha_k}_{N\times 1} F_{t-k-T} + \underbrace{\bar{\xi}_k}_{N\times 1} \overline{M} + \underbrace{\xi_k}_{N\times N} M_{t-k-T} + g_k G_{t-k} \\ + \underbrace{\begin{bmatrix} d_{k,-T+1} & d_{k,-T+2} & \dots & d_{k,0} \end{bmatrix}}_{N\times NT} \underbrace{\begin{bmatrix} D_{t-T-k+1} \\ D_{t-T-k+2} \\ \vdots \\ D_{t-k} \end{bmatrix}}_{\widetilde{D}_{t-k}} \\ + \underbrace{\begin{bmatrix} a_{k,-T+1} & a_{k,-T+2} & \dots & a_{k,0} \end{bmatrix}}_{N\times T} \underbrace{\begin{bmatrix} \epsilon^F_{t-k-T+1} \\ \epsilon^F_{t-k-T+2} \\ \vdots \\ \epsilon^F_{t-k} \end{bmatrix}}_{\widetilde{\epsilon}^F_{t-k}} \\ + \underbrace{\begin{bmatrix} b_{k,-T+1} & b_{k,-T+2} & \dots & b_{k,0} \end{bmatrix}}_{N\times NT} \underbrace{\begin{bmatrix} \epsilon^M_{t-k-T+1} \\ \epsilon^M_{t-k-T+2} \\ \vdots \\ \epsilon^M_{t-k} \end{bmatrix}}_{\widetilde{\epsilon}^M_{t-k}} . \end{aligned} \tag{157}$$

where we use the following notation:

- $\widetilde{D}_{t-k}$ is a stacked vector of dimension NT which includes the payoffs of all N assets from time $t-k-T+1$ to time $t-k$.
- $\widetilde{\epsilon}^F_{t-k}$ is a vector of dimension T containing all the fundamental shocks ϵ^F from time $t-k-T+1$ to time $t-k$.
- $\widetilde{\epsilon}^M_{t-k}$ is a stacked vector of dimension NT containing the supply shocks for all assets from time $t-k-T+1$ to time $t-k$.
- The price coefficients $\bar{\alpha}_k$, α_k, $\bar{\xi}_k$, ξ_k, d_k, g_k, a_k and b_k are scalars/vectors/matrices of conformable dimension.

To understand the price structure (157), notice that any investor i observes at date $t-k$, $k \in \{0,...,T-1\}$:

$$\left\{ \{P_s\}_{s\le t-k}, \{D_s\}_{s\le t-k}, \{F_s\}_{s\le t-k-T}, \{v^i_s\}_{s\le t-k} \right\} . \tag{158}$$

It follows that an investor i also observes $\{\epsilon^F_s\}_{s\le t-k-T}$, hence the vector of innovations $\widetilde{\epsilon}^F_{t-k}$ in the common factor. Furthermore, in this dynamic setup the sequence of past dividends $\{D_{t-k-s}\}_{s=0}^{T-1}$ reveals information regarding past, unobservable factor innovations $\{\epsilon^F_{t-k-s}\}_{s=0}^{T-1}$, hence the vector $\widetilde{D}_{t-k}$. Finally, to understand the structure of the vector of past supply innovations, consider the price at time $t-k-T$. This price reveals a linear combination of $\{\epsilon^F_{t-k-s}\}_{s=0}^{T-1}$ and M_{t-k-T}. Since these factor innovations are observable, so is M_{t-k-T}, hence the vector $\widetilde{\epsilon}^M_{t-k}$ and the presence of F_{t-k-T} and M_{t-k-T} in the equilibrium price.

For the purpose of forecasting future excess returns, it is sufficient to form expectations about the vector $\widetilde{\epsilon}^F$ of innovations in the common factor. We thus first compute an investor i' conditional expectations $\mathbb{E}^i_t[\widetilde{\epsilon}^F_t]$ and conditional variance $\mathbb{V}^i_t[\widetilde{\epsilon}^F_t]$ at each trading date during the periodic announcement cycle.

At time $t-k$, with $k \in \{0, ..., T-1\}$, an investor i's information set contains four sources of information regarding the vector $\widetilde{\epsilon}^F_{t-k}$: past and current prices, private signals, dividends, and public announcements. We isolate the informational part of prices P^a_{t-k} that only contains unobservables:

$$P^a_{t-k} = a_k \widetilde{\epsilon}^F_{t-k} + b_k \widetilde{\epsilon}^M_{t-k}, \quad k \in \{0, ..., T-1\}, \tag{159}$$

which we stack into a single vector $\widetilde{P}^a_{t-k}$ of dimension NT:

$$\widetilde{P}^a_{t-k} \equiv \begin{bmatrix} P^a_{t-k} & P^a_{t-k-1} & \dots & P^a_{t-k-T+2} & P^a_{t-k-T+1} \end{bmatrix}'. \tag{160}$$

Similarly, we collect all past and current private signals that investor i has gathered by time $t-k$ into a vector $\widetilde{V}^i_{t-k}$ of dimension T:

$$\widetilde{V}^i_{t-k} \equiv \begin{bmatrix} V^i_{t-k} & V^i_{t-k-1} & \dots & V^i_{t-k-T+1} \end{bmatrix}'. \tag{161}$$

Furthermore, because we assume that the common factor F becomes public information after T periods, dividends and public announcements also reveal a combination of observable and unobservable information. In particular, the dividend at time $t-k$ can be written as

$$\begin{aligned} D_{t-k} &= \overline{D}\,\mathbf{1}_N + \Phi\left(\kappa_F^T F_{t-k-T} + \sum_{\tau=0}^{T-1} \kappa_F^\tau \epsilon^F_{t-k-\tau}\right) + \epsilon^D_{t-k} & (162)\\ &= \overline{D}\,\mathbf{1}_N + \Phi \kappa_F^T F_{t-k-T} + D^a_{t-k}, & (163) \end{aligned}$$

where D^a_{t-k} represents the informational part of dividends that only contains unobservables, which we stack into a single vector $\widetilde{D}^a_{t-k}$ of dimension NT:

$$\widetilde{D}^a_{t-k} \equiv \begin{bmatrix} D^a_{t-k-T+1} & D^a_{t-k-T+2} & \dots & D^a_{t-k} \end{bmatrix}'. \tag{164}$$

Public announcements G_{t-k} have a similar structure:

$$\begin{aligned} G_{t-k} &= \kappa_F^{k+j(k)} F_{t-k-T} + \sum_{\tau=0}^{k+j(k)-1} \kappa_F^\tau \epsilon^F_{t+j(k)-T-\tau} + \epsilon^G_{t+j(k)-T} & (165)\\ &= \kappa_F^{k+j(k)} F_{t-k-T} + G^a_{t-k} & (166) \end{aligned}$$

where G^a_{t-k} represents the part of public signals that only contains unobservables and the indexing function $j(k)$ equals T if $k=0$ and 0 otherwise.

Overall, an investor i's information at time $t-k$ is fully summarized by $\widetilde{P}^a_{t-k}$, V^i_{t-k}, $\widetilde{D}^a_{t-k}$, and G^a_{t-k}. Since the vector grouping this information and the vector $\widetilde{\epsilon}^F_{t-k}$ are jointly normally distributed, investor i forms her conditional expectations $\mathbb{E}^i_t[\widetilde{\epsilon}^F_t]$ and conditional variance $\mathbb{V}^i_t[\widetilde{\epsilon}^F_t]$ by projecting the former vector on the latter. We write the conditional expectation and conditional variance in Proposition 2.

Proposition 2. *At time $t-k$ an investor i's conditional expectation and variance of the fundamental innovations $\widetilde{\epsilon}^F_{t-k}$ satisfy*

$$\mathbb{E}^i_{t-k}\left[\widetilde{\epsilon}^F_{t-k}\right] = \underbrace{\tau_F^{-1} H_k' \left(\tau_F^{-1} H_k H_k' + r_k\right)^{-1}}_{\equiv m_k} \begin{bmatrix} \widetilde{P}^a_{t-k} \\ V^i_{t-k} \\ \widetilde{D}^a_{t-k} \\ G^a_{t-k} \end{bmatrix} \tag{167}$$

and

$$\mathbb{V}^i_{t-k}\left[\widetilde{\epsilon}^F_{t-k}\right] = \tau_F^{-1}\left(\mathbf{I}_T - m_k H_k\right) \tag{168}$$

where the matrices H_k and r_k are defined below.

Proof. We start by stacking all observable information in a vector. An investor i observes current and past prices, dividends, public announcements and her set of private signals. Starting with prices we first write the vector p_{t-k} in (160) as

$$p_{t-k} = \underbrace{\begin{bmatrix} a_{i(k),-T+1} & a_{i(k),-T+2} & \cdots & a_{i(k),-1} & a_{i(k),0} \\ a_{i(k+1),-T+2} & a_{i(k+1),-T+3} & \cdots & a_{i(k+1),0} & \mathbf{0}_{N\times 1} \\ a_{i(k+2),-T+3} & a_{i(k+2),-T+4} & \cdots & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i(k+T-1),0} & \mathbf{0}_{N\times 1} & \cdots & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} \end{bmatrix}}_{A_k\ (NT\times T)} \bar{\epsilon}^F_{t-k} \tag{169}$$

$$+ \underbrace{\begin{bmatrix} b_{i(k),-T+1} & b_{i(k),-T+2} & \cdots & b_{i(k),-1} & b_{i(k),0} \\ b_{i(k+1),-T+2} & b_{i(k+1),-T+3} & \cdots & b_{i(k+1),0} & \mathbf{0}_{N\times N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{i(k+T-1),0} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \end{bmatrix}}_{B_k\ (NT\times NT)} \bar{\epsilon}^M_{t-k} \tag{170}$$

$$\equiv A_k \bar{\epsilon}^F_{t-k} + B_k \bar{\epsilon}^M_{t-k}. \tag{171}$$

Similarly, we express the vector of past and current private signals in (161) as

$$V^i_{t-k} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{\Omega\ (T\times T)} \bar{\epsilon}^F_{t-k} + \underbrace{\begin{bmatrix} \widehat{\epsilon}^i_{t-k} \\ \widehat{\epsilon}^i_{t-k-1} \\ \vdots \\ \widehat{\epsilon}^i_{t-k-T+2} \\ \widehat{\epsilon}^i_{t-k-T+1} \end{bmatrix}}_{\bar{\epsilon}^i_{t-k}\ \ T\times 1} \tag{172}$$

$$\equiv \Omega \bar{\epsilon}^F_{t-k} + \bar{\epsilon}^i_{t-k} \tag{173}$$

where the vector of investor-specific noise is distributed as

$$\bar{\epsilon}^i_{t-k} \sim N\left(\mathbf{0}_{T\times 1}, \underbrace{\begin{bmatrix} \tau_v^{-1} & 0 & \dots & 0 \\ 0 & \tau_v^{-1}(n_{t-k,1})^{-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \tau_v^{-1}(n_{t-k,T-1})^{-1} \end{bmatrix}}_{S_k \ (T\times T)}\right) \tag{174}$$

To express the informational parts of dividends in vector form, notice that the dividend at time $t-k$ can be written as

$$D_{t-k} = \overline{D}\mathbf{1}_{N\times 1} + \Phi\left(\kappa_F^T F_{t-k-T} + \sum_{\tau=0}^{T-1} \kappa_F^{\tau} \epsilon^F_{t-k-\tau}\right) + \epsilon^D_{t-k} \tag{175}$$

$$\equiv \Phi\kappa_F^T F_{t-k-T} + D^a_{t-k} \tag{176}$$

where D^a_{t-k} represents the part of dividends that only contains unobservables. Likewise, the dividends at time $t-k-1$ satisfies

$$D_{t-k-1} = \Phi\left(\kappa_F^{T-1} F_{t-k-T} + \sum_{\tau=0}^{T-2} \kappa_F^{\tau} \epsilon^F_{t-k-\tau-1}\right) + \epsilon^D_{t-k-1} \tag{177}$$

$$\equiv \Phi\kappa_F^{T-1} F_{t-k-T} + D^a_{t-k-1}. \tag{178}$$

Proceeding iteratively, the dividend at time $t-k-T+1$ satisfies

$$D_{t-k-T+1} = \Phi\left(\kappa_F F_{t-k-T} + \epsilon^F_{t-k-T+1}\right) + \epsilon^D_{t-k-T+1} \tag{179}$$

$$\equiv \Phi\kappa_F F_{t-k-T} + D^a_{t-k-T+1}. \tag{180}$$

Accordingly, we can express the vector of informational dividends in (164) as

$$\overline{D}^a_{t-k} = \underbrace{\begin{bmatrix} \Phi & \mathbf{0}_{N\times 1} & \dots & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} \\ \Phi\kappa_F & \Phi & \dots & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times 1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi\kappa_F^{T-2} & \Phi\kappa_F^{T-3} & \dots & \Phi & \mathbf{0}_{N\times 1} \\ \Phi\kappa_F^{T-1} & \Phi\kappa_F^{T-2} & \dots & \Phi\kappa_F & \Phi \end{bmatrix}}_{\Lambda \ (NT\times T)} \bar{\epsilon}^F_{t-k} + \underbrace{\begin{bmatrix} \epsilon^D_{t-k-T+1} \\ \epsilon^D_{t-k-T+2} \\ \vdots \\ \epsilon^D_{t-k-1} \\ \epsilon^D_{t-k} \end{bmatrix}}_{\bar{\epsilon}^D_{t-k} \ (NT\times 1)} \tag{181}$$

$$\equiv \Lambda\bar{\epsilon}^F_{t-k} + \bar{\epsilon}^D_{t-k}. \tag{182}$$

Finally, we can write the informational part of the public signal in (165) as

$$G^a_{t-k} = \underbrace{\begin{bmatrix} \kappa_F^{k+j(k)-1} & \kappa_F^{k+j(k)-2}\mathbf{1}_{k+j(k)-2\geq 0} & \dots & \kappa_F^{k+j(k)-T}\mathbf{1}_{k+j(k)-T\geq 0} \end{bmatrix}}_{\Xi_k \ (1\times T)} \bar{\epsilon}^F_{t-k} + \epsilon^G_{t+j(k)-T} \tag{183}$$

$$\equiv \Xi_k\bar{\epsilon}^F_{t-k} + \epsilon^G_{t+j(k)-T}. \tag{184}$$

We now stack all observables into a single vector according to:

$$\begin{bmatrix} p_{t-k} \\ V^i_{t-k} \\ \overline{D}^a_{t-k} \\ G^a_{t-k} \end{bmatrix} = \underbrace{\begin{bmatrix} A_k \\ \Omega \\ \Lambda \\ \Xi_k \end{bmatrix}}_{H_k \ (((2N+1)T+1)\times T)} \bar{\epsilon}^F_{t-k} + \underbrace{\begin{bmatrix} B_k & \mathbf{0}_{NT\times T} & \mathbf{0}_{NT\times NT} & \mathbf{0}_{NT\times 1} \\ \mathbf{0}_{T\times NT} & \mathbf{I}_T & \mathbf{0}_{T\times NT} & \mathbf{0}_{T\times 1} \\ \mathbf{0}_{NT\times NT} & \mathbf{0}_{NT\times T} & \mathbf{I}_{NT} & \mathbf{0}_{NT\times 1} \\ \mathbf{0}_{1\times NT} & \mathbf{0}_{1\times T} & \mathbf{0}_{1\times NT} & 1 \end{bmatrix}}_{\Theta_k \ (((2N+1)T+1)\times((2N+1)T+1))} \begin{bmatrix} \bar{\epsilon}^M_{t-k} \\ \bar{\epsilon}^i_{t-k} \\ \bar{\epsilon}^D_{t-k} \\ \epsilon^G_{t+j(k)-T} \end{bmatrix} \tag{185}$$

where the vector of noise is distributed as

$$\begin{bmatrix} \bar{\epsilon}^M_{t-k} \\ \bar{\epsilon}^i_{t-k} \\ \bar{\epsilon}^D_{t-k} \\ \epsilon^G_{t+j(k)-T} \end{bmatrix} \sim N\left(\mathbf{0}_{((2N+1)T+1)\times 1}, \underbrace{\begin{bmatrix} \tau_M^{-1}\mathbf{I}_{NT} & \mathbf{0}_{NT\times T} & \mathbf{0}_{NT\times NT} & \mathbf{0}_{NT\times 1} \\ \mathbf{0}_{T\times NT} & S_k & \mathbf{0}_{T\times NT} & \mathbf{0}_{T\times 1} \\ \mathbf{0}_{NT\times NT} & \mathbf{0}_{NT\times T} & \tau_D^{-1}\mathbf{I}_{NT} & \mathbf{0}_{NT\times 1} \\ \mathbf{0}_{1\times NT} & \mathbf{0}_{1\times T} & \mathbf{0}_{1\times NT} & \tau_G^{-1} \end{bmatrix}}_{\Sigma_k \ (((2N+1)T+1)\times((2N+1)T+1))}\right). \tag{186}$$

Using these matrices, define

$$r_k = \Theta_k \Sigma_k \Theta_k^\top, \tag{187}$$

which we can use to apply the projection theorem. In particular, notice that the vector:

$$\begin{bmatrix} \bar{\epsilon}^F_{t-k} \\ p_{t-k} \\ V^i_{t-k} \\ \overline{D}^a_{t-k} \\ G^a_{t-k} \end{bmatrix} \sim N\left(\mathbf{0}_{((2N+2)T+1)\times 1}, \left[\begin{array}{c|c} \tau_F^{-1}\mathbf{I}_T & \tau_F^{-1}H_k^\top \\ \hline \tau_F^{-1}H_k & \tau_F^{-1}H_kH_k^\top + r_k \end{array}\right]\right) \tag{188}$$

is jointly normally distributed. The projection theorem then implies that:

$$\mathbb{E}^i_{t-k}\left[\bar{\epsilon}^F_{t-k}\right] = \underbrace{\tau_F^{-1}H_k^\top\left(\tau_F^{-1}H_kH_k^\top + r_k\right)^{-1}}_{m_k \ (T\times((2N+1)T+1))} \begin{bmatrix} p_{t-k} \\ V^i_{t-k} \\ \overline{D}^a_{t-k} \\ G^a_{t-k} \end{bmatrix} \tag{189}$$

and

$$\underbrace{\mathbb{V}^i_{t-k}\left[\bar{\epsilon}^F_{t-k}\right]}_{(T\times T)} = \tau_F^{-1}\left(\mathbf{I}_T - m_kH_k\right). \tag{190}$$

This conclude the proof of Proposition 2. □

Using this result, we can now compute $\mathbb{E}^i_{t-k}\left[P_{t-k+1} + D_{t-k+1}\right]$. We first rewrite $P_{t-k+1}+D_{t-k+1}$

using the price conjecture as

$$\begin{aligned}
P_{t-k+1}+D_{t-k+1} &= D_{t-k+1}+\bar{\alpha}_{i(k+T-1)}\overline{D}+\alpha_{i(k+T-1)}F_{t-k-T+1}+\bar{\xi}_{i(k+T-1)}\overline{M}+g_{i(k+T-1)}G_{t-k+1} \\
&+\xi_{i(k+T-1)}M_{t-k-T+1}+d_{i(k+T-1)}\overline{D}_{t-k+1}+a_{i(k+T-1)}\bar{\epsilon}^F_{t-k+1}+b_{i(k+T-1)}\bar{\epsilon}^M_{t-k+1} \\
&= \bar{\alpha}_{i(k+T-1)}\overline{D}+\alpha_{i(k+T-1)}\kappa_F F_{t-k-T}+(\bar{\xi}_{i(k+T-1)}+\xi_{i(k+T-1)}\mathbf{1}_{N\times 1}(1-\kappa_M))\overline{M} \\
&+\xi_{i(k+T-1)}\kappa_M M_{t-k-T}+a_{i(k+T-1)}\bar{\epsilon}^F_{t-k+1}+b_{i(k+T-1)}\bar{\epsilon}^M_{t-k+1}+g_{i(k+T-1)}G_{t-k+1} \\
&+\alpha_{i(k+T-1)}\epsilon^F_{t-k-T+1}+\xi_{i(k+T-1)}\epsilon^M_{t-k-T+1}+D_{t-k+1}+d_{i(k+T-1)}\overline{D}_{t-k+1}.
\end{aligned} \tag{191}$$

Furthermore, defining a new vector $d^\star$ as

$$d_{i(k+T-1)}\overline{D}_{t-k+1}=d_{i(k+T-1),0}D_{t-k+1}+\underbrace{\begin{bmatrix}\mathbf{0}_{N\times N} & d_{i(k+T-1),-T+1} & d_{i(k+T-1),-T+2} & \dots & d_{i(k+T-1),-1}\end{bmatrix}}_{d^\star_{i(k+T-1)}\ (N\times NT)}\overline{D}_{t-k} \tag{192}$$

$$\equiv d_{i(k+T-1),0}\left(\overline{D}\mathbf{1}_{N\times 1}+\Phi\left(\kappa_F^{T+1}F_{t-k-T}+\sum_{\tau=0}^{T}\kappa_F^\tau\epsilon^F_{t-k-\tau+1}\right)+\epsilon^D_{t-k+1}\right)+d^\star_{i(k+T-1)}\overline{D}_{t-k}, \tag{193}$$

we obtain

$$\begin{aligned}
P_{t-k+1}+D_{t-k+1} &= \left(\bar{\alpha}_{i(k+T-1)}+(d_{i(k+T-1),0}+\mathbf{I}_N)\mathbf{1}_{N\times 1}\right)\overline{D}+(\bar{\xi}_{i(k+T-1)}+\xi_{i(k+T-1)}\mathbf{1}_{N\times 1}(1-\kappa_M))\overline{M} \\
&+\kappa_F\left(\alpha_{i(k+T-1)}+\kappa_F^T(d_{i(k+T-1),0}+\mathbf{I}_N)\Phi\right)F_{t-k-T}+\xi_{i(k+T-1)}\kappa_M M_{t-k-T} \\
&+g_{i(k+T-1)}G_{t-k+1}+d^\star_{i(k+T-1)}\overline{D}_{t-k}+b_{i(k+T-1)}\bar{\epsilon}^M_{t-k+1}+\xi_{i(k+T-1)}\epsilon^M_{t-k-T+1} \\
&+a_{i(k+T-1)}\bar{\epsilon}^F_{t-k+1}+\alpha_{i(k+T-1)}\epsilon^F_{t-k-T+1}+(d_{i(k+T-1),0}+\mathbf{I}_N)\Phi\sum_{\tau=0}^{T}\kappa_F^\tau\epsilon^F_{t-k-\tau+1} \\
&+(d_{i(k+T-1),0}+\mathbf{I}_N)\epsilon^D_{t-k+1}.
\end{aligned} \tag{194}$$

Now, importantly, the only time G_{t-k+1} is unobservable is at date $t-1$, right before the announcement. To see this, consider that at time $t-2$ investors attempt to forecast P_{t-1}, which is a function of G_{t-T} that they observe. At time t investors attempt to forecast P_{t+1}, which is a function of G_t that they observe. The only time this property is violated is at time $t-1$, at which investors attempt to forecast P_t, which is a function of G_t that they *do not observe yet.* In this case, we must further decompose G_t as

$$G_t=F_t+\epsilon^G_t=\kappa_F^{T+1}F_{t-1-T}+\sum_{\tau=0}^{T}\kappa_F^\tau\epsilon^F_{t-\tau}+\epsilon^G_t; \tag{195}$$

accordingly, we write

$$
\begin{aligned}
P_{t-k+1}+D_{t-k+1} = & \left(\bar{\alpha}_{i(k+T-1)}+(d_{i(k+T-1),0}+\mathbf{I}_N)\mathbf{1}_{N\times 1}\right)\overline{D}+(\bar{\xi}_{i(k+T-1)}+\xi_{i(k+T-1)}\mathbf{1}_{N\times 1}(1-\kappa_M))\overline{M} \\
& +\kappa_F\left(\alpha_{i(k+T-1)}+\kappa_F^T(d_{i(k+T-1),0}+\mathbf{I}_N)\Phi+g_{i(T)}\kappa_F^{T+1}\mathbf{1}_{k=1}\right)F_{t-k-T}+\xi_{i(k+T-1)}\kappa_M M_{t-k-T} \\
& +g_{i(k+T-1)}G_{t-k+1}\mathbf{1}_{k\neq 1}+d^{\star}_{i(k+T-1)}\overline{D}_{t-k}+b_{i(k+T-1)}\bar{\epsilon}^M_{t-k+1}+\xi_{i(k+T-1)}\epsilon^M_{t-k-T+1} \\
& +a_{i(k+T-1)}\bar{\epsilon}^F_{t-k+1}+\alpha_{i(k+T-1)}\epsilon^F_{t-k-T+1}+\left((d_{i(k+T-1),0}+\mathbf{I}_N)\Phi+g_{i(T)}\mathbf{1}_{k=1}\right)\sum_{\tau=0}^{T}\kappa_F^{\tau}\epsilon^F_{t-k-\tau+1} \\
& +(d_{i(k+T-1),0}+\mathbf{I}_N)\epsilon^D_{t-k+1}+g_{i(T)}\epsilon^G_t\mathbf{1}_{k=1}.
\end{aligned}
\tag{196}
$$

Using this expression we can define a new vector $a^{\star}$ as

$$
\begin{aligned}
& a_{i(k+T-1)}\bar{\epsilon}^F_{t-k+1}+\alpha_{i(k+T-1)}\epsilon^F_{t-k-T+1}+\left((d_{i(k+T-1),0}+\mathbf{I}_N)\Phi+g_{i(T)}\mathbf{1}_{k=1}\right)\sum_{\tau=0}^{T}\kappa_F^{\tau}\epsilon^F_{t-k-\tau+1} \\
& =\underbrace{\begin{bmatrix}\alpha_{i(k+T-1)}+\kappa_F^T\left((d_{i(k+T-1),0}+\mathbf{I}_N)\Phi+g_{i(T)}\mathbf{1}_{k=1}\right)\\ a_{i(k+T-1),-T+1}+\kappa_F^{T-1}\left((d_{i(k+T-1),0}+\mathbf{I}_N)\Phi+g_{i(T)}\mathbf{1}_{k=1}\right)\\ \vdots \\ a_{i(k+T-1),-1}+\kappa_F\left((d_{i(k+T-1),0}+\mathbf{I}_N)\Phi+g_{i(T)}\mathbf{1}_{k=1}\right)\end{bmatrix}^{\top}}_{a^{\star}_{i(k+T-1)}\ (N\times T)}\bar{\epsilon}^F_t+\left(a_0+(d_0+\mathbf{I}_N)\Phi+g_{i(T)}\mathbf{1}_{k=1}\right)\epsilon^F_{t-k+1} \\
& \equiv a^{\star}_{i(k+T-1)}\bar{\epsilon}^F_{t-k}+\left(a_{i(k+T-1),0}+(d_{i(k+T-1),0}+\mathbf{I}_N)\Phi+g_{i(T)}\mathbf{1}_{k=1}\right)\epsilon^F_{t-k+1}
\end{aligned}
\tag{197}
$$

and a new vector $b^{\star}$ as

$$
\begin{aligned}
b_{i(k+T-1)}\bar{\epsilon}^M_{t-k+1}+\xi_{i(k+T-1)}\epsilon^M_{t-k-T+1} = & \underbrace{\begin{bmatrix}\xi_{i(k+T-1)} & b_{i(k+T-1),-T+1} & b_{i(k+T-1),-T+2} & \dots & b_{i(k+T-1),-1}\end{bmatrix}}_{b^{\star}_{i(k+T-1)}\ (N\times NT)}\bar{\epsilon}^M_{t-k} \\
& +b_{i(k+T-1),0}\epsilon^M_{t-k+1} \\
& =b^{\star}_{i(k+T-1)}\bar{\epsilon}^M_{t-k}+b_{i(k+T-1),0}\epsilon^M_{t-k+1}
\end{aligned}
\tag{198}
$$

to finally write

$$
\begin{aligned}
P_{t-k+1}+D_{t-k+1} = & f(\overline{D},\overline{M},M_{t-k-T},F_{t-k-T},\overline{D}_{t-k},G_{t-k}\mathbf{1}_{k\neq 1})+a^{\star}_{i(k+T-1)}\bar{\epsilon}^F_{t-k}+b^{\star}_{i(k+T-1)}\bar{\epsilon}^M_{t-k} \\
& +\left(a_{i(k+T-1),0}+(d_{i(k+T-1),0}+\mathbf{I}_N)\Phi+g_{i(T)}\mathbf{1}_{k=1}\right)\epsilon^F_{t-k+1}+b_{i(k+T-1),0}\epsilon^M_{t-k+1} \\
& +(d_{i(k+T-1),0}+\mathbf{I}_N)\epsilon^D_{t-k+1}+g_{i(T)}\epsilon^G_t\mathbf{1}_{k=1}.
\end{aligned}
\tag{199}
$$

where

$$
\begin{aligned}
& f(\overline{D},\overline{M},M_{t-k-T},F_{t-k-T},\overline{D}_{t-k},G_{t-k}\mathbf{1}_{k\neq 1}):=\left(\bar{\alpha}_{i(k+T-1)}+(d_{i(k+T-1),0}+\mathbf{I}_N)\mathbf{1}_{N\times 1}\right)\overline{D} \\
& +(\bar{\xi}_{i(k+T-1)}+\xi_{i(k+T-1)}\mathbf{1}_{N\times 1}(1-\kappa_M))\overline{M}+\xi_{i(k+T-1)}\kappa_M M_{t-k-T}+g_{i(k+T-1)}G_{t-k+1}\mathbf{1}_{k\neq 1} \\
& +\kappa_F\left(\alpha_{i(k+T-1)}+\kappa_F^T(d_{i(k+T-1),0}+\mathbf{I}_N)\Phi+g_{i(T)}\kappa_F^{T+1}\mathbf{1}_{k=1}\right)F_{t-k-T}+d^{\star}_{i(k+T-1)}\overline{D}_{t-k}.
\end{aligned}
\tag{200}
$$

To compute investor i's conditional expectation and posterior variance, it is convenient to use

$$\bar{\epsilon}^M_{t-k} = B_k^{-1} p_{t-k} - B_k^{-1} A_k \bar{\epsilon}^F_{t-k} \tag{201}$$

to rewrite this expression as

$$\begin{aligned} P_{t-k+1} + D_{t-k+1} &= f(\overline{D}, \overline{M}, M_{t-k-T}, F_{t-k-T}, \overline{D}_{t-k}, G_{t-k}\mathbf{1}_{k\neq 1}) + b^{\star}_{i(k+T-1)} B_k^{-1} p_{t-k} \\ &+ \underbrace{\left(a^{\star}_{i(k+T-1)} - b^{\star}_{i(k+T-1)} B_k^{-1} A_k\right)}_{\psi_k \ (N\times T)} \bar{\epsilon}^F_{t-k} + (d_{i(k+T-1),0} + \mathbf{I}_N)\epsilon^D_{t-k+1} + g_{i(T)}\epsilon^G_t \mathbf{1}_{k=1} \\ &+ \left(a_{i(k+T-1),0} + (d_{i(k+T-1),0} + \mathbf{I}_N)\Phi + g_{i(T)}\mathbf{1}_{k=1}\right)\epsilon^F_{t-k+1} + b_{i(k+T-1),0}\epsilon^M_{t-k+1} \\ &\equiv f(\overline{D}, \overline{M}, M_{t-k-T}, F_{t-k-T}, \overline{D}_{t-k}, G_{t-k}\mathbf{1}_{k\neq 1}) + b^{\star}_{i(k+T-1)} B_k^{-1} p_{t-k} + \psi_k \bar{\epsilon}^F_{t-k} \\ &+ (d_{i(k+T-1),0} + \mathbf{I}_N)\epsilon^D_{t-k+1} + g_{i(T)}\epsilon^G_t \mathbf{1}_{k=1} \\ &+ \left(a_{i(k+T-1),0} + (d_{i(k+T-1),0} + \mathbf{I}_N)\Phi + g_{i(T)}\mathbf{1}_{k=1}\right)\epsilon^F_{t-k+1} + b_{i(k+T-1),0}\epsilon^M_{t-k+1}. \end{aligned} \tag{202}$$

Hence, using our previous projection results we obtain

$$\mathbb{E}^i_{t-k}\left[P_{t-k+1} + D_{t-k+1}\right] = f(\overline{D}, \overline{M}, M_{t-k-T}, F_{t-k-T}, \overline{D}_{t-k}, G_{t-k}\mathbf{1}_{k\neq 1}) + b^{\star}_{i(k+T-1)} B_k^{-1} p_{t-k} + \psi_k \mathbb{E}^i_{t-k}\left[\bar{\epsilon}^F_{t-k}\right] \tag{203}$$

and

$$\begin{aligned} &\mathbb{V}^i_{t-k}\left[P_{t-k+1} + D_{t-k+1}\right] = \tau_F^{-1}\psi_k\left(\mathbf{I}_T - m_k H_k\right)\psi_k^{\top} + (d_{i(k+T-1),0} + \mathbf{I}_N)(d_{i(k+T-1),0} + \mathbf{I}_N)^{\top}\tau_D^{-1} \\ &+ \tau_G^{-1} g_{i(T)} g_{i(T)}^{\top}\mathbf{1}_{k=1} + b_{i(k+T-1),0} b_{i(k+T-1),0}^{\top}\tau_M^{-1} \\ &+ \left(a_{i(k+T-1),0} + (d_{i(k+T-1),0} + \mathbf{I}_N)\Phi + g_{i(T)}\mathbf{1}_{k=1}\right)\left(a_{i(k+T-1),0} + (d_{i(k+T-1),0} + \mathbf{I}_N)\Phi + g_{i(T)}\mathbf{1}_{k=1}\right)^{\top}\tau_F^{-1}. \end{aligned} \tag{204}$$

Equation (167) shows how investors form expectations about the current and past innovations in the common factor F. When forecasting fundamental innovations, investors use all the public information available (including current and past prices, current and past dividends, and the periodic public signal), but also their private information gathered up to time $t-k$. The posterior variance of $\widetilde{\epsilon}^F_{t-k}$ equals $\tau_F^{-1}\mathbf{I}_T$ when investors have no information, but changes as soon as investors learn from private and public signals, as shown in equation (168).

Based on the conditional expectations of Proposition 2, investor i can forecast future excess returns next period, $P_{t-k+1} + D_{t-k+1}$. The conditional expectation and the conditional variance of future excess returns, $\mathbb{E}^i_{t-k}\left[P_{t-k+1} + D_{t-k+1}\right]$ and $\mathbb{V}^i_{t-k}\left[P_{t-k+1} + D_{t-k+1}\right]$, completely determine investor i's portfolio strategy. These expressions are given in Eq. (203) and Eq. (204) respectively. We now aggregate investment strategies over the population of investors to clear the market and obtain equilibrium prices.

We conclude the equilibrium derivation by verifying that the pricing relation in equation (56) is consistent with the price conjecture in equation (16).

Proposition 3. *In equilibrium prices take the linear form in equation* (16) *where the pricing relation in equation* (56) *implies that the coefficients* $\overline{\alpha}$, α, $\bar{\xi}$, ξ, d, g, a *and* b *solve a nonlinear system of equation that we provide below.*

Proof. For the purpose of aggregating individual demands, we use the law of large numbers whereby

$\int_{i\in I}\bar{\epsilon}^i_{t-k}\mathrm{d}i = \mathbf{0}_{T\times 1}$, and obtain that the average expectation satisfies

$$\overline{\mathbb{E}}_{t-k}\left[P_{t-k+1}+D_{t-k+1}\right] = f(\overline{D},\overline{M},M_{t-k-T},F_{t-k-T},\overline{D}_{t-k},G_{t-k}\mathbf{1}_{k\neq 1}) + b^{\star}_{i(k+T-1)}B_k^{-1}p_{t-k}+\psi_k m_k\begin{bmatrix}p_{t-k}\\ \Omega\bar{\epsilon}^F_{t-k}\\ \overline{D}^a_{t-k}\\ G^a_{t-k}\end{bmatrix}. \tag{205}$$

Observing that

$$\overline{D}^a_{t-k} = \begin{bmatrix}D_{t-k-T+1}\\ D_{t-k-T+2}\\ \vdots\\ D_{t-k}\end{bmatrix} - \mathbf{1}_{N\times 1}\overline{D} - \underbrace{\begin{bmatrix}\Phi\kappa_F\\ \Phi\kappa_F^2\\ \vdots\\ \Phi\kappa_F^T\end{bmatrix}}_{L\ \ (NT\times 1)} F_{t-k-T} \tag{206}$$

$$\equiv \overline{D}_{t-k} - \mathbf{1}_{N\times 1}\overline{D} - LF_{t-k-T} \tag{207}$$

and that

$$G^a_{t-k} = G_{t-k} - \kappa_F^{k+j(k)}F_{t-k-T} \tag{208}$$

we can express the vector above as

$$\begin{aligned}\begin{bmatrix}p_{t-k}\\ \Omega\bar{\epsilon}^F_{t-k}\\ \overline{D}^a_{t-k}\\ G^a_{t-k}\end{bmatrix} &= \begin{bmatrix}A_k\bar{\epsilon}^F_{t-k}+B_k\bar{\eta}_{t-k}\\ \Omega\bar{\epsilon}^F_{t-k}\\ \overline{D}_{t-k}-\mathbf{1}_{N\times 1}\overline{D}-LF_{t-k-T}\\ G_{t-k}-\kappa_F^{k+j(k)}F_{t-k-T}\end{bmatrix}\\
&= \underbrace{\begin{bmatrix}A_k\\ \Omega\\ \mathbf{0}_{NT\times T}\\ \mathbf{0}_{1\times T}\end{bmatrix}}_{H^\star_k\ \ (((2N+1)T+1)\times T)}\bar{\epsilon}^F_{t-k} + \underbrace{\begin{bmatrix}B_k\\ \mathbf{0}_{T\times NT}\\ \mathbf{0}_{NT\times NT}\\ \mathbf{0}_{1\times NT}\end{bmatrix}}_{B^\star_k\ \ ((2N+1)T+1\times NT)}\bar{\epsilon}^M_{t-k} + \underbrace{\begin{bmatrix}\mathbf{0}_{NT\times NT}\\ \mathbf{0}_{T\times NT}\\ \mathbf{I}_{NT}\\ \mathbf{0}_{1\times NT}\end{bmatrix}}_{C\ \ ((2N+1)T+1\times NT)}\overline{D}_{t-k}\\
&\quad - \underbrace{\begin{bmatrix}\mathbf{0}_{NT\times 1}\\ \mathbf{0}_{T\times 1}\\ \mathbf{1}_{N\times 1}\\ 0\end{bmatrix}}_{K\ \ ((2N+1)T+1\times 1)}\overline{D} + \underbrace{\begin{bmatrix}\mathbf{0}_{NT\times 1}\\ \mathbf{0}_{T\times 1}\\ \mathbf{0}_{NT\times 1}\\ 1\end{bmatrix}}_{E\ \ (((2N+1)T+1)\times 1)}G_{t-k} - \underbrace{\begin{bmatrix}\mathbf{0}_{NT\times 1}\\ \mathbf{0}_{T\times 1}\\ L\\ \kappa_F^{k+j(k)}\end{bmatrix}}_{L^\star_k\ \ (((2N+1)T+1)\times 1)}F_{t-k-T}\\
&\equiv H^\star_k\bar{\epsilon}^F_{t-k}+B^\star_k\bar{\epsilon}^M_{t-k}+C\overline{D}_{t-k}-K\overline{D}+EG_{t-k}-L^\star F_{t-k-T}.\end{aligned} \tag{209}$$

Substituting this expression in average market expectations and using the pricing relation in (56), we can write:

$$
\begin{aligned}
RP_{t-k} &= f(\overline{D}, \overline{M}, M_{t-k-T}, F_{t-k-T}, \overline{D}_{t-k}, G_{t-k}\mathbf{1}_{k\neq 1}) + b^{\star}_{i(k+T-1)}\bar{\epsilon}^{M}_{t-k} + b^{\star}_{i(k+T-1)}B_k^{-1}A_k\bar{\epsilon}^{F}_{t-k} \\
&+ \psi_k m_k \left(H^{\star}_k\bar{\epsilon}^{F}_{t-k} + B^{\star}_k\bar{\epsilon}^{M}_{t-k} + C\overline{D}_{t-k} - K\overline{D} + EG_{t-k} - L^{\star}_k F_{t-k-T}\right) - \gamma\mathbb{V}_{t-k}[P_{t-k+1} + D_{t-k+1}]M_{t-k} \\
&= f(\overline{D}, \overline{M}, M_{t-k-T}, F_{t-k-T}, \overline{D}_{t-k}, G_{t-k}\mathbf{1}_{k\neq 1}) + \psi_k m_k \left(C\overline{D}_{t-k} - K\overline{D} + EG_{t-k} - L^{\star}_k F_{t-k-T}\right) \\
&+ \left(b^{\star}_{i(k+T-1)}B_k^{-1}A_k + \psi_k m_k H^{\star}_k\right)\bar{\epsilon}^{F}_{t-k} + \left(b^{\star}_{i(k+T-1)} + \psi_k m_k B^{\star}_k\right)\bar{\epsilon}^{M}_{t-k} \\
&- \gamma\mathbb{V}_{t-k}[P_{t-k+1} + D_{t-k+1}]\left(\kappa_M^T M_{t-k-T} + \overline{M}(1-\kappa_M)\sum_{\tau=0}^{T-1}\kappa_M^{\tau}\mathbf{1}_{N\times 1} + \sum_{\tau=0}^{T-1}\kappa_M^{\tau}\epsilon^{M}_{t-k-\tau}\right),
\end{aligned}
\tag{210}
$$

which confirms our initial price conjecture. We finally obtain the equilibrium fixed point by matching this expression with the price conjecture, which yields:

$$
\begin{bmatrix} \bar{\alpha}_k \\ \alpha_k \\ \bar{\xi}_k \\ \xi_k \\ d_k \\ g_k \\ a_k \\ b_k \end{bmatrix} = \frac{1}{R}
\begin{bmatrix}
\left(\bar{\alpha}_{i(k+T-1)} + (d_{i(k+T-1),0} + \mathbf{I}_N)\mathbf{1}_{N\times 1}\right) - \psi_k m_k K \\
\kappa_F\left(\alpha_{i(k+T-1)} + \kappa_F^T(d_{i(k+T-1),0} + \mathbf{I}_N)\Phi + g_{i(T)}\kappa_F^{T+1}\mathbf{1}_{k=1}\right) - \psi_k m_k L^{\star}_k \\
(\bar{\xi}_{i(k+T-1)} + \xi_{i(k+T-1)}\mathbf{1}_{N\times 1}(1-\kappa_M)) - \gamma\mathbb{V}_{t-k}[P_{t-k+1} + D_{t-k+1}]\mathbf{1}_{N\times 1}(1-\kappa_M)\sum_{\tau=0}^{T-1}\kappa_M^{\tau} \\
\xi_{i(k+T-1)}\kappa_M - \gamma\mathbb{V}_{t-k}[P_{t-k+1} + D_{t-k+1}]\kappa_M^T \\
d^{\star}_{i(k+T-1)} + \psi_k m_k C \\
g_{i(k+T-1)}\mathbf{1}_{k\neq 1} + \psi_k m_k E \\
b^{\star}_{i(k+T-1)}B_k^{-1}A_k + \psi_k m_k H^{\star}_k \\
b^{\star}_{i(k+T-1)} + \psi_k m_k B^{\star}_k - \gamma\mathbb{V}_{t-k}[P_{t-k+1} + D_{t-k+1}]\begin{bmatrix}\kappa_M^{T-1}\mathbf{I}_N & \kappa_M^{T-2}\mathbf{I}_N & \dots & \kappa_M\mathbf{I}_N & \mathbf{I}_N\end{bmatrix}
\end{bmatrix}.
\tag{211}
$$

for all $k \in \{0, \dots, T-1\}$. □