Dynamic Fire-Sale Externalities and Rollover Risk Spillovers

Hyunsoo Doh*

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Abstract

This paper studies financial contagion in a short-term debt market by developing a dynamic model with heterogeneous banks in which outside investors of assets are financially constrained and have different asset-management skills. Each bank's creditors make withdrawal decisions at their maturity dates. Each outside investor chooses an optimal time to purchase failed assets. The rollover risks of distressed banks propagate to other banks through the expected changes in a market-clearing liquidation price of assets. In contrast to He and Xiong (2012), the model implies providing more credit support or extending debt maturities alleviates a crisis by increasing the liquidation price.

Keywords: Financial contagion, rollover risk, fire-sales, externalities, liquidity. (JEL: G01, G20, C72)

*University of Chicago, 5807 S. Woodlawn Ave., Chicago, IL 60637. Email: hdoh@chicagobooth.edu.
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1 Introduction

Many economists argue that what caused the collapse of Lehman Brothers and the US financial market in fall 2008 was the spread of disruptions in mortgage-backed securities to other types of assets in the US short-term debt market during 2007 to mid-2008. Specifically, Gorton and Metrick (2012) find that the repurchase agreement (repo) market experienced systemic runs during that period, although most of the securitized loans used as collateral for repo contracts had no direct connection to the US housing market. Covitz, Liang, and Suarez (2013) also find that banks issuing asset-backed commercial papers (ABCP) had difficulty refinancing their maturing debts in the second half of 2007, regardless of which type of assets were used to collateralize those commercial papers.

This empirical evidence raises questions about (i) why rollover risks of financial institutions (or banks) issuing collateralized short-term debts are propagated to other banks, (ii) what kind of aggregate variables matters in generating the systemic rollover risks, (iii) why this financial contagion tends to occur at the initial stage of a crisis even before some catastrophic event occurs, and (iv) how we can mitigate this spillover effect to prevent the meltdown of the debt market. To answer these questions, this paper analyzes dynamic interactions between short-term creditors in the primary debt market and outside investors of failed assets in the secondary market.

The paper uses a general equilibrium approach to develop a model with a continuum of heterogeneous banks in which the rollover risks of distressed banks spread to other banks because of pecuniary externalities driven by asset liquidation. In this economy, the rollover decisions of creditors determine the supply of failed assets, and the asset-takeover timing decisions of outside investors determine the demand for those assets. The (market-wide) liquidation price of assets adjusts to clear the secondary market.[1]

The main friction in this economy is that outside investors have different asset-management skills and have limited capacity in absorbing failed assets, similarly as in Shleifer and Vishny (1992). Indeed, each outside investor cannot hold more than a fixed

[1]We discuss later how the liquidation price is decomposed into an idiosyncratic and aggregate (or market-wide) components. Throughout the paper, the liquidation price means the aggregate component of the liquidation price unless otherwise stated.

amount of assets at each point in time, because of some financial constraints (which we do not explicitly model). In this circumstance, each outside investor chooses an optimal time to buy failed assets by taking the future liquidation price as given. In fact, given any liquidation price path, more skilled investors choose to buy assets earlier, whereas less skilled investors decide to wait until the liquidation price drops to an attractive level. This asset-purchase timing decision of outside investors generates a time-dependent inelastic demand schedule for failed assets.

On the supply side, we build on the model of He and Xiong (2012, hereafter, HX) to develop a dynamic rollover game between creditors within each individual bank. Specifically, each bank finances a long-term asset, exposed to asset-specific idiosyncratic shocks, by issuing short-term debts collateralized by the asset. Each bank has its distinct set of creditors. Also, each creditor of any individual bank has an option to withdraw or roll over her funding at her debt-maturity date. If any bank fails to repay its maturing debts, it is forced to liquidate its asset. In this setting, the quantity of failed assets at each point in time is determined by how aggressively creditors in the market withdraw their funding.

In equilibrium, the liquidation price declines over time endogenously because once highly skilled investors have purchased some assets, only less skilled investors remain in the market. But, importantly, asset liquidation does not merely cause a price impact on the liquidation price only at the liquidation date. To see why, suppose a negative shock has unexpectedly hit a certain number of banks today and thus those banks are expected to default tomorrow. Then an anticipated price impact on tomorrow's liquidation price must be reflected in today's liquidation price; otherwise, those outside investors, who originally planned to buy some assets today, will wait until tomorrow to buy the above troubled assets at a lower price. Thus, to eliminate this intertemporal arbitrage opportunity, the market-clearing liquidation price must be depressed immediately.

This endogenous liquidation price generates a dynamic feedback effect through which the rollover risks of distressed banks are amplified and propagated to other banks. Specifically, if a certain number of banks deteriorate, the creditors of other banks also instantly choose to withdraw their funding more aggressively to protect themselves against both the immediate and anticipated future drops in the liquidation price, even before those troubled banks go bankrupt. Put differently, the increased rollover risk of the distressed banks causes a feedback effect to the creditors of other banks through the expected changes in the liquidation price. This instantly surged withdrawal demand will push down the future liquidation price further by bringing more banks into troubles, which will in turn reinforce the dynamic feedback effect. In this regard, the endogenous liquidation price plays a central role in spawning systemic rollover risks especially at the initial stage of a crisis.

An equilibrium is jointly determined by the rollover strategies of creditors, asset-purchase timing decisions of outside investors, and aggregate variables such as a liquidation price, a cross-sectional distribution of asset qualities (or fundamentals), and a skill-level distribution of outside investors. In particular, each creditor uses a time-dependent rollover strategy by taking into account the fact that the liquidation price varies over time.

The paper characterizes an equilibrium analytically without relying on a closed-form solution for an equilibrium. The main difficulty of this task, however, does not arise from the fact that obtaining such a closed-form solution is almost impossible, but comes from the fact that our economy may exhibit (local) strategic substitutability as well as strategic complementarity, because the liquidation price declines over time. To see the details, suppose some creditors of individual bank i choose to withdraw their funding earlier. This change in their rollover strategies reduces the expected survival time of bank i, without affecting the liquidation price because any single bank is infinitesimally small in this economy. Interestingly, the reduced survival time of the bank can either harm or benefit its creditors. On the one hand, the bank's creditors may not receive the promised payments fully, because the bank is expected to default earlier. Thus, earlier withdrawals of some creditors of bank i can prompt other creditors within the same bank to run earlier as well, which is a strategic complementarity effect.

On the other hand, if bank i is expected to fail earlier, its creditors can sell off their collateral before the liquidation price drops further in the future. Because of this positive effect, earlier runs of some creditors of bank i can actually reduce the withdrawal incentives of other creditors within the same bank, a strategic substitutability effect. If this benefit dominates the above loss, the economy indeed exhibits strategic substitutability.

The presence of these two opposite economic forces generally makes equilibrium characterization hard; see, for instance, Goldstein and Pauzner (2005). In fact, in their model, strategic substitutability arises because when so many depositors run, further withdrawals dilute the payments per each withdrawing creditor, whereas in our model, strategic substitutability can arise because the liquidation price declines over time.

In spite of these technical and conceptual difficulties, the paper shows our economy has at least one equilibrium by using an iterative elimination of dominated strategies from only one side. Unfortunately, whether this economy obtains a unique equilibrium is unknown at the moment.[2]

We now discuss policy implications of the model. One intriguing observation of HX (2012) is that providing more credit support to banks or extending debt maturities may exacerbate a crisis by increasing the withdrawal incentives of creditors, especially when the asset volatility is high. Our model, by contrast, implies such policies tend to mitigate a crisis by increasing the liquidation price. Specifically, in HX (2012), in which the liquidation price (or recovery rate) is exogenously given, extending a bank's survival time through the above policies may hurt the bank's creditors by making a limited liability problem worse.[3] However, in our model, those policies increase the liquidation price by reducing the aggregate default rate, which is a missing channel in HX (2012). According to the numerical results of the model, this positive effect dominates the above negative effect, meaning that those intervention policies can effectively alleviate a crisis. Covitz, Liang, and Suarez (2013) provide some empirical evidence that shows ABCP programs with weaker credit sponsors experienced more runs during the 2008 financial crisis.

The paper contributes to the bank-run or banking-crisis literature as follows. On top of the papers studying a maturity-mismatch problem within a single bank, (e.g., Diamond and Dybvig, 1983; Rochet and Vives, 2004; Goldstein and Pauzner, 2005; Cheng and Milbradt, 2012; He and Xiong, 2012; He and Manela, 2016), another thread of papers

[2] In general, an economy exhibiting a strong feedback effect tends to have multiple equilibria; see, for instance, Angeletos and Werning (2006) and Liu (2016). Providing precise parameter values under which multiple equilibria indeed arise is not available at the moment.

[3] This argument actually implies the economy of HX (2012) may also exhibit strategic substitutability if the asset volatility is high, because earlier withdrawals from some creditors of a bank can reduce the withdrawal incentives of other creditors by shortening the survival time of the bank.

have analyzed the episodes of financial contagion by examining the following transmission channels. First, Allen and Gale (2000), Lagunoff and Schreft (2001), Dasgupta (2004), and Liu (2016) study the direct capital linkages among multiple banks. Second, Goldstein and Pauzner (2004) analyze the wealth effect caused by the behaviors of a common pool of risk-averse investors; see also Kyle and Xiong (2001) and Kodres and Pritsker (2002), who study a similar contagion channel in other asset markets. Third, Chen (1999), Acharya and Yorulmazer (2008b), and Oh (2013) study the role of uncertainty about some economy-wide information. Although all these contagion channels are important, none of these papers study the interaction between creditors and outside investors through the liquidation price of collaterals. To focus on this particular channel, our model assumes (i) balance sheets of different banks are not connected with each other, (ii) all agents are risk neutral, and (iii) all information is publicly observable.

In fact, Choi (2014), Ahnert (2016), and Eisenbach (2016) consider the interplays between creditors across different banks through the liquidation price in their studies on the government's intervention policies and a market-disciplining role of short-term debts. However, those papers consider either a static model with a contemporaneous price impact or a dynamic model with a reduced-form demand function for failed assets. Uhlig (2010) studies systemic bank runs by assuming outside investors are either ambiguity averse or uninformed, but his model assumes each bank has a representative creditor to abstract from a coordination problem within each individual bank. Bernardo and Welch (2004) model preemptive runs in a stock market by analyzing interactions between shareholders and risk-averse market makers. In their model, however, (i) market-makers are myopic and (ii) shareholders receive a liquidity shock exogenously. In our model, (i) outside investors solve a full dynamic problem and (ii) banks liquidate their assets when they fail to meet withdrawal demands.

Another branch of the literature studying the macroeconomic effects of asset fire-sales and collateral constraints includes, for instance, Shleifer and Vishny (1992), Kiyotaki and Moore (1997), Acharya and Yorulmazer (2008a), Brunnermeier and Pedersen (2009), and Caballero and Simsek (2010). Although these papers also consider a liquidity-shortage problem caused by asset fire-sales, the primary interests of these papers lie in studying capital structure, investment patterns, asset price dynamics, portfolio choice, and so on.

In particular, Kiyotaki and Moore (1997) study the interaction between a borrowing constraint and a collateral price to explain how small temporary shocks can be amplified and spill over to other economic sectors. In fact, their model also generates an interesting dynamic feedback effect through the collateral price as in our model. However, a default event never occurs in their model because borrowers cannot borrow money more than their collateral values. That is, such a borrowing constraint effectively protects lenders from the default threat of their borrowers. In this regard, the present paper complements their work by analyzing contagious bank failures and systemic rollover risks more explicitly.

This paper also shares interests with Hackbarth, Miao, and Morellec (2006) and Chen (2010), who study the effects of macroeconomic conditions in the credit market. But both papers assume the recovery rate follows an exogenous regime-shifting process, meaning that their models are more suitable to explain business cycles rather than financial contagion. The present paper is also related to Carlin, Lobo, and Viswanathan (2007), He and Milbradt (2014), and Oehmke (2014) in the sense that these papers study certain frictions in the secondary market. Lastly, this paper makes a methodological contribution to the global-game literature in the way discussed above; see, for instance, Carlsson and van Damme (1993), Morris and Shin (1998), Frankel and Pauzner (2000), and Angeletos, Hellwig, and Pavan (2007) as the references for pioneering work in this field.

The paper is organized as follows. Section 2 develops the model. Section 3 characterizes an equilibrium. Section 4 presents the model results. Section 5 concludes. All technical proofs are included in Appendix.

2 The Model

The economy consists of the primary and secondary debt markets. The primary market is populated by a continuum of banks of measure 1, indexed by $i \in [0, 1]$. Each bank holds one unit of a long-term asset. Each bank finances its asset by issuing short-term debts and equity. The secondary market contains a continuum of outside investors. Time is continuous. All agents are risk neutral and discount future consumptions at a rate of ρ.

All information is publicly observable. There is only one consumption good, called cash.

2.1 Primary Debt Market

2.1.1 *Assets*

The assets of different banks are ex-ante identical but are exposed to idiosyncratic shocks ex post. We use the same index i to denote bank i's asset. Each asset produces constant cash flows cdt at each point in time. Each asset matures at a random time that arrives with a Poisson intensity ϕ, independently of any other assets. If asset i matures, say, at time $\tau(i,\phi)$, it produces a lump-sum final payoff $y^i_{\tau(i,\phi)} = \exp(u^i_{\tau(i,\phi)})$, where $y^i_t = \exp(u^i_t)$ indicates the time-t value of the final payoff. We call both u^i_t and y^i_t the fundamental of bank i at time t. But we will mainly use the log-scale for convenience throughout the paper. The fundamental u^i_t evolves according to

$$du^i_t = \nu dt + \sigma dZ^i_t,$$

where ν measures the drift, $\sigma > 0$ measures the volatility, and Z^i_t is a standard idiosyncratic Brownian motion. Each bank exits the market after its asset matures. We also assume each asset is indivisible to rule out a case in which only a certain fraction of the asset trades in the secondary market.

In this setup, the unlevered (or first-best) value of asset i is given by

$$F(u^i_t) = E_t\left[\int_t^\infty \phi e^{-\phi(s-t)}\left(\int_t^s e^{-\rho(r-t)}cdr + e^{-\rho(s-t)}e^{u^i_s}\right)ds\right] = \frac{c}{\rho+\phi} + \frac{\phi e^{u^i_t}}{\rho+\phi-\nu-\frac{\sigma^2}{2}}.$$

The first term on the right-hand side describes the present value of the interim cash flows and the second term represents the present value of the final output. We assume $\nu + \frac{\sigma^2}{2} < \rho + \phi$ to prevent the explosion of the unlevered asset value.

To deal with the asset heterogeneity, we use $m(t,u)$ to denote the distribution of the asset fundamentals at time t. That is,

$$m(t,u) = \#\{i : u^i_t \in [u, u+du)\},$$

where # represents the canonical measure on $\mathbb{R}$. An initial distribution is arbitrarily given by $m_0(u)$ that satisfies the normalization condition $\int_{-\infty}^{\infty} m_0(u)du = 1$. We discuss later how $m(t, u)$ evolves over time.

2.1.2 *Debt Contracts*

Each bank i issues short-term debts to its own creditors of measure 1, indexed by $(i, j) \in [0, 1] \times [0, 1]$, with equal seniority. The creditors are not overlapped across different banks. Each debt contract of any individual bank matures at a random time with Poisson intensity λ, independently of any other debts. Accordingly, each bank faces maturing debts of a measure λdt at each point in time. Each debt contract pays a coupon payment cdt at each point in time and a lump-sum face value P at the maturity date. Note that each bank transfers all its cash flows, cdt, to its creditors as the coupon payments. We make this assumption to abstract from the bank's cash-management problem. Each bank has limited liability.

In addition, each creditor of any individual bank i can choose whether to withdraw or roll over her funding at her debt-maturity date. If she withdraws (or runs), she receives the face value P unless bank i defaults. If she rolls over, the face value P is not paid today, but her debt contract is renewed under the same terms as the original contract; thus, she keeps receiving the coupons until the next maturity date, at which she can exercise the rollover option again, as long as bank i survives until then. We discuss the default event later.

Lastly, if any individual bank i's asset matures at time $\tau(i, \phi)$, the bank pays its creditors $\min\{\exp(u^i_{\tau(i,\phi)}), P\}$ equally, which reflects the limited liability feature. The bank then exits the market.

2.1.3 *Debt Repayment and Default*

Recall that a fraction λdt of each bank i's creditors reach their maturity dates at each point in time. Then, suppose a fraction ξ^i_t of those maturing creditors choose to run at time t, where $\xi^i_t \in [0, 1]$. (We later assume all creditors use a symmetric rollover strategy so that ξ^i_t will be either 0 or 1.) But, recall bank i does not hold any cash reserves during the interim dates. So each bank i relies on its parent company, which commits at date 0 to provide credit support to bank i to meet the future withdrawal demands. That is, the parent company pays $\xi_t^i \lambda P dt$ to those withdrawing creditors at time t on behalf of bank i, as long as the parent company has enough money. We can interpret the parent companies as credit-line providers, investment banks, insurers, and so on.

But, unfortunately, the parent companies do not have an infinite amount of cash. To model this imperfect credit support in a simple way, we assume bank i's parent company fails to meet the withdrawal demands with an instantaneous probability $\theta \xi_t^i \lambda P dt$ at each time t, independently of any other parent companies. Here, the parameter θ measures the failure probability per unit dollar to be redeemed. Formally speaking, bank i fails at a random time that arrives with a non-stationary Poisson intensity $\theta \xi_t^i \lambda P$. We call θ unreliability of credit support.

If bank i fails to repay its maturing debts, it is forced to liquidate all of its asset holdings immediately in the secondary market, because partial liquidation is not allowed. The bank then distributes the liquidation proceeds to its creditors and exits the market. Again, the bank enjoys the limited liability at the default date. We discuss later how the liquidation price of assets is determined.

Meanwhile, if bank i has successfully repaid any maturing debt, the bank instantly sells the same debt contract to a new creditor at the market price of debt. We make this assumption to avoid having to keep track of the number of remaining creditors of each bank. We discuss the market price of debt later.

2.2 Secondary Debt Market

The secondary market contains an infinite measure of outside investors, indexed by $k \in [0, \infty)$.[4] The secondary market is competitive; that is, all market participants, including failed banks, are price takers. Each outside investor finances her asset with equity only.[5] The main friction in this market is that (i) outside investors have different asset-

[4]We assume the measure of outside investors is infinite only for convenience.

[5]Without loss of generality, we may assume each outside investor purchases her asset with her own money.

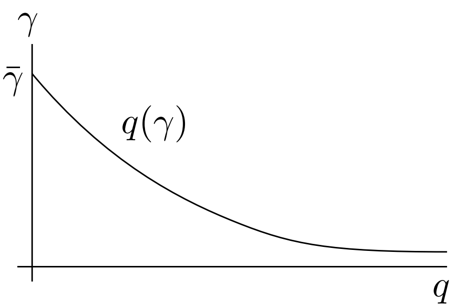

Figure 1: A skill-level distribution function $q(\gamma)$.

management skills and (ii) each outside investor cannot hold more than one asset at each point in time because of some financial constraints (which we do not explicitly model). To clarify, each outside investor cannot hold a certain fraction of an asset, because each asset is indivisible.

Moreover, each outside investor is allowed to resell her asset later and then buy another asset afterwards. Accordingly, in principle, assets in the secondary market can be supplied from either failed banks or some other outside investors who have purchased their assets before. But, in Section 3.2.2, we show that reselling an asset afterwards is not optimal in equilibrium. As a result, we can say only failed banks supply their assets in equilibrium.

Regarding the management skills, we use γ^k to denote outside investor k's skill level, where $\gamma^k \in [0, \bar{\gamma}]$ for some constant $\bar{\gamma}$. We then let $q(\gamma)$ denote the measure of the outside investors whose skill levels are larger than γ as shown in Figure 1. That is, $q(\gamma) = \#\{k : \gamma^k > \gamma\}$. For simplicity, we assume $q(\gamma)$ is continuous and strictly decreasing in γ. We then use $\gamma(q)$ to indicate the inverse function of $q(\gamma)$. In particular, we will use $\gamma(q) = \bar{\gamma}(1+q)^{-\eta}$ for some $0 < \eta$ when simulating the model. We interpret the parameter η as the magnitude of a skill shock (to be discussed).

Importantly, if each outside investor k acquires any asset at time t, she is required to pay $\bar{\gamma} - \gamma^k$ as an investor-specific (lump-sum) fixed cost.[6] Moreover, if any asset i is managed by an outside investor, regardless of who she is, the asset's output level

[6]Even when outside investor k purchases asset i supplied from another outside investor who has bought the asset before, investor k pays the same amount of cost $\bar{\gamma} - \gamma^k$. That is, purchasing an asset supplied from a failed bank does not incur more costs. This assumption simplifies our analysis when considering all admissible (off-equilibrium) trades between outside investors.

reduces by $100 \times (1-\alpha)$ percent, where $\alpha \in (0,1)$ is a common constant. That is, each asset i produces cash flows of $\alpha c dt$ during the interim dates and a final payoff of $\alpha \exp(u^i_{\tau(i,\phi)})$ under the management of any outside investor. We can interpret these fixed and proportional costs as a loss of customers, restructuring charges, agency costs, and so on. Also, the reason why we have introduced the investor-specific cost as a fixed cost is that this specification simplifies our analysis by disentangling the investor-specific cost from the asset fundamental. But we still introduce the proportional cost to make our economy consistent with the frictionless benchmark economy of HX (2012), when $\gamma^k \equiv \bar{\gamma}$ for each k.

In this circumstance, the present value of the future payoffs net of the cost accrued to investor k, who purchases asset i at time t, is given by

$$\alpha F(u^i_t) - \bar{\gamma} + \gamma^k, \tag{1}$$

where the first term reflects the assumption each outside investor finances her asset with equity only. Here, we have also used the presumed fact that investor k will never resell her asset in the future. As mentioned, Section 3.2.2 shows this claim holds true in equilibrium.

Now, expression (1) suggests we can postulate the liquidation price of each asset i at time t has the following form:

$$\alpha F(u^i_t) - \bar{\gamma} + p_t,$$

where p_t indicates an aggregate component of the liquidation price. To avoid complexity, this paper pays attention to only this form of the liquidation price. We can further postulate p_t is deterministic in t, because no aggregate shocks occur in this economy except a one-time unanticipated shock (to be discussed). Put differently, each agent in this economy has perfect foresight of the future liquidation price path $\{p_t\}$. We also assume p_t satisfies the transversality condition such that $\lim_{t\to\infty} e^{-\rho t} p_t = 0$.

Accordingly, the present value of the future profits accrued to outside investor k is given by

$$\alpha F(u^i_t) - \bar{\gamma} + \gamma^k - \big(\alpha F(u^i_t) - \bar{\gamma} + p_t\big) = \gamma^k - p_t.$$

We can interpret this expected profit $\gamma^k - p_t$ as the premium for the investor's waiting option. That is, if the expected profit is too low or negative today, then the investor will wait until the liquidation price drops to an attractive level.

Moreover, the expected profit $\gamma^k - p_t$ does not depend on the idiosyncratic component of the liquidation price, $\alpha F(u_t^i)$. That is, the expected profit would be indifferent, regardless of which asset outside investor k purchases at time t. As a result, each outside investor only needs to care about *when* to buy an asset, not about *which* asset to buy. Put differently, each outside investor needs to specify information only about when she wants to buy an asset, not about which asset she wants to buy, when submitting such information to a presumed auctioneer in the secondary market. In this regard, by saying the secondary market clears, we mean that at each point in time, the quantity of the assets supplied matches the number of outside investors willing to buy some assets, even though those assets supplied may have different fundamentals. This result simplifies our analysis substantially, which is in fact the main reason why the paper disentangles the investor-specific cost from the asset fundamental as mentioned above.

We hereafter call p_t the liquidation price rather than the aggregate component of the liquidation price, unless doing so causes any serious confusion.

Remark 1. Regarding the parameter $\bar{\gamma}$, we can actually remove this parameter from $\alpha F(u_t^i) - \bar{\gamma} + \gamma^k$ (and from $\alpha F(u_t^i) - \bar{\gamma} + p_t$ as well) and interpret γ^k as outside investor k's own valuation on the existing assets in the economy. This alternative interpretation of γ^k might simplify the notations in the sequel, but there is no economically significant difference between the original interpretation and the alternative interpretation.

2.3 Valuations

2.3.1 *Individual Creditor's Problem*

This section describes an individual creditor's problem. To this aim, we first specify admissible rollover strategies of creditors. For tractability, the paper focuses on symmetric time-dependent threshold strategies. That is, we consider a time-dependent deterministic function z_t such that each creditor of any individual bank i chooses to run at her debt-

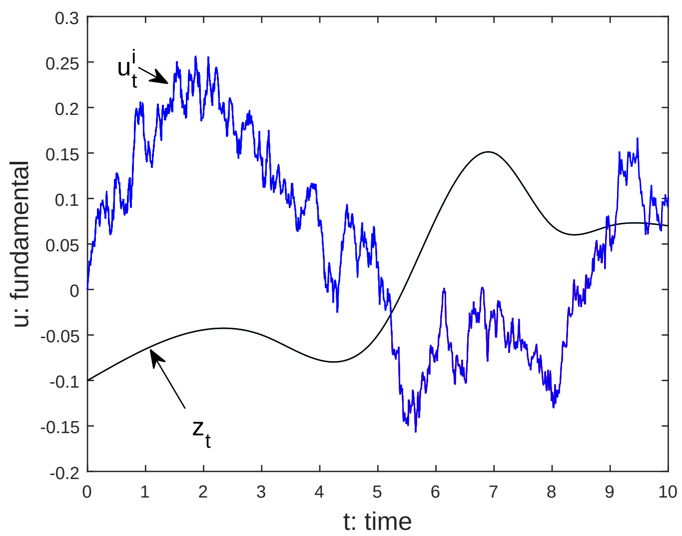

Figure 2: A rollover strategy $\{z_t\}$.

maturity date τ if and only if u_τ^i lies below z_τ. For example, in Figure 2, if the debt claim of a certain creditor of bank i matures at time $\tau = 3$ (resp. $\tau = 7$), she chooses to roll over (resp. withdraw) because $u_\tau^i > z_\tau$ (resp. $u_\tau^i < z_\tau$). Without loss of generality, we assume z_t is continuous in t and has a finite limit as t goes to ∞, because these properties will hold in equilibrium. In addition, we particularly use $z_t \equiv \infty$ to indicate the rollover strategy under which each creditor unconditionally chooses to withdraw her funding at her first maturity date. Similarly, we use $z_t \equiv -\infty$ to indicate the rollover strategy under which each creditor unconditionally chooses to roll over her funding at every maturity date. Under this definition, for any rollover strategy $\{z_t\}$, a fraction λdt of the creditors of each bank i choose to run at each time t if and only if u_t^i lies below z_t. In other words, we have $\xi_t^i = 1_{u_t^i < z_t}$.

Now, assuming the primary debt market is competitive and has a zero entry cost, the market value of debt of bank i at time t, denoted by $V(t, u_t^i)$, is defined as the present value of the future payoffs accrued to each creditor of the bank from time t. Each creditor takes the future liquidation price $\{p_t\}$ and the rollover strategy of other creditors $\{z_t\}$ as given. We will sometimes use $V(t, u; z, p)$ to indicate the dependence of the debt value on

$\{z_t, p_t\}$. In this setting, the debt value satisfies the following recursive equation:

$$V(t,u_t^i) = cdt + \phi dt \min\{e^{u_t^i}, P\} + \theta\lambda P 1_{u_t^i<z_t} dt \min\{\alpha F(u) - \bar{\gamma} + p_t, P\} +$$
$$\lambda dt \max_{\text{run, roll over}} \{P, V(t,u_t^i)\} + (1 - \phi dt - \theta\lambda P 1_{u_t^i<z_t} dt - \lambda dt) e^{-\rho dt} E_t[V(t+dt, u_{t+dt}^i)].$$

The first term on the right-hand side denotes the coupon payment, the second term denotes the final payoff, the third term denotes the liquidation proceeds, the fourth term denotes the rollover option value, and the fifth term denotes the present value of tomorrow's debt value. Before we proceed further, we discuss the fourth term more carefully. If any time-t maturing creditor chooses to run, her expected payoff is equal to

$$P(1 - \theta\lambda P 1_{u_t^i<z_t} dt) + \min\{\alpha F(u_t^i) - \bar{\gamma} + p_t, P\}\theta\lambda P 1_{u_t^i<z_t} dt,$$

where we have used the fact that this individual creditor's rollover decision does not affect the bank's default probability. Meanwhile, if she decides to roll over, her expected payoff is equal to

$$V(t,u_t^i)(1 - \theta\lambda P 1_{u_t^i<z_t} dt) + \min\{\alpha_t F(t,u_t^i), P\}\theta\lambda P 1_{u_t^i<z_t} dt.$$

So, by ignoring the dt-order terms, we can see the rollover option value reduces to $\max\{P, V(t,u_t^i)\}$. In this regard, we can interpret $V(t,u_t^i)$ as the continuation value of debt as well.

Now, from Ito's lemma, we see that $V(t,u)$ satisfies the following HJB equation:

$$\rho V(t,u) = c + \phi\left(\min\{e^u, P\} - V(t,u)\right) + \theta\lambda P 1_{u<z_t}\left(\min\{\alpha F(u) - \bar{\gamma} + p_t, P\} - V(t,u)\right) +$$
$$\lambda \max_{\text{run, roll}} \{P - V(t,u), 0\} + \nu V_u(t,u) + \frac{\sigma^2}{2} V_{uu}(t,u) + V_t(t,u), \quad (2)$$

where the subscripts denote the partial derivatives with respect to t and u. Appendix 6.1 shows the existence and uniqueness of a solution to this equation.

2.3.2 *Levered Asset Value*

In this section, we calculate the levered asset value of each individual bank. The levered value of asset i at time t, denoted by $G(t, u_t^i; z, p)$, is defined as the present value of the future payoffs accrued to bank i's equity holders and creditors from time t, given any rollover strategy $\{z_t\}$ and liquidation price path $\{p_t\}$. Then, $G(t, u; z, p)$ satisfies the following recursive equation:

$$\begin{aligned} G(t,u) = cdt + \phi dte^u + \theta\lambda P dt 1_{u<z_t}(\alpha F(u) - \bar{\gamma} + p_t) + \\ (1 - \phi dt - \theta\lambda P 1_{u<z_t} dt) e^{-\rho dt} E_t[G(t + dt, u_{t+dt})], \end{aligned}$$

where the first term on the right-hand side denotes the interim cash flows, the second term denotes the final payoff, the third term denotes the liquidation proceeds, and the fourth term denotes the present value of tomorrow's levered asset value. Then, Ito's lemma implies $G(t, u)$ satisfies the following HJB equation:

$$\rho G(t,u) = c + \phi(e^u - G(t,u)) + \theta\lambda P 1_{u<z_t}(\alpha F(u) - \bar{\gamma} + p_t - G(t,u)) + \nu G_u + \frac{\sigma^2}{2} G_{uu} + G_t. \quad (3)$$

Again, Appendix 6.1 shows the existence and uniqueness of a solution to this equation.

Now, note that the equity value of bank i at time t is given by $E(t, u) := G(t, u) - V(t, u)$. Thus, to finance asset i at date 0, bank i needs to raise $E(0, u_0^i)$ and $V(0, u_0^i)$ from its equity holders and creditors, respectively. However, the equity holders in our model do not play a significant role, because each bank's parent company repays maturing debts on behalf of the bank itself in the stylized manner described above.

2.3.3 *Individual Outside Investor's Problem*

This section describes an individual outside investor's problem. As mentioned above, each outside investor chooses an optimal time τ to buy an asset by taking the future liquidation price $\{p_t\}$ as given. That is, suppose an outside investor with skill level γ (the index k is omitted) does not hold any asset at time t. Then, we can describe her profit-maximization problem as

$$\max_{\tau \geq t} e^{-\rho(\tau - t)}(\gamma - p_\tau). \tag{4}$$

Assuming p_t is differentiable in t, (which holds true in equilibrium), the first-order condition for this problem is given by[7]

$$\frac{dp_s}{ds}|_{s=\tau} - \rho(\gamma - p_\tau). \tag{5}$$

By rewriting this condition as $\gamma - p_\tau = (\gamma - p_{\tau+dt})/(1 + \rho dt)$, we can say this outside investor has no incentive to locally deviate from the decision to buy an asset at time τ.

Here, the first-order condition (5) may not be sufficient to establish global optimality of the solution, because the liquidation price path $\{p_t\}$ is arbitrarily given here. In Section 3.2, we will show that, in equilibrium, the optimal purchasing time τ satisfying condition (5) is indeed optimal. Also, when describing the maximization problem (4), we have again used the presumed fact that reselling an asset is not optimal. Section 3.2 shows this claim also holds true in equilibrium.

2.4 Aggregation

This section calculates the aggregate demand and supply of failed assets. First, for any $\{p_t\}$, the cumulative quantity of assets demanded up to time t is given by

$$b_t(p) = \#\{k : \tau^k \leq t\},$$

where τ^k denotes the optimal purchasing time for outside investor k. The quantity of assets demanded per unit time is then given by db_t/dt.

To compute the aggregate supply of failed assets, we need to describe how the asset-fundamental distribution $m(t, u)$ evolves. To this aim, recall that (i) a fraction ϕdt of the existing banks exit the market at each time and (ii) each bank i defaults with the non-stationary Poisson intensity $\theta \lambda P 1_{u_t^i < z_t}$. In this setting, the Kolmogorov forward equation

[7] If $\gamma < \min_{\tau \geq t} p_\tau$, the solution is given by $\tau = \infty$. That is, an outside investor, whose skill level is lower than the minimum possible liquidation price, will never purchase an asset. This special case does not matter for our analysis.

implies $m(t,u)$ satisfies

$$m_t(t,u) = -\phi m(t,u) - \theta\lambda P 1_{u<z_t} m(t,u) - \nu m_u(t,u) + \frac{\sigma^2}{2} m_{uu}(t,u). \tag{6}$$

The first two terms on the right-hand side indicate the number of banks exiting the market at each time; the other two terms describe how stochastic changes in the asset fundamentals affect the measure $m(t,u)$. Meanwhile, $m(t,u)$ vanishes in the long run for each u, meaning we need to interpret our model as a short-run behavior of a crisis even though we consider an infinite-horizon model.

The cumulative quantity of failed assets supplied up to time t is then given by

$$q_t(z) = \theta\lambda P \int_0^t \int_{-\infty}^{z_s} m(s,u;z) du ds, \tag{7}$$

whereas the quantity of assets supplied per unit time (or default rate) is equal to dq_t/dt. Note that q_t depends on only $\{z_t\}$; but, of course, q_t will indirectly depend on $\{p_t\}$ in equilibrium through the rollover strategy $\{z_t\}$.

2.5 Equilibrium

Definition 2.1. An equilibrium in this economy is defined as a pair of a rollover strategy and a liquidation price path, $\{z_t^*, p_t^*\}$, such that (i) each creditor maximizes her profit by taking $\{z_t^*, p_t^*\}$ as given, (ii) each outside investor maximizes her profit by taking $\{p_t^*\}$ as given, (iii) the strategy $\{z_t^*\}$ itself is individually optimal for each creditor, and (iv) the supply and demand for assets in the secondary market coincide with each other at each time t, that is, $q_t(z^*) = b_t(p^*)$ for each t.

In fact, we can write condition (iii) as

$$V(t,u^i;z^*,p^*) \begin{cases} < P, & \text{if } u^i < z_t^* \\ = P, & \text{if } u^i = z_t^* \\ > P, & \text{if } u^i > z_t^*, \end{cases} \tag{8}$$

which means each time-t maturing creditor of bank i optimally chooses to run if and only if u_t^i is larger than z_t^*.

3 Equilibrium Characterization

This section solves for an equilibrium of the economy under some parameter restrictions. We first characterize partial equilibria in the primary and secondary markets separately. We then pin down an equilibrium in the whole market. When analyzing a partial equilibrium, we will often use the term "equilibrium" instead of "partial equilibrium".

3.1 Parameter Restrictions

We impose the following restrictions on the model parameters:

$$\nu + \sigma^2/2 < \rho + \phi, \tag{9}$$

$$\rho < c/P < \rho + \phi, \tag{10}$$

$$\alpha < \bar{\alpha} \text{ for some } \bar{\alpha}, \tag{11}$$

$$\bar{\gamma} < \frac{a}{1 - b^{-\eta}}, \tag{12}$$

where $a = \frac{\alpha c}{\rho + \phi}$ and $b = 1 + \frac{\theta \lambda P}{\phi + \theta \lambda P}$.

We have already discussed condition (9). The first inequality in condition (10) assumes the coupon rate c/P is larger than the risk-free rate ρ to ensure the debt value is larger than the face value P when the asset fundamental is sufficiently large, regardless of the strategies of other creditors and the liquidation price. The second inequality in condition (10) assumes the coupon rate c/P is smaller than $\rho + \phi$ to ensure the debt value is lower than the face value P when the asset fundamental is sufficiently small, regardless of the strategies of other creditors and the liquidation price. Lemma 3.1 below justifies these two assertions. Condition (11) says asset liquidation is quite costly. We impose this condition to ensure the existence of a lower dominance region; see Lemma 3.4 for the details. Lastly, we need condition (12) to ensure the liquidation value $\alpha F(u) - \bar{\gamma} + p_t$

remains positive for any (t, u).[8]

3.2 Partial Equilibrium in the Secondary Market

We begin with analyzing the secondary market because this market creates the key friction in our economy. Specifically, we show that for any rollover strategy $\{z_t\}$, there exists a unique liquidation price path $\{p_t\}$ that matches the supply and demand for failed assets.

3.2.1 *Liquidation Price*

We first show that for any (conjectured) liquidation price path $\{p_t\}$, a more skilled outside investor chooses to buy an asset earlier than a less skilled investor. By way of contradiction, suppose τ_1 (resp. τ_2) is an optimal purchasing time for an investor of skill level γ_1 (resp. γ_2), where $\gamma_1 > \gamma_2$ and $\tau_1 > \tau_2$. This assumption means

$$e^{-\rho\tau_1}(\gamma_1 - p_{\tau_1}) \geq e^{-\rho\tau_2}(\gamma_1 - p_{\tau_2}) \quad \text{and} \quad e^{-\rho\tau_2}(\gamma_2 - p_{\tau_2}) \geq e^{-\rho\tau_1}(\gamma_2 - p_{\tau_1}),$$

which implies $e^{-\rho(\tau_1-\tau_2)}(\gamma_1 - \gamma_2) \geq \gamma_1 - \gamma_2$, a contradiction.

We then recall that (i) for any $\{z_t\}$, the cumulative supply of failed assets up to time t is given by formula (7), and (ii) any single outside investor is allowed to buy only one asset. Then, because more skilled investors choose to step in earlier than less skilled investors, for the demand and supply to match each other, a marginal investor who is willing to buy an asset at time t must have the following skill level:

$$\gamma_t := \gamma(q_t), \quad \forall t.$$

The first-order condition (5) then implies the equilibrium liquidation price p_t must satisfy

$$\frac{dp_t}{dt} = -\rho(\gamma_t - p_t), \quad \forall t, \tag{13}$$

[8]Specifically, this condition ensures $\lim_{u\to-\infty} F(u) - \bar{\gamma} + \bar{\gamma}(1 + q_{\max})^{-\eta} > 0$, where $q_{\max}$ is the maximum possible cumulative supply of assets, which is given by $q_{\max} = \theta\lambda P \int_0^\infty e^{-(\phi+\theta\lambda P)t}dt$. The only purpose of imposing condition (12) is to make $\alpha F(u) - \bar{\gamma} + p_t$ positive. We do not need this condition when characterizing an equilibrium.

which can be rewritten as

$$\underbrace{\gamma_t - p_t}_{\text{premium for today's buyer}} = \underbrace{\gamma_t - \gamma_{t+dt}}_{\text{skill-level difference}} + (1-\rho dt) \underbrace{\gamma_{t+dt} - p_{t+dt}}_{\text{premium for tomorrow's buyer}}. \tag{14}$$

That is, the premium for the waiting option of today's buyer, $\gamma_t - p_t$, equals the skill-level difference between today's and tomorrow's buyers plus the discounted premium for the waiting option of tomorrow's buyer.

The solution to equation (13) is given by

$$p_t = \gamma_t + \int_t^{\infty} e^{-\rho(s-t)} \frac{d\gamma_s}{ds} ds = \int_t^{\infty} \rho e^{-\rho(s-t)} \gamma_s ds, \tag{15}$$

where we have used the transversaltiy condition, $\lim_{t\to\infty} e^{-\rho t} p_t = 0$, to pin down the initial price p_0. The first identity in this formula says the premium for the time-t marginal buyer is equal to the discounted sum of the skill-level differences between the time-s and time-$(s+ds)$ marginal buyers, where s spans over $[t, \infty)$. We can view this identity as an expanded version of the recursive expression given by (14). Note that the waiting-option premium is always (weakly) positive because γ_s decreases in s. Meanwhile, the second identity in the formula, which will be useful when characterizing an equilibrium, is easily derived from the first identity.

Moreover, formula (15) implies p_t converges to γ_t as t goes ∞, meaning that the waiting-option premium vanishes in the long run. Lastly, in the frictionless economy of HX (2012), where $\gamma^k \equiv \bar{\gamma}$ for each k, we have $p_t \equiv \bar{\gamma}$ for each t as expected.

3.2.2 *Verification of Optimality*

Given the (partial) equilibrium price path $\{p_t\}$ derived above, we will finally show that for each $t \geq 0$, it is indeed optimal for the outside investor with skill level $\gamma = \gamma_t$ to purchase an asset at date t and not to resell her asset afterwards. For the moment, we assume the second claim is true and focus on proving the first claim. We show the second claim later by using the one-time deviation principle. Recall this outside investor solves the following problem at date 0:

$$\max_s e^{-\rho s}(\gamma_t - p_s).$$

The derivative of this expected profit with respect to s satisfies

$$e^{-\rho s}(-\rho(\gamma_t - p_s) - \frac{dp_s}{ds}) = \rho e^{-\rho s}(\gamma_s - \gamma_t)\begin{cases} \geq 0, & \forall s \leq t \\ \leq 0, & \forall s \geq t, \end{cases}$$

because γ_s is decreasing in s. This result implies the expected profit to the investor with skill level $\gamma = \gamma_t$ is indeed maximized at time t. Here, we have considered only the time-0 problem of the investor without loss of generality. We can easily show her solution is time-consistent.

Regarding the second claim, suppose this investor of skill level γ_t makes a one-time deviation from the above decision. That is, suppose she resells her asset, say, asset i, at time s such that $t < s$. Then, because her skill level γ_t is larger than the price p_{s_2} at any time s_2 such that $s \leq s_2$, she must purchase a new asset at some time r such that $s \leq r$. But then, by the one-time deviation assumption, we assume she will not sell her asset again after time r. The time-s value of the profit earned from this deviation is then given by

$$\alpha F(u_s^i) - \bar{\gamma} + p_s + e^{-\rho(r-s)}(\gamma_t - p_r), \tag{16}$$

where the first three terms represent the resale price of the asset at time s and the fourth term describes the time-s value of the profit earned from buying a new asset at time r. Equation (13) implies the derivative of the profit function (16) with respect to r satisfies

$$-\rho e^{-\rho(r-s)}(\gamma_t - \gamma_r) < 0, \quad \forall r \geq s,$$

which means this investor must buy a new asset immediately at time $r = s$. Then, from the profit function (16), her expected profit is simply given by $\alpha F(u_s^i) - \bar{\gamma} + \gamma_t$. However, if she does not deviate at time s and keeps her asset until the asset matures, she is expected to earn $\alpha F(u_s^i)$ during her lifetime. Therefore, we can conclude reselling the asset at time

s is not optimal because $-\bar{\gamma} + \gamma_t < 0$.

3.3 Partial Equilibrium in the Primary Market

This section pins down a partial equilibrium of the primary market. Specifically, we show that for any liquidation price path $\{p_t\}$, the rollover game within each bank obtains a unique equilibrium strategy $\{z_t^*\}$. Without loss of generality, we assume (i) $\lim_{t\to\infty} p_t$ has a finite limit and (ii) $0 \leq p_t \leq \bar{\gamma}$, because we have already seen these properties hold in partial equilibrium in the secondary market. (We need not assume p_t is decreasing in t.) As discussed earlier, the main obstacle of this task is that our economy may exhibit (local) strategic substitutability as well as strategic complementarity, because when p_t declines sharply or σ is high, earlier runs of some creditors of any individual bank can reduce the withdrawal incentives of other creditors within the same bank by shortening the bank's survival time. To overcome this difficulty, we derive some important properties of the debt value function as follows.

3.3.1 *Properties of the Debt Value Function*

Lemma 3.1. For any $\{z_t, p_t\}$, we have

$$\lim_{u\to-\infty} V(t,u;z,p) < P < \lim_{u\to\infty} V(t,u;z,p), \quad \forall t.$$

Proof. See Appendix 6.2. □

This lemma says that under the parameter restrictions imposed in Section 3.1, we can rule out trivial cases in which the debt value is always less (resp. larger) than the face value and thus $z_t \equiv \infty$ (resp. $z_t \equiv -\infty$) becomes an equilibrium strategy. The proof of this lemma is quite mechanical.

The next lemma provides a sufficient condition that implies the rollover game within each bank exhibits (local) strategic complementarity.

Lemma 3.2. For any two pairs of a rollover strategy and a liquidation price path, $\{z_t^a, p_t^a\}$ and $\{z_t^b, p_t^b\}$, such that $z_t^a \leq z_t^b$ and $p_t^a \geq p_t^b$ for each t, suppose one of the following conditions holds:

$$V(t,u;z^b,p^b) \geq \min\{\alpha F(u) - \bar{\gamma} + p_t^b, P\}, \quad \forall t,u \quad \text{such that} \quad z_t^a < u < z_t^b \tag{17}$$

or

$$V(t,u;z^a,p^a) \geq \min\{\alpha F(u) - \bar{\gamma} + p_t^a, P\}, \quad \forall t,u \quad \text{such that} \quad z_t^a < u < z_t^b. \tag{18}$$

Then, we have

$$V(t,u;z^a,p^a) \geq V(t,u;z^b,p^b), \quad \forall t,u. \tag{19}$$

Proof. See Appendix 6.3. □

Remark 2. In general, strategic complementarity is defined in terms of a utility differential. Such a utility differential in our model is defined as the continuation value of debt minus the face value. So we can rewrite inequality (19) as $V(t,u;z^a,p^a) - P \geq V(t,u;z^b,p^b) - P$ to make our statement consistent with the formal definition of strategic complementarity.

Remark 3. In this lemma, we allow the cases in which $z_t^a \equiv -\infty$, $z_t^a \equiv \infty$, $z_t^b \equiv -\infty$, or $z_t^b \equiv \infty$. In particular, if $z_t^a = z_t^b \equiv \infty$, both conditions (17) and (18) become vacuous; thus, if $p_t^a \geq p_t^b$ for each t, then inequality (19) holds without a need to impose any other conditions. We will consider this special case in the proof of Proposition 4.1.

We can understand this lemma as follows. First, without loss of generality, we focus on the case in which $p_t^a = p_t^b$ for each t, because it is easy to show that the debt value becomes lower if the liquidation price is lowered. So we let $p_t := p_t^a = p_t^b$. We now imagine all creditors of individual bank i (in fact, except one single creditor) have increased their rollover strategy from $\{z_t^a\}$ to $\{z_t^b\}$. This change in the rollover strategy does not affect $\{p_t\}$ because any single bank is minuscule. Then the instantaneous default probability of bank i increases from 0 to $\theta\lambda P dt$, if the bank's fundamental u_t^i lies between z_t^a and z_t^b. The direct effect of this increased default probability on the debt value is given by

$$\theta\lambda P dt 1_{z_t^a < u_t^i < z_t^b} \left(\min\{\alpha F(u_t^i) - \bar{\gamma} + p_t, P\} - V(t,u_t^i;z^b;p) \right), \tag{20}$$

meaning that if u_t^i lies between z_t^a and z_t^b, then (i) the chance to get the liquidation proceeds increases by $\theta\lambda P dt$ and (ii) the chance to keep receiving the debt payments decreases by $\theta\lambda P dt$. As a result, condition (17) implies the effect given by (20) is always negative, and thus the debt value must be lowered, which verifies inequality (19). We can similarly understand condition (18) by considering another scenario in which all creditors of bank i decrease their strategy from $\{z_t^b\}$ to $\{z_t^a\}$.

The next lemma provides a sufficient condition that implies each bank's debt value strictly increases in the bank's fundamental.

Lemma 3.3. For any $\{z_t, p_t\}$ and constant $\epsilon > 0$, suppose

$$V(t, u; z, p) \geq \min\{\alpha F(u) - \bar{\gamma} + p_t, P\}, \quad \forall t, u \quad \text{such that} \quad z_t < u < z_t + \epsilon. \tag{21}$$

Then, $V(t, u; z, p)$ is strictly increasing in u.

Proof. See Appendix 6.4. □

Remark 4. If $z_t \equiv -\infty$ or $z_t \equiv \infty$, condition (21) becomes vacuous, and thus $V(t, u; z, p)$ must be strictly increasing in u.

We explain this lemma as follows. Suppose the initial fundamental of individual bank i increases from u_0^i to $u_0^i + \delta$, where $0 < \delta < \epsilon$.[9] Then, any realized path of the fundamental u_t^i will also increase to $u_t^i + \delta$. Therefore, the instantaneous default probability decreases from $\theta\lambda P dt$ to 0, if $u_t^i + \delta$ lies between z_t and $z_t + \delta$. The direct effect of this decreased default probability on the debt value is given by

$$-\theta\lambda P 1_{z_t < u_t^i + \delta < z_t + \delta} \left(\min\{\alpha F(u_t^i + \delta) - \bar{\gamma} + p_t, P\} - V(t, u_t^i + \delta; z, p) \right), \tag{22}$$

meaning that if $u_t^i + \delta$ lies between z_t and $z_t + \delta$, then (i) the chance to get the liquidation proceeds decreases by $\theta\lambda P dt$ and (ii) the chance to keep receiving the debt payments increases by $\theta\lambda P dt$. As a result, if condition (21) holds, the effect given by (22) always becomes positive, which implies the debt value must increase in the bank's fundamental.

[9] Here, we increase the initial fundamental without loss of generality. We can apply the above argument starting from any other date t.

Meanwhile, the above lemma actually says the debt value is strictly increasing in u. To understand this result, note that the increased fundamental not only reduces the default probability but also increases the final payoff $\min\{e^{u_t^i}, P\}$. Moreover, because (i) $\min\{e^u, P\}$ strictly increases in u over a non-trivial interval and (ii) almost every realized path u_t^i travels over that interval for a non-trivial amount of time due to the assumption $\sigma > 0$, the debt value must strictly increase in u as well.

Lastly, we derive the following lemma, which implies a so-called lower dominance region exists if α is small enough. Here, we use $V(u; z \equiv -\infty)$ to denote the debt value in the case of $z_t \equiv -\infty$. We can use this simple notation because this debt value does not depend on the liquidation price $\{p_t\}$. We also use $V(t, u; z, p, \alpha)$ to indicate the dependence of the debt value on the parameter α.

Lemma 3.4. There is some constant $\bar{\alpha} \in (0, 1)$ such that

$$V(u; z \equiv -\infty) \geq \min\{\bar{\alpha} F(u), P\}, \quad \forall u. \tag{23}$$

Further, there is a constant $\underline{z} \in \mathbb{R}$ that satisfies, for any $\alpha \in (0, \bar{\alpha})$ and $\{z_t, p_t\}$,

$$V(t, u; z, p, \alpha) < P, \quad \forall t, u \quad \text{such that} \quad u < \underline{z}. \tag{24}$$

Proof. See Appendix 6.5. □

Remark 5. The paper does not provide an explicit expression for $\bar{\alpha}$, but we can easily check condition (23) numerically whenever we simulate the model.

First, finding $\bar{\alpha}$ that satisfies condition (23) is easy because the left-hand side in that condition does not depend on $\bar{\alpha}$. We can also easily prove the second assertion of the lemma, using condition (23) and the previous lemmas appropriately. Importantly, inequality (24) says each creditor can rationally eliminate any candidate strategy $\{z_t\}$ such that $z_t < \underline{z}$ for some t, because the debt value is less than the face value whenever $u < \underline{z}$, regardless of which rollover strategy is taken as given. In other words, $[0, \infty) \times (-\infty, \underline{z})$ is a lower dominance region.

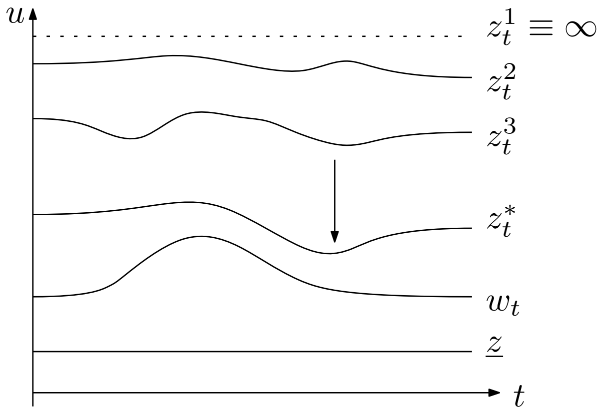

Figure 3

3.3.2 *Existence*

This section shows at least one partial equilibrium exists in this rollover game. The basic idea is as follows. We first consider the largest rollover strategy $z_t^1 \equiv \infty$ and then construct a sequence of strategies $\{z_t^1, z_t^2, \cdots\}$ that satisfies for each n, (i) $z_t^{n+1} \leq z_t^n$ and (ii)

$$V(t, u; z^n, p) \begin{cases} < P & \text{if } u < z_t^{n+1} \\ = P & \text{if } u = z_t^{n+1} \\ > P & \text{if } u > z_t^{n+1}. \end{cases} \tag{25}$$

Here, inequality (25) implies the strategy $\{z_t^{n+1}\}$ is optimal for each creditor of any individual bank i if all other creditors within the same bank use $\{z_t^n\}$ as their strategy. In other words, each creditor is indifferent between withdrawing and rolling over if the bank's current fundamental u_t^i is equal to z_t^{n+1}, given all other creditors use $\{z_t^n\}$ as their strategy. In this regard, we call $\{z_t^{n+1}\}$ the best response to $\{z_t^n\}$. But because Lemma 3.4 implies $z_t^n \geq \underline{z}$ for each (n, t), the decreasing sequence $\{z_t^1, z_t^2, \cdots\}$ must converge to some limit z_t^* for each t. Then, the strategy $\{z_t^*\}$ must be an equilibrium strategy because this limiting strategy becomes a fixed point of the best response. Figure 3 illustrates these arguments and Theorem 3.5 provides a formal proof.[10]

In addition, Theorem 3.5 further shows that if another equilibrium $\{w_t\}$ exists, then

[10]In this section, we do not assume p_t is decreasing in t. Thus, z_t^* does not necessarily increase in t. In equilibrium of the whole market, p_t^* is decreasing in t and z_t^* is increasing in t as expected.

we have $w_t \leq z_t^*$ for each t. In this regard, we can call $\{z_t^*\}$ the upper extremal equilibrium. But, in the next section, we show that $\{z_t^*\}$ constructed above is the only equilibrium.

Theorem 3.5. For any $\{p_t\}$, the rollover game within each bank obtains at least one equilibrium, $\{z_t^*\}$, described above. If another equilibrium $\{w_t\}$ exists, then $w_t \leq z_t^*$ for each t.

Proof. See Appendix 6.6. □

3.3.3 *Uniqueness*

This section shows the uniqueness of partial equilibrium. To this aim, we first discuss the following lemma, which implies if the liquidation price path $\{p_t\}$ is exogenously given, then the rollover game within each bank exhibits weak strategic complementarity.

Lemma 3.6. For any $\epsilon > 0$ and $\{z_t, p_t\}$, we have

$$V(t, u + \epsilon; z + \epsilon, p) > V(t, u; z, p), \quad \forall t, u,$$

where $z + \epsilon$ denotes the rollover strategy $\{z_t + \epsilon\}$.

Proof. See Appendix 6.7. □

The underlying intuition behind this lemma is as follows. Suppose that (i) all creditors of bank i increase their strategy from $\{z_t\}$ to $\{z_t + \epsilon\}$ and (ii) the initial fundamental of the bank also increases from u_0^i to $u_0^i + \epsilon$. Then, the default probability of bank i at any point in time in the future remains unchanged, because the relative position of any realized path $\{u_t^i + \delta\}$ to the threshold curve $\{z_t + \delta\}$ remains the same if both $\{u_t^i\}$ and $\{z_t\}$ increase by the same amount δ. However, the increment in u_t^i additionally increases the final payoff $\min\{e^u, P\}$. Thus, applying the argument behind Lemma 3.3 similarly, we can show that the debt value $V(t, u_t^i; z, p)$ strictly increases if both $\{u_t^i\}$ and $\{z_t\}$ increase by the same amount.

We now show the uniqueness by applying a so-called translation-invariance argument; see, for instance, Frankel and Pauzner (2000) and Frankel, Pauzner, and Morris (2003).

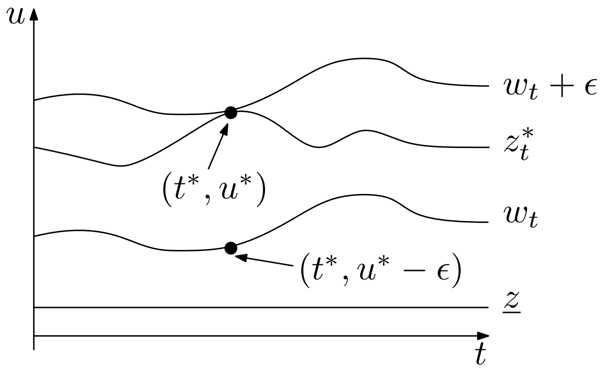

Figure 4

By way of contradiction, suppose there is another equilibrium $\{w_t\}$. Then we have already seen $\underline{z} \leq w_t \leq z_t^*$ for each t in the previous section. We now shift the curve $\{w_t\}$ upward by $\epsilon > 0$, where ϵ is the smallest number such that $w_t + \epsilon \geq z_t^*$ for each t as in Figure 4. Let (t^*, u^*) denote the point where the curve $\{w_t + \epsilon\}$ touches the other curve $\{z_t^*\}$.[11] Then, applying Lemma 3.2 to $z_t^a = z_t^*$ and $z_t^b = w_t + \epsilon$, we have

$$V(t^*, u^*; z^*, p) \geq V(t^*, u^*; w + \epsilon, p).$$

However, Lemma 3.6 shows

$$V(t^*, u^* - \epsilon; w, p) < V(t^*, u^*; w + \epsilon, p).$$

But these two inequalities cannot hold simultaneously because $V(t^*, u^*; z^*, p) = P = V(t^*, u^* - \epsilon; w, p)$ by definition of equilibrium. Hence, $\{z_t^*\}$ must be the only equilibrium.

Theorem 3.7. Given any $\{p_t\}$, the primary market has a unique partial equilibrium.

Proof. See Appendix 6.8. □

3.4 Equilibrium in the Whole Market

This section shows at least one equilibrium exists in the whole market. Whether another equilibrium exists is unknown at the moment. In fact, the above translation-invariance

[11] The proof of Theorem 3.7 deals with a special case in which the curve $\{w_t + \epsilon\}$ does not touch the other curve $\{z_t^*\}$ at a finite point.

argument does not work well if p_t is not exogenously given. In general, as mentioned earlier, an economy exhibiting a strong feedback effect tends to have multiple equilibria. In our model, if creditors choose to run earlier, they not only increase the default probability of their banks but also push down the liquidation price, which generates a feedback effect to the creditors. In this regard, our economy may have multiple equilibria, but, as mentioned above, whether multiple equilibria indeed arise is uncertain at the moment.

To show the existence of equilibrium, recall that for any strategy $\{z_t\}$, we can obtain a unique liquidation price $\{p_t\}$ as a partial equilibrium in the secondary market. We hereafter use $p(t;z)$ to denote the dependence of this liquidation price on the strategy $\{z_t\}$. We also use $V(t,u;z)$ to indicate $V(t,u;z,p(\cdot;z))$ for notational convenience. In this regard, our goal is to find an equilibrium strategy $\{z_t^*\}$ such that $V(t,u;z^*)$ satisfies inequality (8), with p_t^* being replaced by $p(t;z^*)$.

The basic idea of the proof is very similar to the argument in Section 3.3.2. That is, we begin with the largest rollover strategy $z_t^1 \equiv \infty$ and then construct a sequence of strategies $\{z_t^1, z_t^2, \cdots\}$ that satisfies for each n, (i) $z_t^{n+1} \leq z_t^n$ and (ii)

$$V(t,u;z^n) \begin{cases} < P & \text{if } u < z_t^{n+1} \\ = P & \text{if } u = z_t^{n+1} \\ > P & \text{if } u > z_t^{n+1}. \end{cases} \tag{26}$$

Of course, the sequence $\{z_t^1, z_t^2, \cdots\}$ is not the same as the sequence of the strategies obtained in Section 3.3.2, because the liquidation price $p(t;z^n)$ is now endogenously determined. In fact, we have $p(t;z^n) \leq p(t;z^{n+1})$ for each (t,n) because a more aggressive strategy pushes down the liquidation price more. We can then show that the decreasing sequence $\{z_t^1, z_t^2, \cdots\}$ converges to some limit z_t^* for each t and this limit $\{z_t^*\}$ is an equilibrium of the whole market.

In addition, using the fact that $p(t;z^n)$ is decreasing in t for each n, we can show z_t^* is increasing in t. Lastly, as in Section 3.3.2, if there is another equilibrium $\{w_t\}$, we have $w_t \leq z_t^*$ for each t.

Theorem 3.8. Our economy obtains at least one equilibrium, $\{z_t^*, p_t^*\}$, where z_t^* is constructed above and $p_t^* = p(t; z^*)$. Also, z_t^* is increasing in t and p_t^* is decreasing in t. Moreover, if there is another equilibrium w_t, then $w_t \leq z_t^*$ for each t.

Proof. See Appendix 6.9. □

4 Model Results

This section presents the numerical results under reasonable parameter values.

4.1 Parameter Values

The baseline parameter values are summarized in Table 1. We set the risk-free rate as $\rho = 4.0\%$, which corresponds to the annualized one-month Treasury rate in 2007. We choose $\lambda = 10$ because the average maturity of asset-backed commercial papers (ABCP) in the first half of 2007 was 33 days, according to Covitz, Liang, and Suarez (2013). We choose $c = 0.07$ and $P = 1$ because the average mortgage rate between 2000 and 2008 was 6.8% according to the Federal Reserve's statistical release data. We choose $\phi = 0.06$ because this number is consistent with the average duration of 30-year mortgages with a yield of 7% and a coupon rate of 7%. We reasonably choose $\mu = 4.0\%$ because estimating this parameter is hard, as argued by Schroth, Suarez, and Taylor (2014). Both HX (2012) and Schroth, Suarez, and Taylor (2014) choose μ to be close enough to the risk-free rate. We choose $\sigma = 10\%$, which corresponds to the average asset volatility of financial firms in 2008 according to Veronesi and Zingales (2010). We choose $\theta = 1$ because a reasonable estimate for this parameter is around 0.44 according to Schroth, Suarez, and Taylor (2014). We choose $\alpha = 90\%$ because the highest recovery rate for bank loans from 1990 to 2015 was about 85%, according to Moody's Investors Service (2015). We reasonably assume $\bar{\gamma} = 0.9$ and $\eta = 0.8$ because identifying these parameters is difficult. We also reasonably consider that the initial distribution $m_0(u)$ is given by a normal distribution $N(\mu_m, \sigma_m)$, where $\mu_m = -0.2$ and $\sigma_m = 0.5$. We interpret this $m_0(u)$ as a long-run steady-state distribution, although we do not explicitly model the long-run behavior of the economy in this paper.

Parameter	Value	Meaning
ρ	0.04	discount rate
λ	10	1/(average debt maturity)
ϕ	0.06	1/(average asset maturity)
μ	0.04	asset growth rate
σ	0.10	asset volatility
P	1	face value
c	0.07	coupon
θ	1	unreliability of credit support
α	0.9	1-(proportional liquidation cost)
$\bar{\gamma}$	0.9	upper bound of the fixed liquidation cost
η	0.8	magnitude of a skill shock
μ_m	-0.2	mean of the initial distribution
σ_m	0.5	variance of the initial distribution

Table 1: Baseline Parameter Values

Under these basic parameter values, we solve the model numerically by applying the iterative elimination procedure described in Section 3.4. Specifically, we use the finite difference method to solve the HJB equations for the debt-value and levered-asset-value functions; see Barles and Souganidis (1991) and Krylov (2000) as the useful references for this numerical method.

4.2 Economy without the Friction

We first discuss the frictionless economy. In this economy, by definition, the liquidation price p_t^* is given by a constant $\bar{\gamma} = 0.9$, as the dashed line in the left-top panel in Figure 5 shows. The dashed line in the left-bottom panel shows that the rollover strategy $\{z_t^*\}$ is given by a constant -0.93. The rollover strategy remains constant because the liquidation price is fixed constant. Meanwhile, the dashed line in the right panel plots the levered asset value $G(u)$ in the frictionless economy. (In this figure, we use the arithmetic scale represented by $y = \exp(u)$ to illustrate the risk-neutral valuation of the assets.) The levered asset value is lower than the unlevered asset value $F(u)$ (i.e., the dash-dot line in the same panel) because of the inefficient liquidation costs. But as the fundamental $y = \exp(u)$ increases, the levered asset value approaches the unlevered asset value because the chance of default goes to zero.

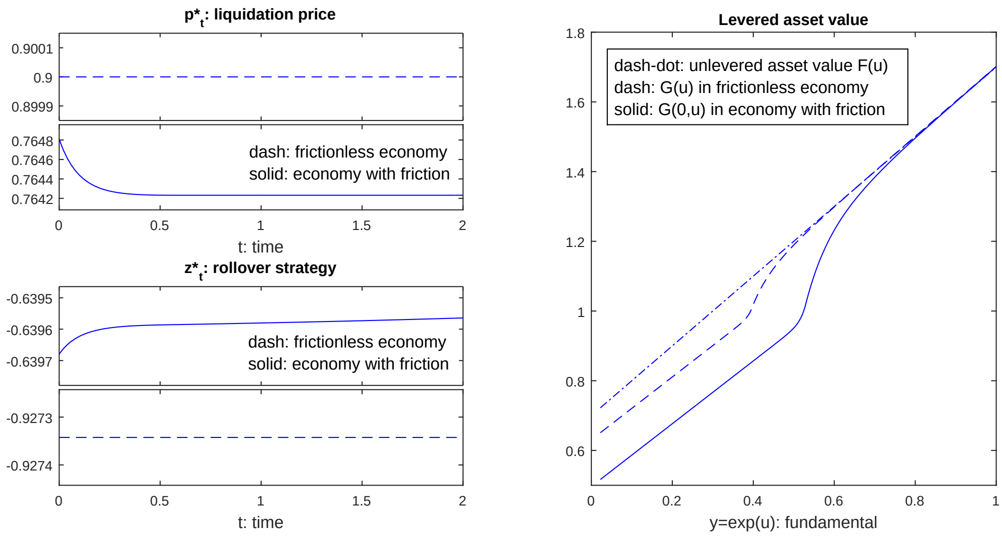

Figure 5: This figure plots the impact of the shock to the management skills. The left-top panel plots the liquidation price $\{p_t^*\}$; the left-bottom panel plots the rollover strategy $\{z_t^*\}$; the right panel plots the levered asset value at time 0.

4.3 Economy with the Friction

We now discuss the model results in the economy with the friction. We can actually view this friction as an outcome of a shock that unexpectedly hits the management skills of the outside investors in the frictionless economy. In this regard, we can understand the parameter η as the magnitude of this so-called skill shock. Without loss of generality, we assume this shock occurs at date 0. Under this interpretation, we now examine the impulse response of the economy to this shock.

The solid line in the left-top panel in Figure 5 plots the liquidation price $\{p_t^*\}$. The liquidation price drops from 0.9 to 0.76 immediately at date 0 and then declines slowly afterwards. Here, even though only a dt-order fraction of the banks default at date 0, the downward jump in the liquidation price occurs immediately, because the anticipated price impact on the future liquidation price is reflected in the liquidation price at date 0.

In addition, for the same reason, the slope of p_t^* must be very small. That is, if p_t^* declines so fast, an intertemporal arbitrage opportunity would arise, so that p_t^* cannot be sustained as an equilibrium. This result may seem inconsistent with empirical data because the liquidation price shows sizable fluctuations over time according to Moody's Report (2015). But, in our model, it is hard to generate such a large variation in the liquidation price after date 0, because no further aggregate shocks hit the economy in the subsequent periods. The author believes introducing time-varying aggregate shocks would be an interesting topic for future research.

The solid line in the left-bottom panel in Figure 5 plots the rollover strategy $\{z_t^*\}$. As expected, the rollover strategy moves in the opposite direction of the liquidation price. That is, z_t^* jumps up immediately from -0.93 to -0.64 at date 0 and then rises slowly afterwards. Again, although the initial size of the upward jump in the rollover strategy is big, the slope of z_t^* is quite small.

We now measure the impact of the skill shock on the levered asset values of individual banks. The solid line in the right panel in Figure 5 plots the levered asset value at date 0, $G(0, u)$. Recall that the dashed line in the same panel plots the levered asset value $G(u)$ in the frictionless economy. For convenience, we denote $G(0, u)$ by $G^f(0, u)$ and $G(u)$ by $G^{fl}(u)$. Then, the total levered asset values in the economies with and without the friction are respectively given by

$$\int_{-\infty}^{\infty} G^f(0, u)m_0(u)du = 1.54 \quad \text{and} \quad \int_{-\infty}^{\infty} G^{fl}(u)m_0(u)du = 1.59,$$

meaning the percentage loss in the total levered asset value is equal to 3.14%.

4.4 Amplification and Spillover Effects

In this section, we conduct another experiment to better understand the amplification and spillover effects in the economy with the friction. Specifically, imagine that a negative shock unanticipatedly hits the asset fundamentals of only a fraction ζ of the banks at date 0, where $0 < \zeta < 1$, so that the fundamentals (represented by $y = \exp(u)$) of those distressed banks reduce by $100 \times \omega$ percent. Accordingly, the initial distribution $m_0(u)$ is distorted to

$$m_0^d(u) = \zeta m_0 \left(u - \log(1 - \omega)\right) + (1 - \zeta)m_0(u).$$

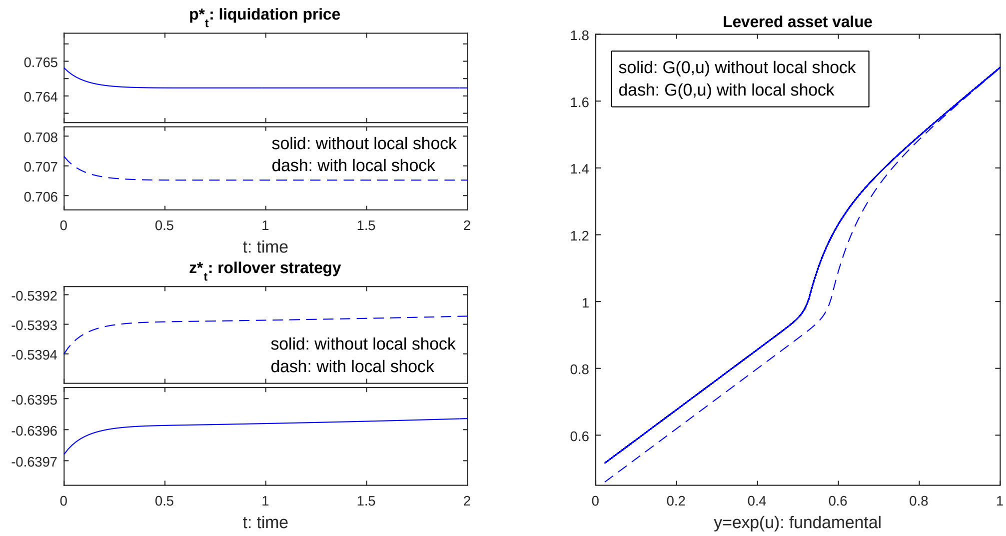

Figure 6: This figure plots the impact of the local shock in the economy with the friction. The left-top panel plots the liquidation price $\{p_t^*\}$; the left-bottom panel plots the rollover strategy $\{z_t^*\}$; the right panel plots the levered asset value at time 0.

We then appropriately choose $\zeta = 0.3$ and $\omega = 0.2$ to examine the impulse response of the economy to this so-called local shock. In the sequel, we consider the economy with the friction for this experiment unless otherwise specified.

The dashed line in the left-top panel in Figure 6 plots the liquidation price $\{p_t^*\}$ in this distorted economy. The solid line in the same panel is taken from the other solid line in the left-top panel in Figure 5 (i.e., the liquidation price path in the economy with the friction but in the absence of the local shock). As expected, the local shock immediately pushes down the liquidation price from 0.76 to 0.71 at date 0. Similarly, the rollover strategy $\{z_t^*\}$ is pushed up from -0.64 to -0.54, as the left-bottom panel in Figure 6 shows. We can interpret these results as the spillover effect of the local shock, because all creditors in the economy choose to run more aggressively even though the shock hits only a certain fraction of the banks.

We now calculate the impact of this local shock on the levered asset values. Let $G^d(t, u)$ denote the levered asset value in the distorted economy. The dashed line in the right panel in Figure 6 plots this levered asset value at date 0, whereas the solid line in the same panel is taken from the other solid line in the right panel in Figure 5 that represents

$G^f(0,u)$. Note that $G^d(0,u)$ is smaller than $G^f(0,u)$ for each u, meaning that the levered asset value of a bank which was not hit by the shock also drops immediately at date 0 because of the spillover effect. The total value of the levered assets in this distorted economy is then given by

$$\int_{-\infty}^{\infty} G^d(0,u) m_0^d(u) du = 1.44.$$

Recall that, in the absence of the local shock, the total levered asset value was given by 1.54. Thus, the percentage loss in the total levered asset value is equal to $1 - \frac{1.44}{1.54} = 6.49\%$.

Meanwhile, in the frictionless economy, the local shock reduces the total levered asset value by

$$1 - \frac{\int_{-\infty}^{\infty} G^{fl}(u) m_0^d(u) du}{\int_{-\infty}^{\infty} G^{fl}(u) m_0(u) du} = 3.22\%,$$

where we have used $G^{fl}(u)$ for both the numerator and denominator because any distortions in the asset-fundamental distribution do not affect the levered asset value in the frictionless economy. The above results imply that the friction in the secondary market amplifies the percentage loss caused by the local shock by 3.27%.

4.5 Comparative Statics

This section discusses policy implications of the model through the comparative statics analysis.

4.5.1 *Effect of Credit Support*

In this section, we numerically show that providing more credit support (i.e., decreasing θ) alleviates the withdrawal incentives. In fact, HX (2012) argue that doing so may exacerbate a crisis, especially when σ is large. The underlying intuition behind their argument is as follows.

Consider the frictionless economy for a moment. In this economy, the liquidation proceeds to each creditor of a failed bank, $\min\{\alpha F(u), P\}$, does not depend on the volatility σ, because the failed asset is acquired by a risk-neutral outside investor who uses equity financing only. Meanwhile, a higher volatility generally decreases the debt value because of a limited liability problem. As a result, if σ is large, the continuation value of debt $V(t,u)$ could be less than the immediate liquidation proceeds $\min\{\alpha F(u), P\}$. Thus, in this case, creditors may want earlier default of their bank to acquire their collateral as soon as possible. For this reason, interestingly, providing more credit guarantees may harm the creditors by extending the bank's survival time and so exacerbating the limited liability problem.

However, such an unintended outcome is not likely to occur in the economy with the friction, because providing more credit support decreases the default probability and thus pushes up the liquidation price. According to the numerical results, this positive effect dominates the above negative effect even when σ is high. Specifically, Figure 7 shows how the change in θ affects the liquidation price and rollover strategy at date 0, that is, $p_0^*(\theta)$ and $z_0^*(\theta)$. Here, we do not plot the entire transition paths of $p_t^*(\theta)$ and $z_t^*(\theta)$ to avoid displaying complicated graphs; those entire paths exhibit similar qualitative patterns.

Figure 7 Here.

The solid line in the right-bottom panel in Figure 7 shows that $z_0^*(\theta)$ increases in θ in the frictionless economy if the volatility is low. The dashed line in the same panel shows $z_0^*(\theta)$ decreases in θ if the volatility is high, as argued by HX (2012). In the economy with the friction, however, $z_0^*(\theta)$ increases in θ, regardless of the magnitude of the prevailing volatility, as the two graphs in the right-top panel show. The reason is that, as mentioned above, increasing θ pushes down the liquidation price $p_0^*(\theta)$. Indeed, the left-top panel shows that $p_0^*(\theta)$ is decreasing in θ, whereas the change in θ does not affect the liquidation price in the frictionless economy, as shown in the left-bottom panel.

4.5.2 *Effect of Debt Maturity*

In this section, we numerically show that extending debt maturities (i.e., decreasing λ) tends to attenuate the incentives to run, again, in contrast to HX (2012). The underlying intuition for this result is similar to that discussed in the previous section. First, extending debt maturities may hurt creditors by locking in those creditors with their debt contracts for a longer period. Moreover, in the frictionless economy, similarly as above, lengthening debt maturities can hurt creditors by deterring their bank's default and so worsening the limited liability problem. However, in the economy with the friction, such a policy can improve the liquidation price by decreasing the default probability. The numerical results of the model again show this positive effect outweighs the negative effects.

Figure 8 Here.

The solid line in the right-bottom panel in Figure 8 shows that, in the frictionless economy, $z_0^*(\lambda)$ increases in λ when σ is low. The dashed line in the same panel shows $z_0^*(\lambda)$ decreases in λ when σ is high. In this economy, the change in λ does not affect the liquidation price, as shown in the left-bottom panel. However, in the economy with the friction, the right-top panel shows $z_0^*(\lambda)$ increases in λ and the left-top panel shows $p_0^*(\lambda)$ decreases in λ, regardless of the magnitude of the volatility.

4.5.3 *Effects of η and $\bar{\gamma}$*

In this section, we analytically show that the increment in either η or $\bar{\gamma}$ increases the withdrawal incentives and so aggravates a crisis. Here, recall η measures the magnitude of a skill shock hitting the outside investors, and $\bar{\gamma}$ describes the upper bound of the fixed liquidation cost. To see the reason for this result, note that for any path of the cumulative supply of failed assets $\{q_t\}$, formula (15) implies

$$-\bar{\gamma} + p_t = -\bar{\gamma}\left(1 - \rho \int_t^\infty e^{-\rho(s-t)} \frac{1}{(1+q_s)^\eta} ds\right),$$

which decreases in η and $\bar{\gamma}$. Using this property, Proposition 4.1 uses the iterative elimination procedure described in Section 3.3.2 to show the rollover strategy z_t^* is increasing in η and $\bar{\gamma}$.[12]

Specifically, the two panels on the top in Figure 9 show that the liquidation price $p_0^*(\eta)$ is decreasing in η and the rollover strategy $z_0^*(\eta)$ is increasing in η. Similarly, the

[12]Meanwhile, as in Remark 1, if we interpret γ^k as the investor-specific valuation on the existing assets in the economy, z_t^* decreases in $\bar{\gamma}$ because outside investors with higher $\bar{\gamma}$ value the assets more. Here, we still use the function $\gamma(q) = \bar{\gamma}(1+q)^{-\eta}$.

other two panels show that the liquidation price $-\bar{\gamma}+p_0^*(\bar{\gamma})$ is decreasing in $\bar{\gamma}$ and the rollover strategy $z_0^*(\bar{\gamma})$ is increasing in $\bar{\gamma}$. Here, we alternatively interpret $-\bar{\gamma}+p_t^*(\bar{\gamma})$ as the liquidation price because the parameter $\bar{\gamma}$ changes under this experiment.

Figure 9 Here.

Proposition 4.1. If $\eta^1 \leq \eta^2$ and $\bar{\gamma}^1 \leq \bar{\gamma}^2$, then $z_t^*(\eta^1, \bar{\gamma}^1) \leq z_t^*(\eta^2, \bar{\gamma}^2)$ for each t.

Proof. See Appendix 6.10. □

5 Conclusion

This paper models a short-term debt market with heterogeneous banks to study the macroeconomic effects of interactions between creditors and outside investors of failed assets. The outside investors in the secondary market have different asset-management skills and have limited capacity in absorbing assets. The supply and demand for failed assets are respectively determined by the rollover decisions of creditors and the asset-takeover timing decisions of outside investors. The liquidation price of assets adjusts to clear the secondary market. This endogenous liquidation price generates a dynamic feedback effect through which the rollover risks of distressed banks are amplified and propagated to other banks. The model implies providing more credit support or extending debt maturities tends to stabilize a crisis by increasing the liquidation price.

For future research, incorporating time-varying aggregate shocks into this model would be interesting. Studying a long-run behavior of the economy by introducing new entrants that replace exiting banks would be important. Analyzing the effects of the presence of a big bank, which causes a sizable price impact on the liquidation price, will provide us meaningful implications. Lastly, studying how the liquidation price affects an individual bank's capital structure or maturity structure would be an interesting topic.

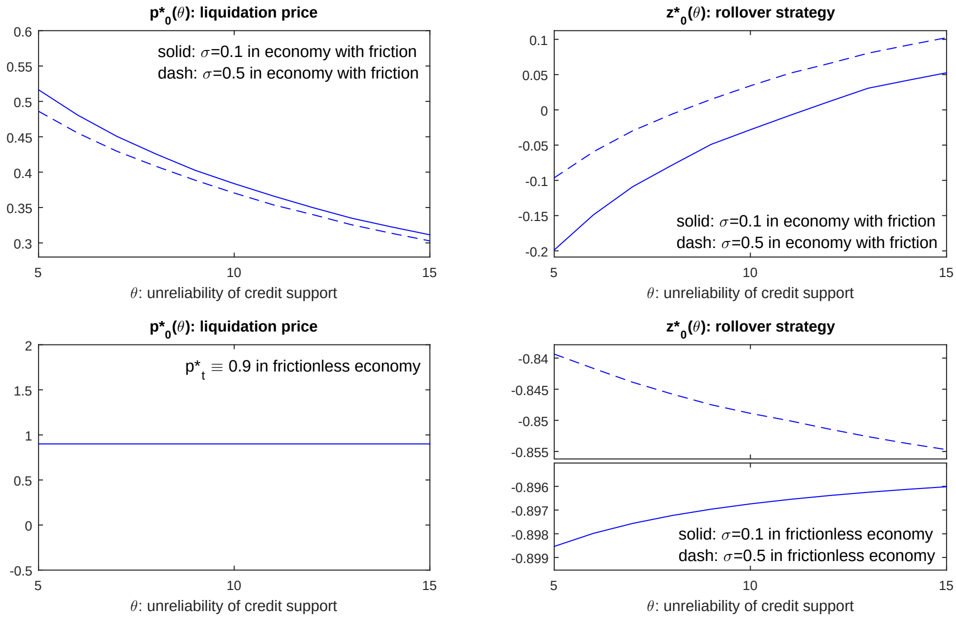

Figure 7: This figure shows the effect of a change in θ on $p_0^*(\theta)$ and $z_0^*(\theta)$ when the asset volatility σ is either 0.1 or 0.5. The two panels on the top (bottom) represent the economy with (without) the friction. We take all other parameter values from Table 1.

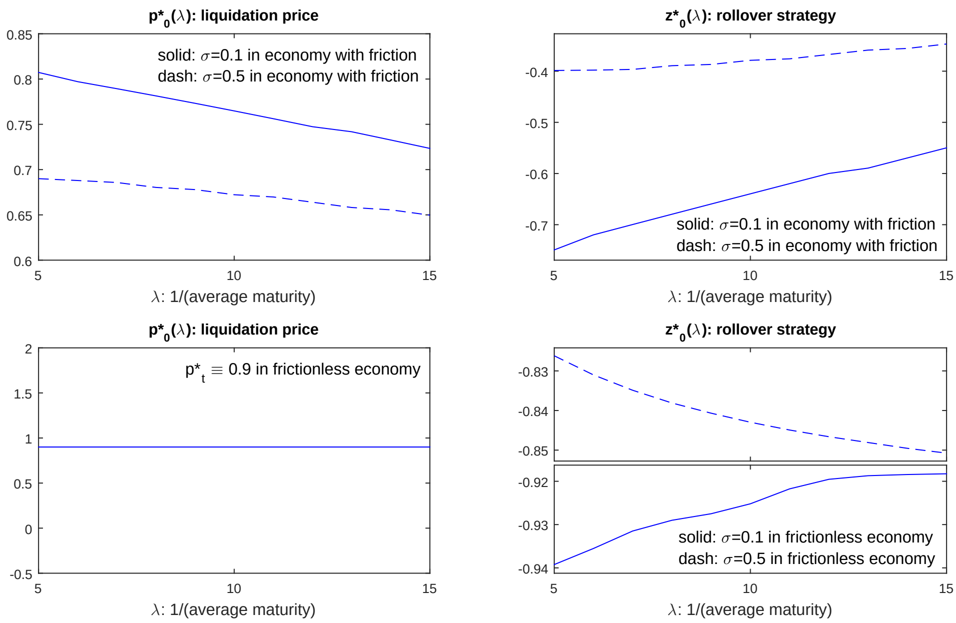

Figure 8: This figure shows the effect of a change in λ on $p_0^*(\lambda)$ and $z_0^*(\lambda)$ when the asset volatility σ is either 0.1 or 0.5. The two panels on the top (bottom) represent the economy with (without) the friction. We take all other parameter values from Table 1.

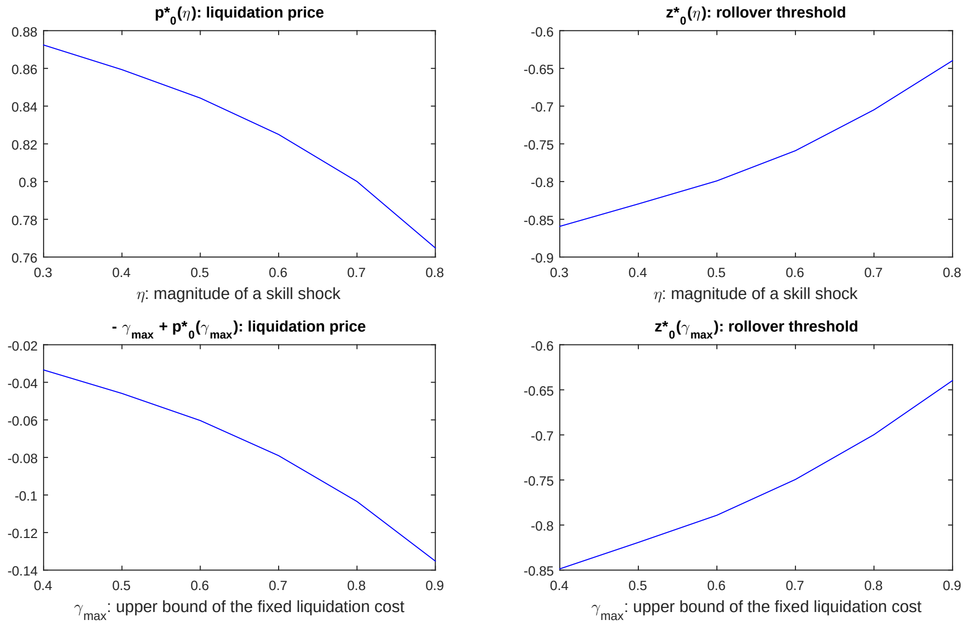

Figure 9: The two panels on the top show the effect of a change in η on $p_0^*(\eta)$ and $z_0^*(\eta)$. The other two panels show the effect of a change in $\bar{\gamma}$ on $-\bar{\gamma}+p_0^*(\bar{\gamma})$ and $z_0^*(\bar{\gamma})$. Here, $\gamma_{\max}$ means $\bar{\gamma}$. We take all other parameter values from Table 1.

6 Appendix

Throughout the appendix, we use $L(t,u;p) = \min\{\alpha F(u) - \bar{\gamma} + p_t, P\}$ for convenience.

6.1 Existence and Uniqueness of a Debt Value Function

We consider the following HJB equation of a general form:

$$0 = f(t,u) + \lambda \max\{P - J(t,u), 0\} - d(t,u)J(t,u) + \nu J_u(t,u) + \frac{\sigma^2}{2} J_{uu}(t,u) + J_t(t,u), \quad (27)$$

for some measurable functions $f(t,u)$ and $d(t,u)$ such that

$$\nu + \frac{\sigma^2}{2} < c, \quad |f(t,u)| \leq Ae^u + B, \quad \forall t,u, \quad \text{and} \quad c < d(t,u), \quad \forall t,u,$$

where $A > 0$, $B > 0$, and $c > 0$ are some constants. We then look for a solution to this equation satisfying the following growth condition:

$$|u(t,u)| \leq De^u + E, \quad (28)$$

for some constants $D > 0$ and $E > 0$. Note that $f(t,u)$ and $d(t,u)$ need not be continuous functions; thus, both HJB equations (2) and (3) can be considered examples of the above HJB equation.

In the math literature, Crandall, Kocan, and Święch (2000, hereafter, CKŚ) show the existence and uniqueness of a solution to an HJB equation of which the non-homogeneous term and the coefficients except for the second-order term are allowed to be discontinuous. (Wang [1992] also shows a similar result under a less general setup.) But, because CKŚ (2000) consider a problem with a bounded domain, we need to extend their result by using a so-called penalization method; see Theorem 6 in Section 2.3 in Evans (2010).

Throughout the paper, we use the concept of a viscosity solution for HJB equations; see Section 1 in CKŚ (2000) for the formal definition. First, the existence of a solution to HJB equation (27) directly comes from the Feynman-Kac formula, which obviously satisfies the growth condition (28). To show the uniqueness, we use the following lemma, a so-called maximum principle.

Lemma 6.1. Suppose $J(t,u)$ is a viscosity subsolution to the following HJB equation:

$$\mathcal{L}J(t,u) := \lambda \max\{-J(t,u),0\} - d(t,u)J(t,u) + \nu J_u(t,u) + \frac{\sigma^2}{2}J_{uu}(t,u) + J_t(t,u) = 0. \quad (29)$$

Or, we can formally say $\mathcal{L}J(t,u) \geq 0$. Also, suppose $J(t,u)$ satisfies the growth condition (28). Then, $J(t,u) \leq 0$ for each (t,u).

Proof. For any $\epsilon > 0$, define

$$K(t,u) = J(t,u) - \epsilon(e^{a_1 u} + e^{a_2 u} + e^{ct}),$$

where

$$a_1 = \frac{-\nu + \sqrt{\nu^2 + 2c\sigma^2}}{\sigma^2} \quad \text{and} \quad a_2 \frac{-\nu - \sqrt{\nu^2 + 2c\sigma^2}}{\sigma^2}.$$

Note that a_1 and a_2 are the solutions to

$$\frac{\sigma^2}{2}a^2 + \nu a - c = 0.$$

Here, it is easy to check $a_1 > 1$ and $a_2 < 0$, using condition (9). Then, it is also easy to see

$$\mathcal{L}K(t,u) \geq \mathcal{L}J(t,u) + (d(t,u) - c)\epsilon(e^{a_1 u} + e^{a_2 u} + e^{ct}) \geq 0, \quad \forall t, u. \quad (30)$$

We now consider a bounded domain $Q(R) = \{(t,u) : 0 \leq t \leq R^2,\ |u| \leq R\}$ for some $R > 0$. The parabolic boundary of $Q(R)$ is defined as $\partial Q(R) = \{(t,u) \in Q(R) : |u| = R \text{ or } t = R^2\}$. Then, by using condition (28) and the fact that $a_1 > 1$ and $a_2 < 0$, we can show there is some $R^* > 0$, (which decreases in ϵ), such that for each $R > R^*$,

$$K(t,u) \leq 0, \quad \forall (t,u) \in \partial Q(R). \quad (31)$$

Then, Proposition 2.6 in CKŚ (2000), combined with inequalities (30) and (31), implies for each $R > R^*$,

$$K(t,u) \leq 0, \quad \forall (t,u) \in Q(R).$$

Letting $\epsilon \to 0$, we conclude $J(t,u) \leq 0$, $\forall (t,u) \in (0,\infty) \times \mathbb{R}$. $\square$

We now use this lemma to show the uniqueness of a solution to HJB equation (27).

Theorem 6.2. HJB equation (27) has a unique viscosity solution satisfying condition (28).

Proof. Suppose $V(t,u)$ and $W(t,u)$ are two solutions to HJB equation (27), both of which satisfy the growth condition. Then, using the fact that $\mathcal{L}V(t,u) = \mathcal{L}W(t,u)$ and

$$\max\{P - V(t,u), 0\} - \max\{P - W(t,u), 0\} \leq \max\{W(t,u) - V(t,u), 0\},$$

we have $\mathcal{L}(V - W)(t,u) \geq 0$. Then, because $V - W$ also satisfies the growth condition, Lemma 6.1 implies $V(t,u) - W(t,u) \leq 0$ for each (t,u). Switching the roles of V and W, we conclude $V(t,u) \geq W(t,u)$ for each (t,u) as well. $\square$

6.2 Proof of Lemma 3.1

Proof. We first consider the case in which $-\infty < z_t < \infty$ for each t. We then suppose u is sufficiently small, that is, $u \approx -\infty$. Then, the fact that

$$F(-\infty) = \frac{c}{\rho + \phi} < P$$

implies the debt value $V(t, -\infty)$ satisfies the following first-order HJB equation:

$$0 = c + \theta\lambda PL(t, -\infty; p) + \lambda \max\{P - V, 0\} - (\rho + \phi + \theta\lambda P)V + V_t. \tag{32}$$

However, observe

$$J(t) := \int_t^\infty e^{-(\rho+\phi+\theta\lambda P+\lambda)(s-t)} \left(c + \theta\lambda PL(s, -\infty; p) + \lambda P\right) ds \leq \frac{c + \theta\lambda P\frac{\alpha c}{\rho+\phi} + \lambda P}{\rho + \phi + \theta\lambda P + \lambda} < P,$$

where we have used $c/P < \rho + \phi$. But then, we can easily see $V(t, -\infty; z, p) = J(t)$ solves HJB equation (32), which verifies $V(t, -\infty) < P$.

Now we suppose u is sufficiently large, that is, $u \approx \infty$. In this case, the debt value $V(t, \infty)$ satisfies

$$0 = c + \phi P + \lambda \max\{P - V, 0\} - (\rho + \phi)V + V_t. \tag{33}$$

We can then show $V(t, \infty) > P$ by observing the solution to this equation is given by

$$V(t, \infty) \equiv \frac{c + \phi P}{\rho + \phi} > P, \tag{34}$$

where we have used the fact that $c/P > \rho$.

We now consider an extreme rollover strategy given by $z_t \equiv -\infty$. First, when $u \approx -\infty$, by removing the term $\theta \lambda P(L(t, \infty; p) - V)$ from equation (32), we can see

$$V(t, -\infty) \equiv \frac{c + \lambda P}{\rho + \phi + \lambda} < P.$$

Next, when $u \approx \infty$, the debt value $V(t, \infty)$ satisfies the equation that is the same as (33), and thus $V(t, \infty) > P$ as we have already seen.

Lastly, we consider the other extreme rollover strategy given by $z_t \equiv \infty$. When $u \approx -\infty$, the debt value $V(t, -\infty)$ satisfies the equation that is the same as (32), and thus $V(t, -\infty) < P$ as we have already shown. When $u \approx \infty$, we note that $\min\{\alpha F(\infty) - \bar{\gamma} + p_t, P\} = P$. Thus, the debt value $V(t, \infty)$ satisfies the following HJB equation:

$$0 = c + \phi P + \theta \lambda P^2 + \lambda \max\{P - V, 0\} - (\rho + \phi + \theta \lambda)V + V_t.$$

We can then show $V(t, \infty) > P$ by observing the solution to this equation is given by

$$V(t, \infty) \equiv \frac{c + \phi P + \theta \lambda P^2}{\rho + \phi + \theta \lambda P} > P,$$

where we have used the fact that $c/P > \rho$. $\square$

6.3 Proof of Lemma 3.2

Proof. We first consider the case in which condition (17) holds. By subtracting HJB equation (2) corresponding to $\{z_t^a, p_t^a\}$ from the same HJB equation but corresponding to $\{z_t^b, p_t^b\}$, we have

$$0 = \theta\lambda P 1_{u<z_t^a}(L(t,u;p^b) - L(t,u;p^a)) + \theta\lambda P 1_{z_t^a \le u < z_t^b}(L(t,u;p^b) - V^b) +$$
$$\lambda \max\{P - V^b, 0\} - \lambda \max\{P - V^a, 0\} - (\rho + \phi + \theta\lambda P 1_{u<z_t^a})G + \nu G_u + \frac{\sigma^a}{2} G_{uu} + G_t,$$

where $V^b = V(t,u;z^b,p^b)$, $V^a = V(t,u;z^a,p^a)$, and $G(t,u) = V^b(t,u) - V^a(t,u)$. Here, the term

$$\theta\lambda P 1_{u<z_t^a}(L(t,u;p^b) - L(t,u;p^a)) + \theta\lambda P 1_{z_t^a \le u < z_t^b}(L(t,u;p^b) - V^b)$$

is the direct effect and the term $-\theta\lambda P 1_{u<z_t^a} G(t,u)$ is the indirect effect of the changes in the rollover strategy and liquidation price path. But the fact that $L(t,u;p^b) \le L(t,u;p^a)$ and condition (17) imply the above direct effect is positive for each (t,u). Moreover, from the fact

$$\max\{P - V^b, 0\} - \max\{P - V^a, 0\} \le \max\{-V^b + V^a, 0\},$$

we have

$$0 \le \lambda \max\{-G, 0\} - (\rho + \phi + \theta\lambda P 1_{u<z_t^a})G + \nu G_u + \frac{\sigma^a}{2} G_{uu} + G_t.$$

Thus, from Lemma 6.1, we conclude $G(t,u) \le 0$ for each (t,u).

Next, we assume condition (18) holds. Then, similarly as above, we have

$$0 = \theta\lambda P 1_{u<z_t^b}(L(t,u;p^b) - L(t,u;p^a)) + \theta\lambda P 1_{z_t^a \le u < z_t^b}(L(t,u;p^a) - V^a) +$$
$$\lambda \max\{P - V^b, 0\} - \lambda \max\{P - V^a, 0\} - (\rho + \phi + \theta\lambda P 1_{u<z_t^b})G + \nu G_u + \frac{\sigma^a}{2} G_{uu},$$

which again leads to $G(t,u) \le 0$ for each (t,u). $\square$

6.4 Proof of Lemma 3.3

Proof. To prove this lemma, we will show that for any δ such that $0 < \delta < \epsilon$,

$$V(t, u; z, p) < V(t, u + \delta; z, p), \quad \forall t, u.$$

Note that $V^\delta(t, u) := V(t, u + \delta; z, p)$ satisfies

$$0 = c + \phi \min\{e^{u+\delta}, P\} + \theta\lambda P 1_{u+\delta<z_t}(L(t, u + \delta; p) - V^\delta) + \lambda \max\{P - V^\delta, 0\}- \\ (\rho + \phi)V^\delta + \nu V_u^\delta + \frac{\sigma^2}{2} V_{uu}^\delta + V_t^\delta.$$

Subtracting this equation from HJB equation (2) corresponding to $\{z_t, p_t\}$, we have

$$0 = \phi \left(\min\{e^u, P\} - \min\{e^{u+\delta}, P\}\right) + \theta\lambda P 1_{u<z_t} \left(L(t, u; p) - L(t, u + \delta; p)\right) + \\ \theta\lambda P 1_{z_t<u+\delta<z_t+\delta} \left(L(t, u + \delta; p) - V^\delta\right) + \lambda \max\{P - V, 0\} - \lambda \max\{P - V^\delta, 0\}- \\ (\rho + \phi + \theta\lambda P 1_{u<z_t})G + \nu G_u + \frac{\sigma^2}{2} G_{uu} + G_t,$$

where $G(t, u) = V(t, u; z, p) - V(t, u + \delta; z, p)$. Here, the term

$$\phi \left(\min\{e^u, P\} - \min\{e^{u+\delta}, P\}\right) + \theta\lambda P 1_{u<z_t} \left(L(t, u; p) - L(t, u + \delta; p)\right) + \\ \theta\lambda P 1_{z_t\le u+\delta<z_t+\delta} \left(L(t, u + \delta; p) - V^\delta\right)$$

is the direct effect and the term $-\theta\lambda P 1_{u<z_t} G(t, u)$ is the indirect effect of the increment in the fundamental. But condition (21) implies this direct effect is positive for each (t, u). Thus, together with the fact

$$\max\{P - V, 0\} - \max\{P - V^\delta, 0\} \le \max\{V^\delta - V, 0\},$$

we obtain

$$0 \le f(u) + \lambda \max\{-G, 0\} - (\rho + \phi + \theta\lambda P 1_{u<z_t})G + \nu G_u + \frac{\sigma^2}{2} G_{uu} + G_t, \tag{35}$$

where $f(u) = \phi(\min\{e^u, P\} - \min\{e^{u+\delta}, P\})$. Lemma 6.1 then shows $G(t,u) \leq 0$ for each (t,u). But, recall our goal is to show $G(t,u) < 0$ for each (t,u). We will show this strict inequality by using the fact that $f(u)$ is strictly negative over a non-trivial interval.

Indeed, there are some constant $\eta > 0$ and interval $(a,b) \subseteq \mathbb{R}$ such that

$$f(u) \leq -\eta, \quad \forall u \in (a,b).$$

We then consider a function $W(t,u)$ that satisfies

$$0 = -\eta 1_{a<u<b} + \lambda \max\{-W, 0\} - (\rho + \phi + \theta\lambda P 1_{u<z_t})W + \nu W_u + \frac{\sigma^2}{2} W_{uu} + W_t. \tag{36}$$

The solution to this equation is explicitly given by

$$W(t,u) = -E_t \left[\int_t^\infty e^{-\int_t^s (\rho+\phi+\lambda+\theta\lambda P 1_{u_r<z_r})dr} \eta 1_{u_s \in (a,b)} ds \right],$$

which is strictly less than 0 because almost every realized path u_t travels over the interval (a,b) for a non-trivial amount of time due to the assumption $\sigma > 0$. Then, subtracting (36) from (35), we can see that $H(t,u) = G(t,u) - W(t,u)$ satisfies

$$0 \leq \lambda \max\{-H, 0\} - (\rho + \phi + \theta\lambda P 1_{u<z_t})H(t,u) + \nu H_u + \frac{\sigma^2}{2} H_{uu} + H_t.$$

Lemma 6.1 then implies $G(t,u) \leq W(t,u) < 0$ for each (t,u). □

6.5 Proof of Lemma 3.4

Proof. Lemma 3.3 shows that the debt value $V(u; z \equiv -\infty)$ is strictly increasing in u. Lemma 3.1 then implies there is a point $\underline{z}$ such that

$$V(u; z \equiv -\infty) \begin{cases} < P & \text{if } u < \underline{z} \\ = P & \text{if } u = \underline{z} \\ > P & \text{if } u > \underline{z}. \end{cases} \tag{37}$$

We can then find $\bar{\alpha} \in (0,1)$ such that

$$\bar{\alpha} F(\underline{z}) \leq V(-\infty; z \equiv -\infty) = \frac{c + \lambda P}{\rho + \phi + \lambda} < P.$$

But, because $F(u)$ is also increasing in u, we obtain inequality (23).

Moreover, once condition (23) holds, by applying Lemma 3.2 to $z_t^a \equiv -\infty$ and $z_t^b = z_t$, we can say that for each $\alpha < \bar{\alpha}$,

$$V(u; z \equiv -\infty) \geq V(t, u; z, p, \alpha), \quad \forall t, u.$$

Then, from inequality (37), we have $V(t, u; z, p, \alpha) < P$ for each (t, u) such that $u < \underline{z}$. $\square$

6.6 Proof of Theorem 3.5

Proof. We start with the strategy $z_t^1 \equiv \infty$. Applying Lemmas 3.1 and 3.3 to $\{z_t^1\}$, we can obtain $\{z_t^2\}$ such that (i) $z_t^2 \leq z_t^1$ for each t and (ii)

$$V(t, u; z^1, p) \begin{cases} < P & \text{if } u < z_t^2 \\ = P & \text{if } u = z_t^2 \\ > P & \text{if } u > z_t^2. \end{cases} \tag{38}$$

These two properties imply condition (17) in Lemma 3.2 holds with z_t^a and z_t^b being replaced by z_t^2 and z_t^1, respectively. Thus, we have $V(t, u; z^2, p) \geq V(t, u; z^1, p)$ for each (t, u). This fact then, combined with inequality (38), implies $V(t, u; z^2, p) \geq P$ for each (t, u) such that $z_t^2 \leq u$. Hence, by applying Lemma 3.3 to $\epsilon = 1$ and $z_t = z_t^2$, we can say $V(t, u; z^2, p)$ is strictly increasing in u.

Next, applying the above arguments to $\{z_t^2\}$ in place of $\{z_t^1\}$, we can find $\{z_t^3\}$ such that (i) $z_t^3 \leq z_t^2$ for each t and (ii) inequality (25) holds with $n = 2$. Then, similarly as above, we can show that condition (17) in Lemma 3.2 holds with $\{z_t^3\}$ and $\{z_t^2\}$ in place of $\{z_t^a\}$ and $\{z_t^b\}$, respectively, which implies $V(t, u; z^3, p) \geq V(t, u; z^2, p)$ for each (t, u). This fact together with inequality (25) with $n = 2$ implies $V(t, u; z^3) \geq P$ for each (t, u) such that $z_t^3 \leq u$. Lemma 3.3 then shows $V(t, u; z^3, p)$ is strictly increasing in u.

Repeatedly applying these arguments to $\{z_t^3\}$ and so on, we can construct a sequence of strategies $\{z_t^1, z_t^2, \cdots\}$ such that (i) $z_t^{n+1} \leq z_t^n$ for each (n,t) and (ii) inequality (25) holds for each n. But the decreasing sequence $\{z_t^1, z_t^2, \cdots\}$ must converge to some limit z_t^* for each t because Lemma 3.4 assures $\underline{z} \leq z_t^n$ for each (n,t). This result implies the limiting strategy $\{z_t^*\}$ is a fixed point of the best response, which shows that $\{z_t^*\}$ is an equilibrium. But, in fact, to rigorously prove $\{z_t^*\}$ is a fixed point of the best response, we need to show

$$\lim_{n\to\infty} V(t,u;z^n,p) = V(t,u;z^*,p), \quad \forall t,u. \tag{39}$$

We postpone to show this technical statement for a moment.

We now suppose there is another equilibrium $\{w_t\}$. Then, applying Lemma 3.2 to $z_t^a = w_t$ and $z_t^b = z_t^1$, we have $w_t \leq z_t^2$ for each t. We then apply the same lemma to $z_t^a = w_t$ and $z_t^b = z_t^2$ to show $w_t \leq z_t^3$ for each t. Repeating this procedure, we can show $w_t \leq z_t^n$ for each (n,t); therefore, $w_t \leq z_t^*$ for each t.

Lastly, we show identity (39). We let $W(t,u) = \lim_{n\to\infty} V(t,u;z^n,p)$ and suppose an arbitrary number $R > 0$ is given. Then, note that (i) $V(t,u;z^n,p)$ is bounded uniformly and (ii) all the data in HJB equation (2) corresponding to $\{z_t^n, p_t\}$ are bounded uniformly. As a result, assertion (i) in Theorem 9.2 in CKŚ (2000) implies $V(t,u;z^n,p)$ is Hölder continuous with a Hölder constant independent of n. That is, there are some constants $0 < \xi < 1$ and $K > 0$, both of which are independent of n, such that

$$|V(s,w;z^n,p) - V(t,u;z^n,p)| \leq K(|s-t|^{(1+\xi)/2} + |w-u|), \quad \forall (s,w,t,u) \in Q(R)\times Q(R),\ \forall n.$$

Then, the Arzelà-Ascoli theorem implies the sequence $\{V(t,u;z^n,p)\}_n$ has a uniformly convergent subsequence in $Q(R)$. Obviously, the limit of this subsequence coincides with $W(t,u)$. But, recall that the original sequence $\{V(t,u;z^n,p)\}_n$ monotonically converges to $W(t,u)$ for each (t,u). The Dini theorem then implies $V(t,u;z^n,p)$ uniformly converges to $W(t,u)$ in $Q(R)$. We can then use Theorem 6.1 in CKŚ (2000) to show $W(t,u)$ satisfies HJB equation (2) for $(t,u) \in Q(R)$ with $\{z_t\}$ being replaced by $\{z_t^*\}$. But then, because we can choose R arbitrarily large, we conclude $W(t,u) = V(t,u;z^*,p)$ for each $(t,u) \in [0,\infty)\times\mathbb{R}$. $\square$

6.7 Proof of Lemma 3.6

Proof. Let $V(t,u) = V(t,u;z,p)$, $V^\epsilon(t,u) = V(t,u+\epsilon;z+\epsilon,p)$, and $G(t,u) = V(t,u) - V^\epsilon(t,u)$. Then, $V^\epsilon(t,u)$ satisfies the following HJB equation:

$$0 = c + \phi\min\{e^{u+\epsilon}, P\} + \theta\lambda P 1_{u<z_t}(L(t,u+\epsilon;p) - V^\epsilon) + \lambda\max\{P - V^\epsilon, 0\} - (\rho+\phi)V^\epsilon + \nu V^\epsilon_u + \frac{\sigma^2}{2}V^\epsilon_{uu} + V^\epsilon_t.$$

The funtion $G(t,u)$ then satisfies

$$0 = \phi(\min\{e^u, P\} - \min\{e^{u+\epsilon}, P\}) + \theta\lambda P 1_{u<z_t}(L(t,u;p) - L(t,u+\epsilon;p)) + \lambda\max\{P-V,0\} - \lambda\max\{P-V^\epsilon,0\} - (\rho+\phi+\theta\lambda P 1_{u<z_t})G + \nu G_u + \frac{\sigma^2}{2}G_{uu} + G_t.$$

Then, using the fact that (i) $\min\{e^u,P\} \le \min\{e^{u+\epsilon},P\}$ with inequality strict over a non-trivial interval, (ii) $L(t,u;p) \le L(t,u+\epsilon;p)$, and (iii) $\max\{P-V,0\} - \max\{P-V^\epsilon,0\} \le \max\{-G,0\}$, we can show

$$G(t,u) < 0, \quad \forall t,u,$$

as in the proof of Lemma 3.3. □

6.8 Proof of Theorem 3.7

Proof. Suppose another equilibrium $\{w_t\}$ exists. Then, we have already seen $\underline{z} \le w_t \le z^*_t$ for each t. We now define

$$\epsilon = \inf\{\delta \ge 0 : w_t + \delta \ge z^*_t,\ \forall t\}.$$

We first show ϵ is finite. To this aim, we consider a case in which $z_t \equiv \infty$ and $p_t \equiv 0$; that is, we have chosen the largest rollover strategy and the lowest liquidation price. Then, Lemmas 3.1 and 3.3 imply there is some constant $\bar{z}$ such that

$$V(t, u; z_t \equiv \infty, p_t \equiv 0) \begin{cases} < P & \text{if } u < \bar{z} \\ = P & \text{if } u = \bar{z} \\ > P & \text{if } u > \bar{z}. \end{cases}$$

Then, from Lemma 3.2, we have $z_t^n \leq \bar{z}$ for each (t, n), where $\{z_t^n\}$ is the sequence of the strategies constructed in Section 3.3.2. Thus, $z_t^* \leq \bar{z}$ for each t as well, which implies ϵ is finite.

Now, with respect to a general case in which $w_t + \epsilon$ indeed touches z_t^* at some point (t^*, u^*), we have already derived a contradiction in the body-text. So, we focus on a special case in which $w_t + \epsilon$ converges to z_t^* as t goes to ∞ without touching the curve $\{z_t^*\}$ at a finite point. To handle this case, we note that for any $\{z_t, p_t\}$, $J(u) := \lim_{t \to \infty} V(t, u; z, p)$ solves the following stationary HJB equation:

$$\begin{aligned} 0 = c + \phi(\min\{e^u, P\} - J(u)) + \theta\lambda P 1_{u < z_\infty} (\min\{\alpha F(u) - \bar{\gamma} + p_\infty, P\} - J(u)) + \\ \lambda \max\{P - J(u), P\} - \rho J(u) + \nu J_u(u) + \frac{\sigma^2}{2} J_{uu}(u), \end{aligned}$$

where $z_\infty = \lim_{t \to \infty} z_t$ and $p_\infty = \lim_{t \to \infty} p_t$. But then, similarly as in Lemma 3.6, we can show

$$J(u + \epsilon; w_\infty + \epsilon, p_\infty) > J(u; w_\infty, p_\infty), \quad \forall u.$$

This result leads to a contradiction because

$$J(w_\infty; w_\infty, p_\infty) = \lim_{t \to \infty} V(t, w_\infty; w, p) = P = \lim_{t \to \infty} V(t, z_\infty^*; z^*, p) = J(w_\infty + \epsilon; w_\infty + \epsilon, p_\infty),$$

by definition of equilibrium. □

6.9 Proof of Theorem 3.8

Proof. To prove the existence, it suffices to show $p(t; z) \geq p(t; x)$ for any rollover strategies $\{z_t\}$ and $\{x_t\}$ such that $z_t \leq x_t$ for each t, because we can then proceed in the same way as in the proof of Theorem 3.5 by replacing p_t in that proof with either $p(t; z^n)$ or $p(t; z^*)$ appropriately.

To show this claim, we let $m^z(t,u) = m(t,u;z)$, $m^x(t,u) = m(t,u;x)$, and $n(t,u) = m^z(t,u) - m^x(t,u)$. Subtracting equation (6) corresponding to $\{x_t\}$ from the same equation but corresponding to $\{z_t\}$, we have

$$n_t(t,u) = \theta\lambda P 1_{z_t<u<x_t} m^x(t,u) - \phi n(t,u) - \theta\lambda P 1_{u<z_t} n(t,u) - \nu n_u(t,u) + \frac{\sigma^2}{2} n_{uu}(t,u),$$

Then, because the non-homogeneous term $\theta\lambda P 1_{z_t<u<x_t} m^x(t,u)$ is positive, the probabilistic formula for $n(t,u)$ implies $n(t,u) \geq 0$ for each (t,u). (Or, we can slightly modify Lemma 6.1 to show this property.)

We now note that the cumulative supply of failed assets under the strategies $\{z_t\}$ and $\{x_t\}$ are respectively given by

$$\begin{aligned} q_t(z) &= 1 - \int_{-\infty}^{\infty} m^z(t,u)du - \int_0^t \int_{-\infty}^{\infty} \phi m^z(s,u)duds, \\ q_t(x) &= 1 - \int_{-\infty}^{\infty} m^x(t,u)du - \int_0^t \int_{-\infty}^{\infty} \phi m^x(s,u)duds, \end{aligned}$$

which implies $q_t(z) \leq q_t(x)$ for each t. As a result, formula (15) implies

$$p(t;z) = \rho \int_t^{\infty} e^{-\rho(s-t)} \gamma(q_s(z))ds \geq \rho \int_t^{\infty} e^{-\rho(s-t)} \gamma(q_s(x))ds = p(t;x), \quad \forall t,$$

because $\gamma(q)$ is a decreasing function.

Now we will use the induction argument to show that z_t^* is increasing in t; that is, we will show z_t^n is increasing in t for each n, which then implies z_t^* is increasing in t as well. To this aim, we fix $n \geq 2$ for a moment and assume z_t^n is increasing in t. To avoid abuse of notation, we let $z_t = z_t^n$ and $p_t = p(t; z^n)$. We then show that $V(t,u;z)$ is decreasing in t for each u as follows. For any $\epsilon > 0$, let $G(t,u) = V(t+\epsilon;u) - V(t,u;z)$. Then, $G(t,u)$

satisfies the following HJB equation:

$$0 = \theta\lambda P 1_{u<z_{t+\epsilon}} \left(L(t+\epsilon, u; p) - L(t, u; p)\right) + \theta\lambda P 1_{z_t<u<z_{t+\epsilon}} \left(L(t, u; p) - V(t, u; z)\right) +$$

$$\lambda \max\{P - V(t+\epsilon, u; z), 0\} - \lambda \max\{P - V(t, u; z), 0\} - (\rho + \phi + \theta\lambda P 1_{u<z_{t+\epsilon}})G -$$

$$\nu G_u + \frac{\sigma^2}{2} G_{uu} + G_t.$$

Then, we have (i) $L(t+\epsilon, u; p) \leq L(t, u; p)$ and (ii) $L(t, u; p) \leq V(t, u; z)$ for each (t, u) such that $z_t < u$ by construction of $\{z_t\}$. As a result, we have

$$0 \leq \lambda \max\{-G, 0\} - (\rho + \phi + \theta\lambda P 1_{u<z_{t+\epsilon}})G - \nu G_u + \frac{\sigma^2}{2} G_{uu} + G_t,$$

which implies $G(t, u) \leq 0$ for each (t, u). But then, the fact that $V(t, u; z^n)$ decreases in t implies z_t^{n+1} increase in t as desired. Now, to complete the induction argument, we need to show z_t^2 is increasing in t. But we can simply show this property by applying the above argument to $z_t \equiv \infty$.

Lastly, we can show $w_t \leq z_t^*$ for each t similarly as in the proof of Theorem 3.5. □

6.10 Proof of Proposition 4.1

Proof. Without loss of generality, we focus on the case in which $\eta^1 \leq \eta^2$ and $\bar{\gamma}^1 = \bar{\gamma}^2 = \bar{\gamma}$, because we can simply modify the arguments below to handle the general case in which $\bar{\gamma}^1 \leq \bar{\gamma}^2$. Then, for each parameter η^i, where $i = 1, 2$, we use $\{z_t^n(\eta^i)\}$ to denote the convergent sequence of rollover strategies constructed in Section 3.3.2. Our goal is to show $z_t^n(\eta^1) \leq z_t^n(\eta^2)$ for each (t, n), which essentially proves our proposition.

First, note that formula (15) implies the liquidation price under the rollover strategy $\{z_t^n(\eta^i)\}$ is given by

$$p_t^n(\eta^i) := p^n(t; z^n(\eta^i)) = \rho\bar{\gamma} \int_t^\infty e^{-\rho(s-t)} \frac{1}{(1 + q_s^n(\eta^i))^{\eta^i}} ds,$$

where

$$q_t^n(\eta^i) = \theta\lambda P \int_0^t \int_{-\infty}^{z_s^n(\eta^i)} m(s, u; z^n(\eta^i)) du ds.$$

We then use the induction argument as follows. When $n = 1$, we have $z_t^n(\eta^1) = z_t^n(\eta^2) \equiv \infty$, which implies $q_t^n(\eta^1) = q_t^n(\eta^2)$ and thus $p_t^n(\eta^1) \geq p_t^n(\eta^2)$ for each t. Then, applying Lemma 3.2 to $z_t^a = z_t^b \equiv \infty$, we have

$$V(t, u; z^n(\eta^1)) \geq V(t, u; z^n(\eta^2)), \quad \forall (t, u),$$

which implies $z_t^{n+1}(\eta^1) \leq z_t^{n+1}(\eta^2)$ for each t.

We now fix $n \geq 1$ and assume $z_t^n(\eta^1) \leq z_t^n(\eta^2)$ for each t. Then, as in the proof of Theorem 3.8, we can show $m(t, u; z^n(\eta^1)) \geq m(t, u; z^n(\eta^2))$ for each (t, u), which implies $q_t^n(\eta^1) \leq q_t^n(\eta^2)$ and $p_t^n(\eta^1) \geq p_t^n(\eta^2)$ for each t. Moreover, by construction of the sequence $\{z_t^n(\eta^i)\}$, we have

$$V(t, u; z^n(\eta^1)) \geq P, \quad \forall (t, u) \text{ such that } u \geq z_t^n(\eta^1).$$

Then, because $z_t^n(\eta^1) \leq z_t^n(\eta^2)$ for each t, condition (18) holds if we apply Lemma 3.2 to $z_t^a = z_t^n(\eta^1)$ and $z_t^b = z_t^n(\eta^2)$. Hence, this lemma shows $z_t^{n+1}(\eta^1) \leq z_t^{n+1}(\eta^2)$ for each t, which proves our desired goal by induction.

Lastly, we briefly explain how to handle the general case in which $\bar{\gamma}^1 \leq \bar{\gamma}^2$. Note that in Lemma 3.2, $\bar{\gamma}$ is a fixed parameter. However, this lemma still holds even if we replace (i) the condition $p_t^a \geq p_t^b$ by $-\bar{\gamma}^a + p_t^a \geq -\bar{\gamma}^b + p_t^b$, (ii) $-\bar{\gamma}$ in condition (17) by $\bar{\gamma}^b$, and (iii) $-\bar{\gamma}$ in condition (18) by $\bar{\gamma}^a$. Using this modified version of the lemma, we can deal with the general case similarly as above. $\square$

References

[1] Acharya, V. V., and T. Yorulmazer (2008a) Cash-in-the-Market Pricing and Optimal Resolution of Bank Failures, Review of Financial Studies 21:2705-2742.

[2] Acharya, V. V., and T. Yorulmazer (2008b) Information Contagion and Bank Herding, Journal of Money, Credit and Banking 40:215-231.

[3] Ahnert, T. (2016) Rollover Risk, Liquidity and Macroprudential Regulation, Journal of Money, Credit and Banking 48:1753-85.

[4] Allen, F. and D. Gale (2000) Financial Contagion, Journal of Political Economy 108:1-33.

[5] Angeletos, G.-M., C. Hellwig, and A. Pavan (2007) Dynamic Global Games of Regime Change: Learning, Multiplicity, and the Timing of Attacks, Econometrica 75:711-756.

[6] Angeletos, G.-M. and I. Werning (2006) Crises and Prices: Information Aggregation, Multiplicity, and Volatility, American Economic Review 96:1720-1736.

[7] Barles, G., and P. E. Souganidis (1991) Convergence of Approximation Schemes for Fully Nonlinear Second Order Equations, Asymptotic Analysis 4: 271283.

[8] Bernardo A. E. and I. Welch (2004) Liquidity and Financial Market Runs, The Quarterly Journal of Economics 119:135-158.

[9] Brunnermeier, M. K. and L. H. Pedersen (2009) Market Liquidity and Funding Liquidity, Review of Financial Studies 22:2201-2238.

[10] Caballero, R. J. and A. Simsek (2013) Fire Sales in a Model of Complexity, Journal of Finance 68:2549-2587.

[11] Carlin, B. I., M. S. Lobo, and S. Viswanathan (2007) Episodic Liquidity Crises: Cooperative and Predatory Trading, Journal of Finance 62:2235-2274.

[12] Carlsson, H. and E. van Damme (1993) Global Games and Equilibrium Selection, Econometrica 61:989-1018.

[13] Chen, H. (2010) Macroeconomic Conditions and the Puzzles of Credit Spreads and Capital Structure, Journal of Finance LXV:2171-2212.

[14] Chen, Y. (1999) Banking Panics: The Role of the First-come, First-served Rule and Information Externalities, Journal of Political Economy 107:946-968.

[15] Cheng, Ing-Haw and K. Milbradt (2012) The Hazard of Debt: Rollover Freezes, Incentives, and Bailouts, Review of Financial Studies 25:1070-1110.

[16] Choi, D. (2014) Heterogeneity and Stability: Bolster the Strong, Not the Weak, Review of Financial Studies 27:1830-1867.

[17] Covitz, D., N. Liang, and G. A. Suarez (2013) The Evolution of a Financial Crisis: Collapse of the Asset-Backed Commercial Paper Market, Journal of Finance 68:815-848.

[18] Crandall, M. G., M. Kocan, and A. Święch (2000) L^p-Theory for Fully Nonlinear Uniformly Parabolic Equations, Communications in Partial Differential Equations 25:1997-2053.

[19] Dasgupta, A. (2004) Financial Contagion Through Capital Connections: A Model of the Origin and Spread of Banks Panics, Journal of the European Economic Association 2:1049-1084.

[20] Diamond, D. W. and P. H. Dybvig (1983) Bank Runs, Deposit Insurance, and Liquidity, Journal of Political Economy 91:401-419.

[21] Evans, L. C. (2010) Partial Differential Equations, American Mathematical Society.

[22] Eisenbach, T. M. (2016) Rollover Risk as Market Discipline: A Two-Sided Inefficiency, forthcoming, Journal of Financial Economics.

[23] Frankel, D. M., and A. Pauzner (2000) Resolving Indeterminacy in Dynamic Settings: The Role of Shocks, The Quarterly Journal of Economics 115:285-304.

[24] Frankel, D. M., S. Morris, and A. Pauzner (2003) Equilibrium Selection in Global Games with Strategic Complementarities, Journal of Economic Theory 103:1-44.

[25] Goldstein, I., and A. Pauzner (2004) Contagion of Self-fulfilling Financial Crises due to Diversification of Investment Portfolios, Journal of Economic Theory 119:151-183.

[26] Goldstein, I. and A. Pauzner (2005) Demand-deposit Contracts and the Probability of Bank Runs, Journal of Finance 60:151-183.

[27] Gorton, G. and A. Metrick (2012) Securitized Banking and the Run on Repo, Journal of Financial Economics 104:425-451.

[28] Hackbarth, D., J. Miao, and E. Morellec (2006) Capital Structure, Credit Risk, and Macroeconomic Conditions, Journal of Financial Economics 82:519-550.

[29] He, Z. and A. Manela (2016) Information Acquisition in Rumor-Based Bank Runs, Journal of Finance 71:1113-1158.

[30] He, Z. and K. Milbradt (2014) Endogenous Liquidity and Defaultable Debt, Econometrica 82:1443-1508.

[31] He, Z. and W. Xiong (2012) Dynamic Debt Runs, Review of Financial Studies 25:1799-1843.

[32] Kiyotaki, N. and J. Moore (1997) Credit Cycles, Journal of Political Economy 105:21148.

[33] Kodres, L. and M. Pritsker (2002) A Rational Expectations Model of Financial Contagion, Journal of Finance 57:769-799.

[34] Krylov, N. V. (2000) On the Rate of Convergence of Finite-Difference Approximations for Bellman's Equations with Variable Coefficients, Probability Theory and Related Fields 117:1-16.

[35] Kyle, A. S. and W. Xiong (2001) Contagion as a Wealth Effect, Journal of Finance LVI:1401-1440.

[36] Lagunoff, R. and S. Schreft (2001) A Model of Financial Fragility, Journal of Economic Theory 99:220-264.

[37] Liu, X. (2016) The Interbank Market Run and Creditor Runs, Review of Financial Studies 29:1860-1910.

[38] Moody's (2015) Annual Default Study: Corporate Default and Recovery Rates, 1920-2015, Moody's Investor Service.

[39] Morris, S. and H. S. Shin (1998) Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks, American Economic Review 88:587-597.

[40] Oehmke, M. (2014) Liquidating illiquid collateral, Journal of Economic Theory 149:183-210.

[41] Oh, F. (2013) Contagion of a Liquidity Crisis Between Two Firms, Journal of Financial Economics 107:386-400.

[42] Rochet, Jean-Charles and X. Vives (2004) Coordination Failures and The Lender of Last Resort: Was Bagehot Right After All?, Journal of the European Economic Association 2:1116-1147.

[43] Schroth, E., G. A. Suarez, and L. A. Taylor (2014) Dynamic Debt Runs and Financial Fragility: Evidence from the 2007 ABCP Crisis, Journal of Financial Economics 112:164-189.

[44] Shleifer, A. and R. Vishny (1992) Liquidation Values and Debt Capacity: A Market Equilibrium Approach, Journal of Finance 47:1343-1366.

[45] Uhlig, H. (2010) A Model of a Systemic Bank Run, Journal of Monetary Economics 57:78-96.

[46] Veronesi, P. and L. Zingales (2010) Paulson's Gift, Journal of Financial Economics 97:339-368.

[47] Wang, L. (1992) On the Regularity Theory of Fully Nonlinear Parabolic Equations: I, Communications on Pure and Applied Mathematics 45:27-76.