Dynamic Adverse Selection and Liquidity*

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Abstract

Does a larger fraction of informed trading generate more illiquidity, as measured by the bid-ask spread? We answer this question in the negative in the context of a dynamic dealer market where the fundamental value follows a random walk, provided we consider the long run (stationary) equilibrium. More informed traders tend to generate more adverse selection and hence larger spreads, but at the same time cause faster learning by the market makers and hence smaller spreads. These two effects offset each other in the long run.

KEYWORDS: Learning, stationary distribution.

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1 Introduction

A traditional view of market liquidity, going at least as far back as Bagehot (1971), posits that one of the causes of illiquidity is adverse selection: *"The essence of market making, viewed as a business, is that in order for the market maker to survive and prosper, his gains from liquidity-motivated transactors must exceed his losses to information motivated transactors. [...] The spread he sets between his bid and asked price affects both: the larger the spread, the less money he loses to information-motivated, transactors and the more he makes from liquidity-motivated transactors."*

This intuition has later been made precise by models such as Glosten and Milgrom (1985, henceforth GM85), in which a competitive risk-neutral dealer sequentially sets bid and ask prices in a risky asset, and makes zero expected profits in each trading round. Traders are selected at random from a population that contains a fraction ρ of informed traders, and must trade at most one unit of the asset. The asset liquidates at a value v that is constant and is either zero or one. In equilibrium, the bid-ask spread is wider when the informed share ρ is higher: there is more adverse selection, hence the dealer must set a larger bid-ask spread to break even.

This intuition, however, must be modified once we consider the dynamics of the bid-ask spread. A larger informed share also means that orders carry more information, which over time reduces the uncertainty about v and thus puts downward pressure on the bid-ask spread. We call this last effect *dynamic efficiency*. This effect is already present in GM85, who observe that a larger informed share causes initially a larger bid-ask spread, but also causes the bid-ask spread to decrease faster to its eventual value, which is zero (when v is fully learned).

A natural question is then: to what extent does dynamic efficiency reduce the traditional adverse selection? To answer this question, we extend the framework of GM85 to allow v to move over time according to a random walk v_t.[1] To obtain closed-form results, we assume that the increments of v_t are normally distributed with volatility σ_v, called the *fundamental*

[1] For simplicity, if an informed trader sees v_t outside the bid-ask spread, she exits the model and is immediately replaced with an uninformed trader who submits either a buy or a sell order with equal probability. This assumption is made to avoid having no order submitted at t, which would complicate the model by introducing the time between trades as a state variable.

volatility. We chose a moving value for two reasons. First, this is a realistic assumption in modern financial markets, where relevant information arrives essentially at a continuous rate. Second, we want to study the long-term evolution of the bid-ask spreads, and this long-term analysis is trivial when v_t is constant, as the dealer eventually fully learns v. Note that we are interested in the long run because (as we show later) in the short run the equilibrium is similar to GM85, but in the long run it converges to the *stationary* equilibrium, which has novel properties.

The first property of the stationary equilibrium is that the dealer's uncertainty about v_t is constant. More precisely, let the *efficient density* at t be the dealer's posterior density of v_t after observing the previous order flow, and let the *efficient mean* and *efficient volatility* be, respectively, its mean and standard deviation.[2] Thus, in a stationary equilibrium, the efficient volatility (which is a measure of the dealer's uncertainty) is constant. The second property of the stationary equilibrium is that the informed share is inversely related to the efficient volatility. The intuition is simple: when the informed share is low, the order flow carries little information, and thus the efficient volatility is large.[3]

A surprising property of the stationary equilibrium is that the informed share has no effect on the bid-ask spread. To understand this result, consider a small informed share, say 1%. Suppose a buy order arrives, and the dealer estimates how much to update the efficient mean (in equilibrium this update is half of the bid-ask spread). There are two opposite effects. First, it is very unlikely that the buy order comes from an informed trader (with only 1% chance). This is the *adverse selection effect*: a low informed share makes the dealer less concerned about adverse selection, which leads to a smaller update of the efficient mean, and hence decreases the bid-ask spread. But, second, if the buy order does come from an informed trader, a large efficient volatility translates into the dealer knowing that, on average, the informed trader must have observed a value far above the efficient price. This is the *dynamic efficiency effect*: a low informed share leads to a larger update of the efficient mean, and hence increases the

[2] To simplify the analysis, we assume that the dealer always considers the efficient density as normal: after observing the order flow at t, the dealer computes correctly the first two moments of the efficient density at $t+1$, but not the higher moments, and therefore considers the new density as normal as well. The analysis in the Introduction follows this simplifying assumption.

[3] What is perhaps less obvious is that, when the informed share is small, it is not possible for the efficient volatility to become infinite, but instead it converges to a constant value.

bid-ask spread. At the other end, a large informed share means that the dealer learns well about the asset value (the efficient volatility is small), and therefore the bid-ask spread tends to be small.

It turns out that the two opposite effects exactly offset each other. This result depends crucially on the equilibrium being stationary. More generally, we show that, in any stationary filtration problem, the efficient mean and the asset value must evolve with the same volatility (see Appendix C). Denote by h the amount by which the efficient mean increases after a buy order; by symmetry, the efficient mean decreases by the same h after a sell order. This implies that the efficient mean evolves with volatility h, and the bid-ask spread is $2h$. But the asset value evolves with volatility σ_v. It follows that the equilibrium bid-ask spread is twice the fundamental volatility, and hence is independent of the informed share.

Our next result is that, for any initial efficient volatility, the equilibrium converges to the stationary equilibrium. In particular, consider a wide initial efficient volatility. Then, as the order flow starts providing information to the dealer, the efficient volatility starts decreasing toward its stationary value. The same is true for the bid-ask spread, which in a non-stationary equilibrium is always proportional to the efficient volatility. This phenomenon is similar to the GM85 equilibrium, except that there the stationary efficient volatility and bid-ask spread are both zero. This illustrates the statement made above, that the non-stationary equilibrium (the "short run") resembles GM85, while the stationary equilibrium (the "long run") is different and produces novel insights.

Studying the equilibrium behavior after various types of shocks provides a few testable implications. First, consider a positive shock to the informed share (e.g., the stock is now studied by more hedge funds). Then, the adverse selection effect suddenly becomes stronger, and as a result the bid-ask spread temporarily increases. In the long run, though, the bid-ask spread reverts to its stationary value, which does not change. At the same time, the efficient volatility gradually decreases to its new level, which is lower due to the increase in the informed share. Second, consider a negative shock to the current efficient volatility (e.g., public news about the current asset value). Then, the bid-ask spread follows the efficient volatility and drops immediately, after which it increases gradually to its old stationary level. Third, consider a positive shock to the fundamental volatility (e.g., all future uncertainty about the asset increases). Then, the bid-ask spread follows the efficient volatility and increases gradually to its new stationary level.

Based on our results, the picture on dynamic adverse selection that emerges is that liquidity is more strongly affected not by the informed share (the intensive margin), but by the fundamental volatility (the extensive margin). By contrast, price discovery (measured by the efficient volatility) is strongly affected by both informed share and fundamental volatility. This suggests that the presence of privately informed traders can be more precisely identified by proxies of the current level of uncertainty, rather than by illiquidity measures such as the bid-ask spread (which is used by Collin-Dufresne and Fos, 2015).

A surprising outcome of our theory is that a lower level of uncertainty (lower efficient volatility) can occur if either the informed share becomes larger (more privately informed traders arrive), or more precise public news arrives.[4] We can disentangle the two scenarios, however, by examining the effect on the bid-ask spread: more precise public news should reduce it, while a larger informed share should have no effect.

Our paper contributes to the literature of dynamic models of adverse selection.[5] To our knowledge, this paper is the first to study the effect of stationarity in dealer models of the Glosten and Milgrom (1985) type.[6] By contrast, there are several stationary models of the Kyle (1985) type, e.g., Chau and Vayanos (2008) and Caldentey and Stacchetti (2010). The focus of these models, however, is not liquidity but price discovery: it turns out that in the limit the market in this models becomes strong-form efficient, as the insider trades very aggressively.

The paper speaks to the literature on the identification of informed trading and in particular on the identification of insider trading. Collin-Dufresne and Fos (2015, 2016) show both empirically and theoretically that times when insiders trade coincide with times when liquidity is actually stronger (and in particular bid-ask spreads decline). They attribute this

[4]A simple extension of our model with public news can be solved to show that more precise public news translates into a lower efficient volatility.

[5]See for instance the survey of Foucault, Pagano, and Röell (2013) and the references therein.

[6]Glosten and Putnins (2016) study the welfare effect of the informed share in the Glosten and Milgrom (1985) model, but they do not consider the effect of stationarity.

finding to the action of discretionary insiders who trade when they expect a larger presence of liquidity (noise) traders. During those times the usual positive effect of noise traders on liquidity dominates, and thus bid-ask spreads decline despite there being more informed trading. By contrast, our effect works even when the noise trader activity is constant over time, as long as there is enough time for the equilibrium to become stationary.

The paper is organized as follows. Section 2 describes the model (in which the value follows a random walk). Section 3 shows how to compute the equilibrium when the dealer is fully Bayesian. Section 4 studies in detail the model in which the dealer is approximately Bayesian, and describes the stationary and non-stationary equilibria. Section 5 verifies how well the approximate equilibrium approaches the exact equilibrium. Section 6 concludes. All proofs are in the Appendix. The Internet Appendix contains a discussion of general dealer models, and an application to a model in which the fundamental value switches randomly between zero and one.

2 Environment

The model is similar to GM85, except that the fundamental value moves according to a random walk:

$$v_{t+1} \;=\; v_t + \varepsilon_{t+1}, \quad \text{with} \quad \varepsilon_t \overset{IID}{\sim} \mathcal{N}\left(\cdot, 0, \sigma_v\right). \tag{1}$$

There is a single risky asset, and time is discrete and infinite. Trading in the risky asset is done on an exchange, where before each time $t = 0, 1, 2, \ldots$ a dealer posts two quotes: the *ask price* (or simply *ask*) A_t, and the *bid price* (or simply *bid*) B_t. Thus, a buy order at t executes at A_t, while a sell order at t executes at B_t. The dealer (referred to in the paper as "she") is risk neutral and competitive, and therefore makes zero expected profits from each trade.

The buy or sell orders are submitted by a trading population with a fraction $\rho \in (0, 1)$ of informed traders and a fraction $1 - \rho$ of uninformed traders. At each $t = 0, 1, \ldots$ a trader is selected at random from the population, and can trade at most one unit of the asset. If the trader at t is uninformed, then he is equally likely to buy or to sell. If the trader at t is informed and observes the value v_t, then she submits either (i) buy order if $v_t > A_t$,

(ii) sell order if $v_t < B_t$, (iii) no order if $v_t \in [B_t, A_t]$, in which case she exits the model and is immediately replaced with an uninformed trader who submits either a buy or a sell order with equal probability.[7]

The dealer's uncertainty about the fundamental value is summarized by the *efficient density*, which is the density of v_t just before trading at t, conditional on all the order flow available at t, that is, the sequence of orders submitted at times $0, 1, \ldots, t-1$. Denote by ϕ_t the efficient density, and let μ_t be its mean (called the *efficient mean*) and σ_t its standard deviation (called the *efficient volatility*). The initial density ϕ_0 is assumed to be rapidly decaying at infinity.[8] In the rest of the paper, by "density" we typically include the requirement that the density be rapidly decaying. To avoid cumbersome language, we make this requirement explicit only when we need to be more precise, such as when we state the formal results.

3 Equilibrium

We prove the existence of an equilibrium of the model in two steps. First, for each $t = 0, 1, 2, \ldots$ we start with an efficient density ϕ_t, an ask A_t, a bid $B_t < A_t$, and compute the efficient density ϕ_{t+1} after a buy or sell order. Second, for any efficient density ϕ_t we show that there exists an *ask-bid pair* (A_t, B_t), meaning that the ask A_t and the bid B_t satisfy the dealer's pricing conditions which require that her expected profit from trading at t is zero. The ask-bid pair (A_t, B_t) is not necessarily unique, and we choose the pair with the ask closest to the efficient mean.

3.1 Evolution of the Efficient Density

Let ϕ_t be the efficient density of v_t before trading at t, and let $A_t > B_t$ be, respectively, the ask and bid at t (not necessarily satisfying the dealer's pricing conditions). Suppose a buy or sell order $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$ arrives at t. Let $\mathbf{1}_P$ be the indicator function, which is one if P is true

[7] This assumption is made to avoid having no order submitted at t, which would complicate the model by introducing the time between trades as a state variable.

[8] A function f is rapidly decaying (at infinity) if it is smooth and satisfies $\lim_{v \to \pm\infty} |v|^M f_0^{(N)}(v) = 0$, where $f^{(N)}$ is the N-th derivative of f. The space $\mathcal{S}$ of rapidly decaying functions is called the Schwartz space. Any normal density belongs to $\mathcal{S}$, and the convolution of two densities in $\mathcal{S}$ also belongs to $\mathcal{S}$.

and zero if P is false. Conditional on $v_t = v$, the probability of observing the a buy order at t is

$$g_t(\mathrm{B}, v) \;=\; \rho \mathbf{1}_{v > A_t} + \tfrac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \tfrac{1-\rho}{2}. \tag{2}$$

Indeed, a buy order is submitted either by (i) an informed trader (with probability ρ) who sees v_t above A_t, (ii) by an informed trader (with probability ρ) who sees v_t within the spread $[B_t, A_t]$, exits the model and gets replaced with an uninformed buyer (with probability $\frac{1}{2}$), or (iii) by an uninformed trader (with probability $\frac{1-\rho}{2}$) who is a buyer (with probability $\frac{1}{2}$). Similarly, the probability of observing a sell order at t is

$$g_t(\mathrm{S}, v) \;=\; \rho \mathbf{1}_{v < B_t} + \tfrac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \tfrac{1-\rho}{2}. \tag{3}$$

The next result shows the evolution of the efficient density.

Proposition 1. *Consider a rapidly decaying efficient density ϕ_t, and an ask-bid pair with $A_t > B_t$. After observing an order $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$, the density of v_t is $\psi_t(v|\mathcal{O}_t)$, where*

$$\begin{aligned} \psi_t(v|\mathrm{B}) \;&=\; \frac{\left(\rho \mathbf{1}_{v > A_t} + \frac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \frac{1-\rho}{2}\right) \cdot \phi_t(v)}{\frac{\rho}{2}(1 - \Phi_t(A_t)) + \frac{\rho}{2}(1 - \Phi_t(B_t)) + \frac{1-\rho}{2}}, \\ \psi_t(v|\mathrm{S}) \;&=\; \frac{\left(\rho \mathbf{1}_{v < B_t} + \frac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \frac{1-\rho}{2}\right) \cdot \phi_t(v)}{\frac{\rho}{2}\Phi_t(A_t) + \frac{\rho}{2}\Phi_t(B_t) + \frac{1-\rho}{2}}, \end{aligned} \tag{4}$$

where Φ_t is the cumulative density function corresponding to ϕ_t. The efficient density at $t+1$ is rapidly decaying, and satisfies

$$\phi_{t+1}(w|\mathcal{O}_t) \;=\; \int_v \psi_t(v|\mathcal{O}_t) \mathcal{N}(w - v, 0, \sigma_v) \mathrm{d}v \;=\; \Big(\psi_t(\cdot|\mathcal{O}_t) \,*\, \mathcal{N}(\cdot, 0, \sigma_v)\Big)(w), \tag{5}$$

where "$*$" denotes the convolution of two densities.

Proposition 1 describes how the efficient density evolves once a particular order (buy or sell) is submitted at t. Note, however, that this result does not assume anything about the ask and bid other than $A_t > B_t$, so in principle these can be chosen arbitrarily. In equilibrium, however, these prices must satisfy the dealer's pricing conditions, namely that the dealer's expected profits at t must be zero.

In the next section (Section 3.2) we impose this conditions and we show how to determine the equilibrium ask and bid. Once we resolve this issue, Proposition 1 allows us to describe the whole evolution of the efficient density, conditional on the initial density ϕ_0 and the sequence of orders $\mathcal{O}_0, \mathcal{O}_1, \ldots$ that have been submitted.

3.2 Ask and Bid Prices

Let ϕ_t be the efficient density of v_t before trading at t. We define an *ask-bid pair* (A_t, B_t) as a pair of ask and bid satisfying the pricing conditions of the dealer. As the dealer is risk neutral and competitive, the pricing conditions are: (i) the ask A_t is the expected value of v_t conditional on a buy order at t, and (ii) the bid B_t is the expected value of v_t conditional on a sell order at t. Using the previous notation, the dealer's pricing conditions are that A_t is the mean of $\psi_t(v|\mathrm{B})$, the posterior density of v_t after observing a buy order; and B_t is the mean of $\psi_t(v|\mathrm{S})$, the posterior density after observing a sell order. Thus, the dealer's pricing conditions are equivalent to

$$A_t \;=\; \int_{-\infty}^{+\infty} v\psi_t(v|\mathrm{B}), \qquad B_t \;=\; \int_{-\infty}^{+\infty} v\psi_t(v|\mathrm{S}). \tag{6}$$

For future use, we record the following straightforward result.

Corollary 1. *The pair (A_t, B_t) is an ask-bid pair if and only if the following equations are satisfied:*

$$A_t \;=\; \mu_{t+1,\mathrm{B}}, \quad B_t \;=\; \mu_{t+1,\mathrm{S}}, \quad \textit{with} \quad \mu_{t+1,\mathcal{O}_t} \;=\; \int_{-\infty}^{+\infty} w\phi_{t+1}(w|\mathcal{O}_t), \quad \mathcal{O}_t = \{\mathrm{B},\mathrm{S}\}. \tag{7}$$

The next result shows that the existence of an ask-bid pair is equivalent to solving a 2×2 system of nonlinear equations. Suppose μ_t is the mean of ϕ_t. For $(A,B) \in (\mu_t,\infty)\times(-\infty,\mu_t)$, define the functions:

$$\begin{aligned} F(A,B) \;&=\; \frac{\Theta_t(A)+\Theta_t(B)}{A-\mu_t} - \frac{1+\rho}{\rho} + \Phi_t(A) + \Phi_t(B), \\ G(A,B) \;&=\; \frac{\Theta_t(A)+\Theta_t(B)}{\mu_t-B} - \frac{1-\rho}{\rho} - \Phi_t(A) - \Phi_t(B), \end{aligned} \tag{8}$$

where Φ_t is the cumulative density associated to ϕ_t, and Θ_t is defined by

$$\Theta_t(v) \;=\; \int_{-\infty}^{v} (\mu_t - w)\phi_t(w)\mathrm{d}w. \tag{9}$$

The function Θ_t is strictly positive everywhere and approaches zero at infinity on both sides.[9]

Proposition 2. *Consider a rapidly decaying efficient density ϕ_t, with mean μ_t. Then, the existence of an ask-bid pair is equivalent to finding a solution $(A, B) \in (\mu_t, \infty) \times (-\infty, \mu_t)$ of the system of equations:*

$$F(A, B) \;=\; 0, \qquad G(A, B) \;=\; 0. \tag{10}$$

A solution of (10) always exists. Among the set of ask-bid pairs (A, B) there is a unique one for which A is closest to μ_t.

The last statement in Proposition 2 shows that one can choose a unique ask-bid pair based on the criterion that the ask A be the closest to the efficient mean μ_t. Denote this pair by (A_t, B_t). In the rest of the paper, we assume that this is indeed the ask-bid pair chosen by the dealer.[10]

4 Equilibrium with Approximate Bayesian Inference

In this section, we assume that at each step the dealer approximates the efficient density with a normal density such that the first two moments are correctly computed. Specifically, suppose that the dealer regards v_t to be distributed as

$$\phi_t^a(v) \;=\; \mathcal{N}(v, \mu_t, \sigma_t). \tag{11}$$

[9] As ϕ_t is rapidly decaying, $\Theta_t(-\infty)$ is equal to zero. The definition of μ_t implies that $\Theta_t(+\infty) = \int_{-\infty}^{+\infty}(\mu_t - w)\phi_t(w) = \mu_t - \int_{-\infty}^{+\infty} w\phi_t(w) = 0$. Also, $\Theta_t'(v) = (\mu_t - v)\phi_t(v)$, hence $\Theta_t(v)$ is increasing below μ_t and decreasing above μ_t. As $\Theta_t(\pm\infty) = 0$, the function Θ_t is strictly positive everywhere.

[10] In principle, the equations in (10) might have multiple solutions, meaning that one could manufacture an efficient density ϕ_t for which there is more that one corresponding ask-bid pair. Numerically, we have computed the sequence of efficient densities that starts with a normal density ϕ_0 and is associated by an arbitrary sequence of orders, but we have not yet been able to encounter a non-unique ask-bid pair. Nevertheless, we must account for the possibility that such non-uniqueness may in fact arise.

After the dealer observes an order $\mathcal{O}_t$ at t, denote by $\phi_{t+1}(w|\mathcal{O}_t)$ the exact density of v_{t+1} conditional on $\mathcal{O}_t$, and by $\mu_{t+1,\mathcal{O}_t}$ and $\sigma_{t+1,\mathcal{O}_t}$ its mean and standard deviation, respectively. Thus, after observing $\mathcal{O}_t$ the dealer regards v_{t+1} to be distributed as

$$\phi^a_{t+1}(w|\mathcal{O}_t) \;=\; \mathcal{N}\big(w, \mu_{t+1,\mathcal{O}_t}, \sigma_{t+1,\mathcal{O}_t}\big). \tag{12}$$

In Section 5, we discuss the accuracy of the approximation. For now, however, we just assume that the dealer continues to make this approximation at each step. Thus, in this section, $\phi_t(v)$ is the *efficient density*, μ_t is the *efficient mean*, and σ_t is the *efficient volatility*.

4.1 Evolution of the Efficient Density

Proposition 3. *Suppose the efficient density at* $t = 0, 1, 2, \ldots$ *is* $\phi_t(v) = \mathcal{N}(v, \mu_t, \sigma_t)$*. After observing* $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$*, the posterior mean and volatility at* $t+1$ *satisfy*

$$\mu_{t+1,\mathrm{B}} \;=\; \mu_t + \delta\sigma_t, \quad \mu_{t+1,\mathrm{S}} \;=\; \mu_t - \delta\sigma_t, \quad \sigma_{t+1,\mathrm{B}} \;=\; \sigma_{t+1,\mathrm{S}} \;=\; \sqrt{(1-\delta^2)\sigma_t^2 + \sigma_v^2}. \tag{13}$$

where δ *is defined by:*

$$\delta \;=\; g^{-1}(2\rho), \quad \textit{with} \quad g(x) \;=\; \frac{x}{\mathcal{N}(x, 0, 1)}. \tag{14}$$

There is a unique ask: $A_t = \mu_t + \delta\sigma_t$ *and unique bid:* $B_t = \mu_t - \delta\sigma_t$*, and the bid-ask spread is*

$$s_t \;=\; A_t - B_t \;=\; 2\delta\sigma_t. \tag{15}$$

We now investigate whether the efficient density reaches a steady state, in the sense that its shape converges to a particular density. As the mean μ_t evolves according to a random walk, we must demean the efficient density and focus on its standard deviation σ_t. The next result shows that the efficient volatility σ_t converges to a particular value, σ_*, regardless of the initial value σ_0.

Proposition 4. *For any $t = 0, 1, 2, \ldots$ the efficient volatility satisfies*

$$\sigma_t^2 \;=\; \sigma_*^2 + \left(\sigma_0^2 - \sigma_*^2\right)\left(1 - \delta^2\right)^t, \tag{16}$$

where

$$\sigma_* \;=\; \frac{\sigma_v}{\delta} \;=\; \frac{\sigma_v}{g^{-1}(2\rho)}. \tag{17}$$

For any initial value σ_0 and any sequence of orders, the efficient volatility σ_t monotonically converges to σ_*, and the bid-ask spread monotonically converges to

$$s_* \;=\; 2\sigma_v. \tag{18}$$

Proposition 4 shows that in the long run the equilibrium approaches a particular stationary equilibrium, which we analyze next.

4.2 Stationary Equilibrium

We define a *stationary equilibrium* an equilibrium in which the efficient volatility σ_t is constant. According to Proposition 4, if the initial density is $\phi_0(v) = \mathcal{N}(v, \mu_0, \sigma_*)$, then all subsequent efficient densities have the same volatility, namely the stationary volatility σ_*. We now analyze the properties of the stationary equilibrium.

Corollary 2. *In the stationary equilibrium, the efficient volatility σ_* is decreasing in the fraction of informed trading ρ, while the bid-ask spread s_* does not depend on ρ. Both σ_* and s_* are increasing in the fundamental volatility σ_v.*

Intuitively, an increase in the fundamental volatility σ_v raises the efficient volatility as the dealer's knowledge about the fundamental value becomes more imprecise. It also increases the adverse selection overall for the dealer, hence she increases the bid-ask spread. Moreover, a decrease in the fraction of informed trading ρ means that the order flow becomes less informative, and therefore the dealer's knowledge about the fundamental value is more imprecise (σ_* is large).

The surprising result is that the stationary bid-ask spread is independent of ρ. This is equivalent to the update in efficient mean being independent of ρ. Indeed, the efficient mean evolves according to

$$\mu_{t+1,\mathrm{B}} = \mu_t + \sigma_v, \qquad \mu_{t+1,\mathrm{S}} = \mu_t - \sigma_v. \tag{19}$$

Thus, the bid-ask spread is $s_* = (\mu_t + \sigma_v) - (\mu_t - \sigma_v) = 2\sigma_v$. To understand the intuition behind this result, consider the case when ρ is low. Suppose the dealer observes a buy order at t. As ρ is low, there are two effects on the size of the update in efficient mean. The first effect is negative: the trader at t is unlikely to be informed, which decreases the size of the update. This is the traditional *adverse selection effect* from models such as GM85. The second effect is positive: when the trader at t *is* informed, he must have observed a large fundamental value v_t, as the uncertainty in v_t (measured by the efficient volatility σ_*) is also large. This we call the *dynamic efficiency effect*: more informed traders create over time a more precise knowledge about the fundamental value, and thus reduce the effect of informational updates.

It turns out that the dynamic efficiency effect exactly cancels the adverse selection effect in a stationary setup. To understand why the two effects are equal, consider an equilibrium which is not necessarily stationary. If there was no order flow at t, then the uncertainty in (the efficient volatility) would increase from t to $t+1$ because the fundamental value diffuses. But there *is* order flow at t, which reduces the uncertainty. In a stationary equilibrium the two effects must cancel each other. As the increase in uncertainty due to value diffusion is independent of the fraction of informed trading ρ, the decrease in uncertainty due to order flow should also be independent of ρ.

Formally, the decrease in uncertainty due to the order $\mathcal{O}_t$ at t can be evaluated by comparing the prior efficient density $\phi_t(v)$ and the posterior density $\psi_t(v|\mathcal{O}_t)$. One measure of the decrease in uncertainty is how much the efficient mean is updated after a buy or sell order (which are equally likely). But (19) implies that this update is $\pm\sigma_v$, which from the point of view of the information at t is a binary distribution, with standard deviation σ_v which is indeed independent of ρ. Note that we have also essentially proved the following result.

Corollary 3. *In the stationary equilibrium, the volatility of the change in efficient mean is constant and equal to* σ_v.

This result is in fact true quite generally. Indeed, in Appendix C we prove that for any filtration problem in which the variance remains constant over time the volatility of the change in efficient mean must equal the fundamental volatility.

4.3 Liquidity Dynamics

In this section we analyze the evolution of the efficient volatility and the bid-ask spread after a shock to either the efficient volatility σ_t, the fundamental volatility σ_v, or the fraction of informed trading ρ. We are also interested in how quickly the equilibrium converges to the stationary equilibrium. In general, the speed of convergence of a sequence x_t that converges to a limit x_* is defined as the limit ratio

$$S = \lim_{t \to \infty} \frac{|x_t^2 - x_*^2|}{|x_{t+1}^2 - x_*^2|}, \tag{20}$$

provided that the limit exists. The next result computes the speed of convergence for several variables of interest.

Corollary 4. *The efficient volatility, efficient variance and bid-ask spread have the same speed of convergence:*

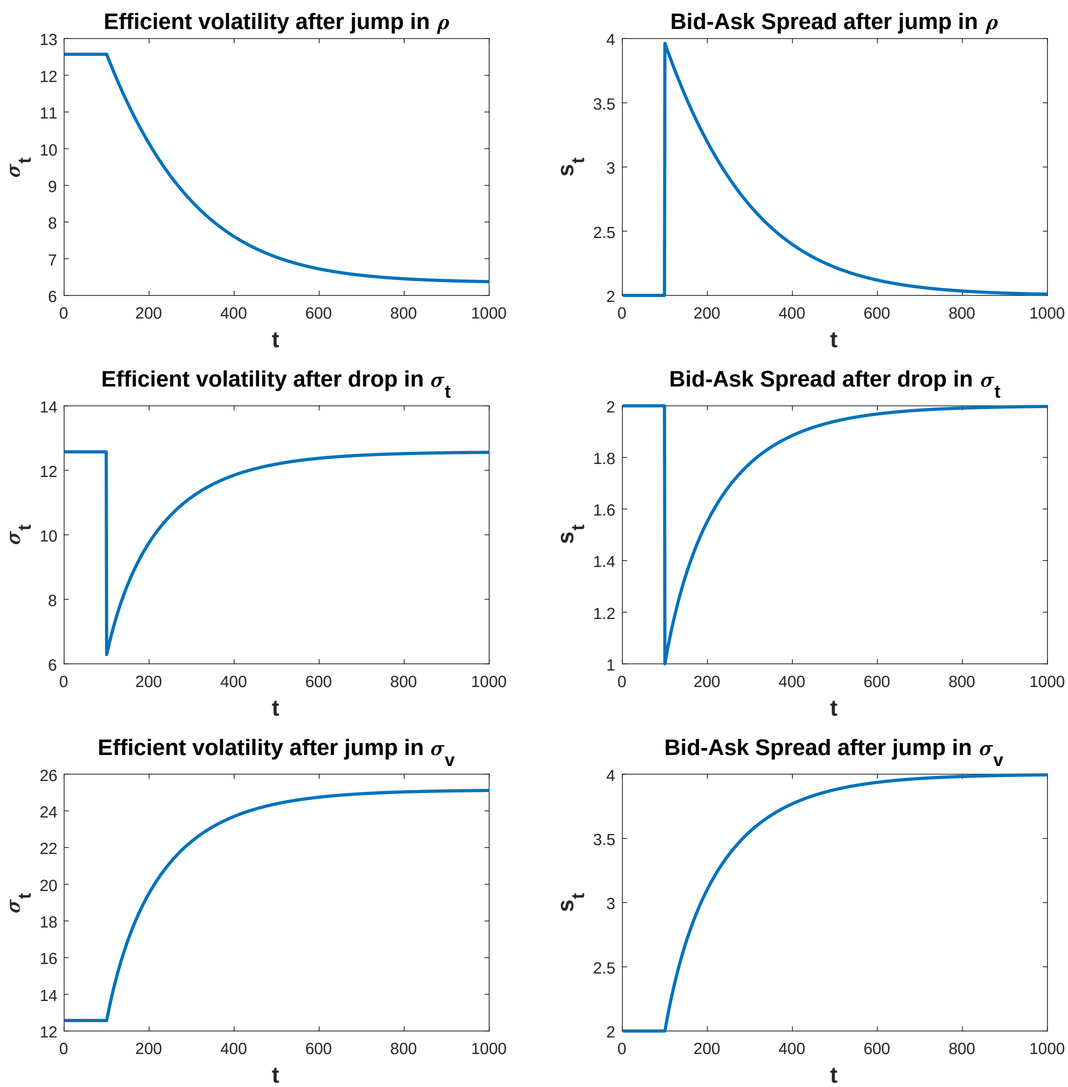
$$S = \frac{1}{1 - \delta^2}. \tag{21}$$

Moreover, S is increasing in the fraction of informed trading ρ.

Corollary 4 shows that the variables of interest have the same speed of convergence S, and we can thus call S simply as the *convergence speed* of the equilibrium. Another result of Corollary 4 is that a larger fraction of informed trading ρ implies a faster convergence speed of the equilibrium to its stationary value. This is intuitive, as more informed trading helps the dealer make quicker dynamic inferences. Note that when $\rho = 1$, equation (14) implies that $\delta = g^{-1}(2) \approx 0.647$, thus the maximum value of δ is less than one. Therefore, the maximum convergence speed is finite.

We now consider the effect of various types of shocks to our stationary equilibrium. In the first row of Figure 1 we plot the effects of a positive shock to the fraction of informed trading,

Figure 1: Efficient Volatility and Bid-Ask Spread after Shocks. This figure plots the effect of three types of shocks on the efficient volatility σ_t, and on the bid-ask spread s_t (each shock occurs at $t_0 = 100$). The initial parameters are: $\sigma_v = 1$, and $\rho = 0.1$ (hence $\delta = 0.0795$, $\sigma_* = 12.573$, $s_* = 2$). In the first row, the fraction of informed trading ρ jumps from 0.1 to 0.2 (hence σ_* drops from 12.573 to 6.345). In the second row, the efficient volatility drops from $\sigma_* = 12.573$ to half of its value (6.286). In the third row, the fundamental volatility jumps from 1 to 2.



meaning that ρ suddenly jumps to a higher value ρ'. This generates an increase in δ, which jumps to its new value $\delta' = g^{-1}(\rho')$, and it also generates a drop in the stationary efficient volatility, which is now $\sigma'_* = \sigma_v/\delta'$. Nevertheless, as there is no new information above the fundamental value, the current efficient volatility σ_t remains equal to its old stationary value, $\sigma_* = \sigma_v/\delta$. Proposition 4 shows that the efficient volatility starts decreasing monotonically toward its stationary value σ'_*. Note that according to Corollary 4 the speed of convergence to the new stationary equilibrium is $S' = 1/(1-\delta'^2)$, which is higher than the old convergence speed. We also describe the evolution of the bid-ask spread, which according to Proposition 3 satisfies $s_t = 2\delta'\sigma_t$. Initially, the bid-ask spread jumps to reflect the jump to δ'. But then, as σ_t converges to $\sigma'_* = \sigma_v/\delta'$, the bid-ask spread starts decreasing to $s_* = 2\sigma_v$, which does not depend on ρ.

To summarize, after a positive shock to ρ, the efficient volatility starts decreasing monotonically to its now lower stationary value, while the bid-ask spread initially jumps and then decreases monotonically to the same stationary value (that does not depend on ρ). Intuitively, a positive shock to the fraction of informed trading leads to a sudden increase in adverse selection for the dealer, reflected in an initially larger bid-ask spread, after which the bid-ask spread reverts to its fundamental value, which is independent of informed trading. At the same time, more informed trading leads to more precision for the dealer in the long run, which is reflected in a smaller efficient volatility.

In the second row of Figure 1 we plot the effects of a negative shock to the efficient volatility, meaning that σ_t suddenly drops from the stationary value σ_* to a lower value. This drop can be caused for instance by public news about the value of the asset v_t. Then, according to Proposition 4, the efficient volatility increases monotonically back to the stationary value. The bid-ask spread is always proportional to the efficient density: $s_t = 2\delta\sigma_t$, hence s_t also drops initially and then increases monotonically toward the stationary value s_*. Intuitively, public news has the effect of helping the dealer initially to get a more precise understanding about the fundamental value. This brings down the bid-ask spread, as temporarily the dealer faces less adverse selection. But this decrease is only temporary, as the value diffuses and the same forces increase the efficient volatility and the bid-ask spread toward their corresponding stationary values, which are the same as before.

In the third row of Figure 1 we plot the effects of a positive shock to the fundamental volatility, meaning that σ_v suddenly jumps to a higher value σ'_v. This implies that every value increment $v_{t+1} - v_t$ now has higher volatility, but the uncertainty in v_t, which is measured by the efficient volatility σ_t, stays the same.[11] Proposition 4 shows that the stationary efficient volatility changes to $\sigma'_* = \sigma'_v/\delta$, and the stationary bid-ask spread changes to $s'_* = 2\sigma'_v$. Therefore, the efficient density increases monotonically from the initial stationary value to the new stationary value, and the same is true for the bid-ask spread. Intuitively, a larger fundamental volatility increases overall adverse selection for the dealer, and as a result both the efficient density and the bid-ask spread eventually increase.

4.4 Public News

We now analyze an extension of the model in Section 2, in which the dealer receives news every period. Specifically, suppose that before each $t = 1, 2, \ldots$ the dealer receives a signal $\Delta s_t = s_t - s_{t-1}$ about the increment $\Delta v_t = v_t - v_{t-1}$:[12]

$$\Delta s_t \;=\; \Delta v_t + \Delta \eta_t, \quad \text{with} \quad \Delta \eta_t \;=\; \eta_t - \eta_{t-1} \overset{IID}{\sim} \mathcal{N}\big(\cdot, 0, \sigma_\eta\big). \tag{22}$$

Denote, respectively, by μ_t and σ_t the efficient mean and efficient volatility just before trading at $t = 0, 1, 2, \ldots$ (but after the signal Δs_t is observed). Note that this extension generalizes the model in Section 2: when σ_η approaches infinity, it is as if the dealer receives no signal at t. The next result generalizes Proposition 4.

Proposition 5. *For any $t = 0, 1, 2, \ldots$ the efficient mean and volatility satisfy*

$$\mu_{t+1} \;=\; \mu_t \pm \delta\sigma_t + \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2}\Delta s_{t+1}, \qquad \sigma_t^2 \;=\; \sigma_*^2 + \left(\sigma_0^2 - \sigma_*^2\right)(1 - \delta^2)^t, \tag{23}$$

[11] One can mix this type of shock with a shock to the efficient volatility σ_t, which was already analyzed.

[12] Alternatively, but perhaps less realistically, the dealer receives in each period t a signal about the level v_t. In this case, there is an upper bound for the dealer's uncertainty even if the informed share is very small. Hence, the efficient volatility is no longer increasing indefinitely with the informed share, and therefore the dynamic efficiency effect is reduced. As a result, the adverse selection effect dominates the dynamic efficiency effect, and the stationary bid-ask spread is increasing in the informed share.

where $\delta = g^{-1}(2\rho)$, as in equation (14), and

$$\sigma_* \;=\; \frac{\sigma_{v\eta}}{\delta} \quad \textit{with} \quad \sigma_{v\eta} \;=\; \frac{\sigma_v \sigma_\eta}{\sqrt{\sigma_v^2 + \sigma_\eta^2}}. \tag{24}$$

For any initial value σ_0 and any sequence of orders, the efficient volatility σ_t monotonically converges to σ_*, and the bid-ask spread monotonically converges to

$$s_* \;=\; 2\sigma_{v\eta}. \tag{25}$$

Proposition 5 is essentially the same result as Proposition 4, except that the fundamental volatility σ_v is replaced here by $\sigma_{v\eta}$. The parameter $\sigma_{v\eta}$ represents the increase in dealer uncertainty from t to $t+1$, conditional on her receiving the signal Δs_{t+1}.[13] When σ_η is zero, the dealer learns perfectly the increment Δv, hence even if the dealer does not know the initial value v_0, she ends up by learning v_t almost perfectly (she also learns about v_t from the order flow). When σ_η approaches infinity, the dealer receives uninformative signals, $\sigma_{v\eta}$ approaches σ_v, and the equilibrium behavior is described as in Proposition 4.

The stationary bid-ask spread s_* is twice the parameter $\sigma_{v\eta}$. Thus, the bid-ask spread is increasing in the news uncertainty parameter σ_η, and ranges from zero (when $\sigma_\eta = 0$) to $2\sigma_v$ (when $\sigma_\eta = \infty$). The relation between the bid-ask spread and σ_η is intuitive: with more imprecise news, the dealer is more uncertain about the asset value, and sets a larger stationary bid-ask spread.

Note that even in this more general context the stationary bid-ask spread s_* does not depend on the informed share ρ. The intuition is the same as for Proposition 4, and is discussed at the end of the proof of Proposition 5. This intuition is based on the general result (proved in Appendix C) that for any filtration problem in which the variance remains constant over time, the variance of the change in efficient mean must equal the fundamental variance. But the latter variance is independent of ρ, as is the variance of the signal Δs, hence the bid-ask spread is also independent of ρ.

[13] Indeed, its square $\sigma_{v\eta}^2$ is equal to the conditional variance $\mathsf{Var}(\Delta v_{t+1} | \Delta s_{t+1})$.

5 Equilibrium with Exact Bayesian Inference

In this section, we describe the evolution of the efficient density ϕ_t if the dealer is fully Bayesian and can therefore correctly compute the posterior density. We are interested in describing the average shape of the efficient density. To use the methods developed above, we assume that the initial density ϕ_0 is rapidly decaying. This implies by induction that the efficient density ϕ_t is always rapidly decaying. To simplify notation, we normalize the variables and densities involved in the previous formulas (normalization by the efficient volatility $\sigma_* = \sigma_v/\delta$):

$$\begin{aligned} \tilde{A}_t &= \frac{A_t - \mu_t}{\sigma_*}, \qquad \tilde{B}_t = \frac{B_t - \mu_t}{\sigma_*}, \qquad \tilde{v}_t = \frac{v_t - \mu_t}{\sigma_*}, \qquad \tilde{\phi}_t(\tilde{v}) = \sigma_* \phi_t(\mu_t + \sigma_* \tilde{v}), \\ \tilde{\Phi}_t(\tilde{v}) &= \int_{-\infty}^{\tilde{v}} \tilde{\phi}_t(w) \mathrm{d}w, \qquad \tilde{\psi}_t(\tilde{v}|\mathcal{O}_t) = \sigma_* \psi_t(\mu_t + \sigma_* \tilde{v}|\mathcal{O}_t). \end{aligned} \tag{26}$$

We call $\tilde{\phi}_t$ the *normalized efficient density*. We are interested in computing the average normalized efficient density over all possible future paths of the game.[14] We show that, at least numerically, this average is well defined and is approximately normal.

From equations (4) and (5), the normalized efficient density after a buy or a sell order is

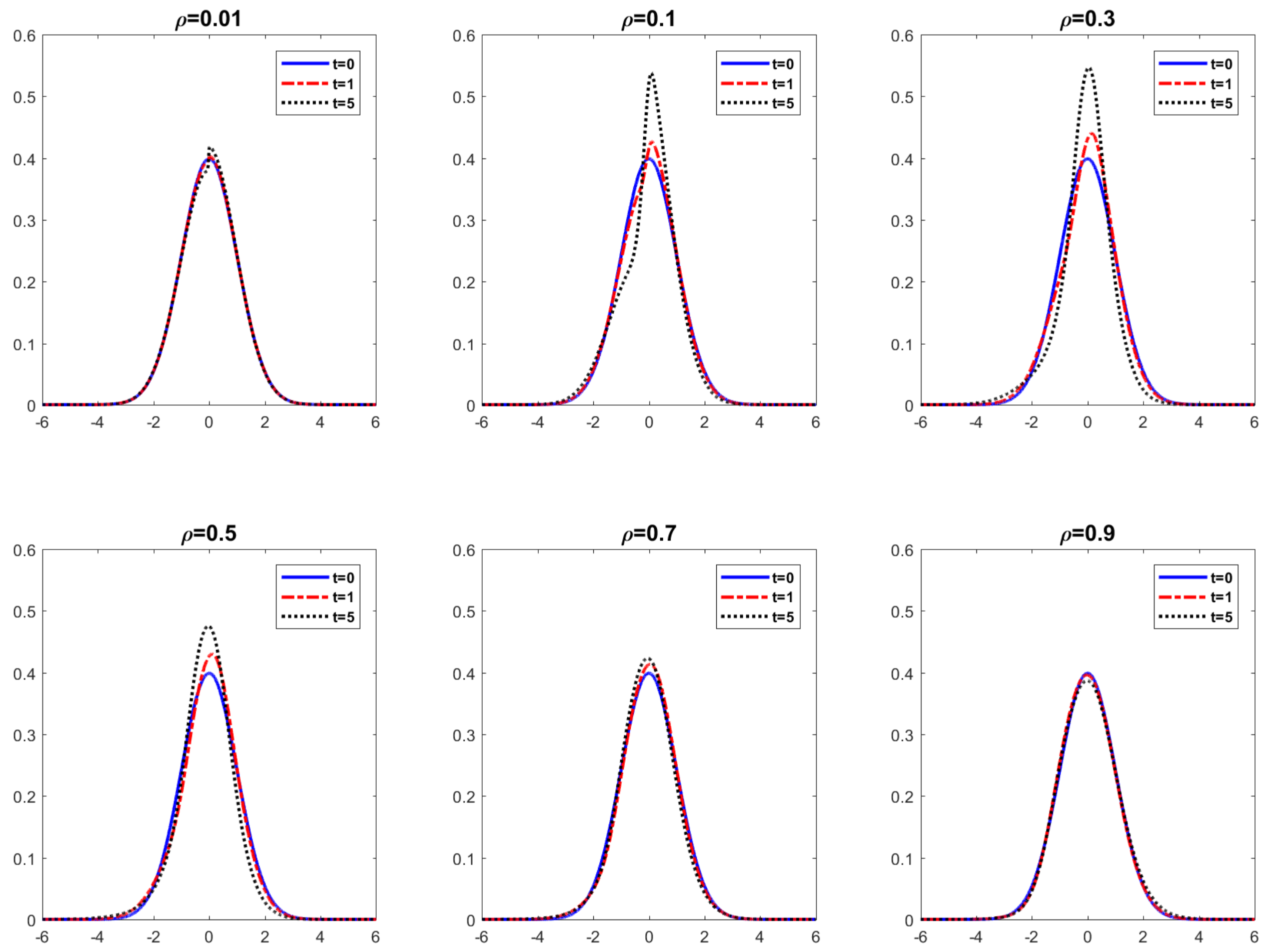
$$\begin{aligned} \tilde{\phi}_{t+1}(\tilde{w}|\mathrm{B}) &= \int_{-\infty}^{+\infty} \mathcal{N}\Big(\frac{\tilde{w} - \tilde{v} + \tilde{A}_t}{\delta}\Big) \frac{\big(\rho \mathbf{1}_{\tilde{v} > \tilde{A}_t} + \frac{\rho}{2} \mathbf{1}_{\tilde{v} \in [\tilde{B}_t, \tilde{A}_t]} + \frac{1-\rho}{2}\big) \cdot \tilde{\phi}_t(\tilde{v})}{\frac{\rho}{2}(1 - \tilde{\Phi}_t(\tilde{A}_t)) + \frac{\rho}{2}(1 - \tilde{\Phi}_t(\tilde{B}_t)) + \frac{1-\rho}{2}} \mathrm{d}\tilde{v}, \\ \tilde{\phi}_{t+1}(\tilde{w}|S) &= \int_{-\infty}^{+\infty} \mathcal{N}\Big(\frac{\tilde{w} - \tilde{v} + \tilde{B}_t}{\delta}\Big) \frac{\big(\rho \mathbf{1}_{\tilde{v} < \tilde{B}_t} + \frac{\rho}{2} \mathbf{1}_{\tilde{v} \in [\tilde{B}_t, \tilde{A}_t]} + \frac{1-\rho}{2}\big) \cdot \tilde{\phi}_t(\tilde{v})}{\frac{\rho}{2}\tilde{\Phi}_t(\tilde{A}_t) + \frac{\rho}{2}\tilde{\Phi}_t(\tilde{B}_t) + \frac{1-\rho}{2}} \mathrm{d}\tilde{v}, \end{aligned} \tag{27}$$

where the ask A_t and bid B_t are computed as in Proposition 2.

Figure 2 displays the efficient density after $t = 0$, $t = 1$, and $t = 5$ buy orders for various values of the informed share ρ. We notice by visual inspection that the efficient density is close to the standard normal density even after a sequence of 5 buy orders (this sequence happens with probability 2^{-5}, which is approximately 3.13%). The deviation of the efficient densities from the standard normal density is at its smallest level when the fraction of informed trading

[14] As we see in Internet Appendix (Sections 1 and 2), one expects a well defined stationary density for the continuous time Markov chain associated to our game. The only problem is that the fundamental value v_t is no longer stationary in our case, but follows a random walk. One can show that it is still possible to define a stationary density as long as one does not require it to integrate to one over v. But we are interested in the simpler problem of computing the marginal stationary density of $v_t - \mu_t$, which we solve numerically.

Figure 2: Exact Efficient Density after Series of buy orders. Each of the 6 plots represents the evolution of the normalized efficient density $\tilde{\phi}_t$ after $t = 0$, $t = 1$ and $t = 5$ buy orders. The initial normalized efficient density in all cases (at $t = 0$) is the standard normal density with mean zero and volatility one. The 6 plots correspond to the fraction of informed trading $\rho \in \{0.01, 0.1, 0.3, 0.5, 0.7, 0.9\}$.



ρ is either small or large, and it peaks for an intermediate value ρ near 0.2. When ρ is small, the order flow is uninformative, hence the posterior is not far from the prior. When ρ is large, the order flow is very informative, hence the posterior depends strongly on the increment, which is normally distributed.

Additionally, we compute the average *density* over 200 random series of length $t = 5$. The result for all ρ is almost indistinguishable from the standard normal density, giving another indication that the dealer is reasonable in making the standard normal approximation.

6 Conclusion

In this paper we have presented a dealer model in which the asset value follows a random walk. The stationary equilibrium of the model has novel properties. Our main finding is that the stationary bid-ask spread no longer depends on the informed share (the fraction of traders that are informed). This result is driven by two offsetting effects: (i) the traditional adverse selection effect: the dealer sets higher bid-ask spreads to protect from a larger number of informed traders, and (ii) the dynamic efficiency effect: the dealer learns faster from the order flow when there are more informed traders, and this reduces the bid-ask spread.

The non-stationary equilibria converge to the stationary equilibrium, regardless of the initial state. The evolution of the non-stationary equilibrium after various types of shocks provides additional testable implications of our model. For instance, after a positive shock to the informed share (e.g., if more informed investors start trading in that stock) the bid-ask spread jumps but then it decreases again to its stationary level. This type of liquidity resilience occurs purely for informational reasons, without any additional market maker jumping in to provide liquidity.

Appendix A. Formulas for Normal Densities

We show how to compute

$$I_n \;=\; \int_u u^n \phi(\alpha u + \beta)\phi(u), \qquad J_n \;=\; \int_u u^n \Phi(\alpha u + \beta)\phi(u). \tag{A1}$$

Note that $\phi'(u) = -u\phi(u)$. We now perform (i) direct computation for $n = 0$, and (ii) integration by parts to obtain recursive formulas for I_n and J_n.[15] We get

$$
\begin{aligned}
I_0 &= \frac{1}{\sqrt{\alpha^2+1}}\,\phi\left(\frac{\beta}{\sqrt{\alpha^2+1}}\right), \qquad & I_n &= \frac{(n-1)I_{n-2} - \alpha\beta I_{n-1}}{\alpha^2+1}, \\
J_0 &= \Phi\left(\frac{\beta}{\sqrt{\alpha^2+1}}\right), \qquad & J_n &= (n-1)J_{n-2} + \alpha I_{n-1},
\end{aligned}
\tag{A2}
$$

where $I_{-1} = J_{-1} = 0$. The formulas above imply

$$
I_1 = -\frac{\alpha\beta}{\alpha^2+1} I_0, \qquad J_1 = \alpha I_0, \qquad J_2 = J_0 - \frac{\alpha^2\beta}{\alpha^2+1} I_0. \tag{A3}
$$

We obtain

$$
J_0 = \Phi\left(\frac{\beta}{\sqrt{\alpha^2+1}}\right), \quad J_1 = \frac{\alpha}{\sqrt{\alpha^2+1}}\,\phi\left(\frac{\beta}{\sqrt{\alpha^2+1}}\right), \quad J_2 = J_0 - \frac{\alpha\beta}{\alpha^2+1} J_1. \tag{A4}
$$

Appendix B. Proofs of Results

Proof of Proposition 1. Using Bayes' rule, the posterior density of v_t after observing $\mathcal{O}$ is

$$
\psi_t(v|\mathcal{O}) = \frac{\mathsf{P}(\mathcal{O}_t = \mathcal{O} \mid v_t = v) \cdot \mathsf{P}(v_t = v)}{\int_v \mathsf{P}(\mathcal{O}_t = \mathcal{O} \mid v_t = v) \cdot \mathsf{P}(v_t = v)} = \frac{g_t(\mathcal{O}, v) \cdot \phi_t(v)}{\int_v g_t(\mathcal{O}, v) \cdot \phi_t(v)}. \tag{B1}
$$

Substituting $g_t(\mathcal{O}, v)$ from (2) and (3) in the above equation, we obtain (4).

Let $f(w, v) = \mathsf{P}(v_{t+1} = w | v_t = v) = \mathcal{N}(w - v, 0, \sigma_v)$ be the transition density of v_t. To compute the posterior density of v_t after observing $\mathcal{O}_t = \mathcal{O}$, note that

$$
\begin{aligned}
\phi_{t+1}(w|\mathcal{O}) &= \int_v \mathsf{P}(v_{t+1} = w \mid v_t = v, \mathcal{O}_t = \mathcal{O}) \cdot \mathsf{P}(v_t = v \mid \mathcal{O}_t = \mathcal{O}) \\
&= \int_v \mathsf{P}(v_{t+1} = w \mid v_t = v) \cdot \mathsf{P}(v_t = v \mid \mathcal{O}_t = \mathcal{O}) = \int_v f(w, v) \cdot \psi_t(v|\mathcal{O}),
\end{aligned}
\tag{B2}
$$

which proves (5).

[15] The formula for I_0 is computed by noticing that $\phi(u)$ and $\phi(\alpha u + \beta)$ are log-quadratic in u. The formula for J_0 is obtained by noticing that I_0 is the differential of J_0 with respect to β.

To simplify notation, we omit conditioning on the order $\mathcal{O}_t$. From (4), it follows that the posterior density ψ_t is equal to ϕ_t multiplied by a piecewise constant function. The prior density ϕ_t is rapidly decaying, hence it is bounded. Therefore ψ_t is also bounded and continuous, although it is no longer smooth. Nevertheless, when we convolute $\psi_t(\cdot)$ with $\mathcal{N}(\cdot, 0, \sigma_v)$ the result ϕ_{t+1} becomes smooth. Indeed, the N'th derivative $\mathrm{d}^N \phi_{t+1}(w)/\mathrm{d}w^N$ involves differentiating the smooth function $\mathcal{N}(w - v, 0, \sigma_v)$ under the integral sign. As the remaining term $\psi_t(v)$ is bounded, the integrals are well defined, and hence ϕ_{t+1} is a smooth function. The fact that ϕ_{t+1} is also rapidly decaying can be seen in the same way, using again the fact that ψ_t is bounded. □

Proof of Corollary 1. By definition of the ask-bid pair, A_t is the mean of the posterior density of v_t after observing a buy order at t. But the increment $v_{t+1} - v_t$ has zero mean and is independent of the previous variables until t. Therefore, A_t is also the mean of the posterior density of v_{t+1} after observing a buy order at t. Similarly, B_t is the mean of the posterior density of v_{t+1} after observing a sell order at t. This proves the equations in (7). □

Proof of Proposition 2. Define the following function:[16]

$$H_t(v) \ = \ \int_{-\infty}^{v} w\phi_t(w)\mathrm{d}w \ = \ v\Phi_t(v) - \int_{-\infty}^{v} \Phi_t(w)\mathrm{d}w. \tag{B3}$$

Note that $H_t(-\infty) = 0$ and $H_t(+\infty) = \int_{-\infty}^{\infty} w\phi_t(w)\mathrm{d}w = \mu_t$. Also, note that

$$\Theta_t(v) \ = \ \mu_t\Phi_t(v) - H_t(v). \tag{B4}$$

To prove the desired equivalence, start with an ask-bid pair (A_t, B_t). This pair must satisfy the dealer's pricing conditions: A_t is the mean of $\psi_t(\cdot|\mathrm{B})$, and B_t is the mean of $\psi_t(\cdot|\mathrm{S})$. Using

[16] In the formula for H_t we use integration by parts, and also the fact that $\lim_{v\to-\infty} v\Phi_t(v) = 0$. To prove this last fact, suppose $v = -x$ with $x > 0$. Since ϕ_t is rapidly decaying, $\phi_t(-x) < Cx^{-3}$ for some constant C. Then $x\Phi_t(-x) = x\int_{-\infty}^{-x} \phi_t(w)\mathrm{d}w < x\frac{Cx^{-2}}{2}$, which implies $\lim_{x\to\infty} x\Phi_t(-x) = 0$.

the formulas in (4) for $\psi_t(v|\mathcal{O})$, we compute

$$\begin{aligned} A_t &= \frac{\rho\big(\mu_t - H_t(A_t)\big) + \frac{\rho}{2}\big(H_t(A_t) - H_t(B_t)\big) + \frac{1-\rho}{2}\mu_t}{\frac{\rho}{2}(1-\Phi_t(A_t)) + \frac{\rho}{2}(1-\Phi_t(B_t)) + \frac{1-\rho}{2}}, \\ B_t &= \frac{\rho H_t(B_t) + \frac{\rho}{2}\big(H_t(A_t) - H_t(B_t)\big) + \frac{1-\rho}{2}\mu_t}{\frac{\rho}{2}\Phi_t(A_t) + \frac{\rho}{2}\Phi_t(B_t) + \frac{1-\rho}{2}}. \end{aligned} \tag{B5}$$

Using (B4), we compute the following differences:

$$\begin{aligned} A_t - \mu_t &= \frac{\frac{\rho}{2}\Theta_t(A_t) + \frac{\rho}{2}\Theta_t(B_t)}{\rho(1-\Phi_t(A_t)) + \frac{\rho}{2}(\Phi_t(A_t) - \Phi_t(B_t)) + \frac{1-\rho}{2}}, \\ \mu_t - B_t &= \frac{\frac{\rho}{2}\Theta_t(A_t) + \frac{\rho}{2}\Theta_t(B_t)}{\rho\Phi_t(B_t) + \frac{\rho}{2}(\Phi_t(A_t) - \Phi_t(B_t)) + \frac{1-\rho}{2}}. \end{aligned} \tag{B6}$$

As Θ_t is strictly positive everywhere (see Footnote 9), we have the following inequalities: $A_t > \mu_t > B_t$, or equivalently $A_t \in (\mu_t, +\infty)$ and $B_t \in (-\infty, \mu_t)$. The equations (B6) can be written as

$$F(A_t, B_t) \;=\; 0, \qquad G(A_t, B_t) \;=\; 0, \tag{B7}$$

where the functions F and G are defined in (8). Conversely, suppose we have a solution (A_t, B_t) of (B7), with $A_t > \mu_t > B_t$. Then, this pair satisfies the equations in (B6), which are the dealer's pricing conditions. Thus, (A_t, B_t) is an ask-bid pair.

We now show that a solution of (B7) exists. The partial derivatives of F and G are

$$\begin{aligned} \frac{\partial F}{\partial A} &= -\frac{\Theta_t(A) + \Theta_t(B)}{(A-\mu_t)^2}, \qquad & \frac{\partial F}{\partial B} &= \frac{A-B}{A-\mu_t}\,\phi_t(B), \\ \frac{\partial G}{\partial A} &= -\frac{A-B}{\mu_t - B}\,\phi_t(A), \qquad & \frac{\partial G}{\partial B} &= \frac{\Theta_t(A) + \Theta_t(B)}{(\mu_t - B)^2}. \end{aligned} \tag{B8}$$

From (8) we see that $F(A,B)$ has well defined limits at $B = \pm\infty$, which follows from the formulas: $\Theta_t(\pm\infty) = 0$, $\Phi_t(-\infty) = 0$, and $\Phi_t(+\infty) = 1$. Thus we extend the definition of F for all $B \in \bar{\mathbb{R}} = [-\infty, +\infty]$. Now fix $B \in \bar{\mathbb{R}}$. We show that there is a unique solution $A = \alpha(B)$ of the equation $F(A,B) = 0$. From (B8) we see that $\frac{\partial F}{\partial A} < 0$ for all $A \in (\mu_t, \infty)$. From (8) we see that when $A \searrow \mu_t$, $F(A,B) \nearrow \infty$; while when $A \nearrow \infty$, $F(A,B) \searrow -\frac{1+\rho}{\rho} + 1 + \Phi_t(B) = -\frac{1}{\rho} + \Phi_t(B) < 0$ (recall that $\rho \in (0,1)$). Thus, for any B there is a unique solution of $F(A,B) = 0$ for $A \in (\mu_t, \infty)$. Denote this unique solution by $\alpha(B)$. Differentiating the equation $F\big(\alpha(B), B\big) = 0$ implies that for all B the derivative of $\alpha(B)$ is $\alpha'(B) = -\frac{\partial F}{\partial B}(\alpha(B), B) / \frac{\partial F}{\partial A}(\alpha(B), B) > 0$. Define $\underline{A} = \alpha(-\infty)$ and $\overline{A} = \alpha(\mu_t)$. The results above imply that both $\underline{A}$ and $\overline{A}$ belong to (μ_t, ∞), and α is a bijective function between $[-\infty, \mu_t]$ and $[\underline{A}, \overline{A}]$.

A similar analysis shows that for all $A \in \bar{\mathbb{R}}$, there is a unique solution $B = \beta(A)$ of the equation $G(A,B) = 0$. Moreover, the function β is increasing, and if we define $\underline{B} = \beta(\mu_t)$ and $\overline{B} = \beta(\infty)$, it follows that both $\underline{B}$ and $\overline{B}$ belong to $(-\infty, \mu_t)$, and the function α is bijective between $[\mu_t, \infty]$ and $[\underline{B}, \overline{B}]$.

Next, define the function $f : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ by

$$f(A) \;=\; \alpha\big(\beta(A)\big). \tag{B9}$$

Consider the set

$$S \;=\; \{(A,B) \mid A - f(A) \;=\; 0 \;,\; B \;=\; \beta(A)\}. \tag{B10}$$

It is straightforward to show that S coincides with the set of all ask-bid pairs. Indeed, $(A,B) \in S$ is equivalent to $A = \alpha(B)$ and $B = \beta(A)$, which, from the discussion above, is equivalent to $F(A,B) = 0$ and $G(A,B) = 0$. Therefore, the existence of an ask-bid pair is equivalent to there being at least one solution of $A - f(A) = 0$.

We now show that the equation $A - f(A) = 0$ has at least one solution. The function $f(A)$ is increasing and bijective between $[\mu_t, \infty]$ and $[\alpha(\underline{B}), \alpha(\overline{B})]$. As $\underline{B}, \overline{B} \in (-\infty, \mu_t)$, it follows that $[\alpha(\underline{B}), \alpha(\overline{B})] \subset (\underline{A}, \overline{A}) \subset (\mu_t, \infty)$. When $A \searrow \mu_t$, $A - f(A) \to \mu_t - \alpha(\underline{B}) < 0$, while when $A \nearrow \infty$, $A - f(A) \to \infty - \alpha(\overline{B}) > 0$. Thus, there exists a solution of $A - f(A) = 0$ on (μ_t, ∞).

Finally, define

$$A_t \;=\; \inf\Big\{ A \in (\mu_t, \infty) \;\Big|\; A - f(A) = 0 \Big\}, \qquad B_t \;=\; \beta(A_t). \tag{B11}$$

As f is continuous, A_t also satisfies $A_t - f(A_t) = 0$, hence among all possible ask-bid pairs the ask closest to μ_t is attained at A_t. $\square$

Proof of Proposition 3. Consider the following useful formulas, that are true for any v, w, μ_t, σ_t, σ_v:

$$\begin{aligned}\mathcal{N}(w-v,0,\sigma_v)\cdot\mathcal{N}(v,\mu_t,\sigma_t) &= \mathcal{N}(v,M_t(w),S_t)\cdot\mathcal{N}(w,\mu_t,s_t), \quad \text{with} \\ M_t(w) = \frac{\mu_t\sigma_v^2+w\sigma_t^2}{\sigma_t^2+\sigma_v^2}, \qquad S_t &= \frac{\sigma_t\sigma_v}{\sqrt{\sigma_t^2+\sigma_v^2}}, \qquad s_t = \sqrt{\sigma_t^2+\sigma_v^2},\end{aligned} \tag{B12}$$

From equation (5), the exact density is $\phi_{t+1}(w|\mathrm{B}) = \int_v \mathcal{N}(w-v,0,\sigma_v)\cdot\psi_t(v|\mathrm{B})$. Using equations (4) and (B12), we compute

$$\begin{aligned}\phi_{t+1}(w|\mathrm{B}) &= \mathcal{N}(w,\mu_t,s_t)\,\frac{\int_v\left(\rho\mathbf{1}_{v>A_t}+\frac{\rho}{2}\mathbf{1}_{v\in[B_t,A_t]}+\frac{1-\rho}{2}\right)\cdot\mathcal{N}(v,M_t(w),S_t)}{\frac{\rho}{2}(1-\Phi_t(A_t))+\frac{\rho}{2}(1-\Phi_t(B_t))+\frac{1-\rho}{2}} \\ &= \frac{1}{s_t}\,\phi\Big(\frac{w-\mu_t}{s_t}\Big)\,\frac{\frac{\rho}{2}\Phi\big(-\frac{A_t-M_t(w)}{S_t}\big)+\frac{\rho}{2}\Phi\big(-\frac{B_t-M_t(w)}{S_t}\big)+\frac{1-\rho}{2}}{\frac{\rho}{2}\Phi\big(-\frac{A_t-\mu_t}{\sigma_t}\big)+\frac{\rho}{2}\Phi\big(-\frac{B_t-\mu_t}{\sigma_t}\big)+\frac{1-\rho}{2}} \\ &= \frac{1}{s_t}\,\phi\Big(\frac{w-\mu_t}{s_t}\Big)\,\frac{\Phi\big(\frac{M_t(w)-A_t}{S_t}\big)+\Phi\big(\frac{M_t(w)-B_t}{S_t}\big)+\frac{1-\rho}{\rho}}{\Phi\big(\frac{\mu_t-A_t}{\sigma_t}\big)+\Phi\big(\frac{\mu_t-B_t}{\sigma_t}\big)+\frac{1-\rho}{\rho}}.\end{aligned} \tag{B13}$$

Similarly, we compute

$$\phi_{t+1}(w|\mathrm{S}) = \frac{1}{s_t}\,\phi\Big(\frac{w-\mu_t}{s_t}\Big)\,\frac{\Phi\big(\frac{A_t-M_t(w)}{S_t}\big)+\Phi\big(\frac{B_t-M_t(w)}{S_t}\big)+\frac{1-\rho}{\rho}}{\Phi\big(\frac{A_t-\mu_t}{\sigma_t}\big)+\Phi\big(\frac{B_t-\mu_t}{\sigma_t}\big)+\frac{1-\rho}{\rho}}. \tag{B14}$$

We now compute the first and second moments of the posterior efficient density $\phi_{t+1}(w|\mathrm{B})$. We have the following formulas with the change of variables:

$$u = \frac{w-\mu_t}{s_t} \quad \Longrightarrow \quad \frac{M_t(w)-A_t}{S_t} = \alpha u+\beta, \quad \text{with} \quad \alpha = \frac{\sigma_t}{\sigma_v}, \quad \beta = \frac{\mu_t-A_t}{S_t}. \tag{B15}$$

We observe that the moments of $\phi_{t+1}(w,\mathrm{B})$ are all of the form

$$I_n = \int_u u^n\phi(\alpha u+\beta)\phi(u), \qquad J_n = \int_u u^n\Phi(\alpha u+\beta)\phi(u), \tag{B16}$$

where $\phi(\cdot)$ is the standard normal density, and $\Phi(\cdot)$ is the standard cumulative normal density.

In Appendix A, we show that

$$J_0 \;=\; \Phi\left(\frac{\beta}{\sqrt{\alpha^2+1}}\right), \quad J_1 \;=\; \frac{\alpha}{\sqrt{\alpha^2+1}}\,\phi\left(\frac{\beta}{\sqrt{\alpha^2+1}}\right), \quad J_2 \;=\; J_0 - \frac{\alpha\beta}{\alpha^2+1}J_1. \qquad \text{(B17)}$$

Equation (B15) implies the following formulas

$$\frac{1}{\sqrt{\alpha^2+1}} \;=\; \frac{\sigma_v}{s_t}, \qquad \frac{\beta}{\sqrt{\alpha^2+1}} \;=\; \frac{\mu_t - A_t}{\sigma_t}. \qquad \text{(B18)}$$

Define the normalized ask and bid:

$$a_t \;=\; \frac{A_t - \mu_t}{\sigma_t}, \qquad b_t \;=\; \frac{B_t - \mu_t}{\sigma_t}. \qquad \text{(B19)}$$

The mean and the variance of the posterior efficient density $\phi_{t+1,\mathrm{B}}$ are

$$\mu_{t+1,\mathrm{B}} \;=\; \int w\phi_{t+1}(w|\mathrm{B})\mathrm{d}w, \qquad \sigma^2_{t+1,\mathrm{B}} \;=\; \int (w-\mu_{t+1,\mathrm{B}})^2\phi_{t+1}(w|\mathrm{B})\mathrm{d}w. \qquad \text{(B20)}$$

After the change of variables $w = \mu_t + s_t u$, the posterior mean satisfies

$$\mu_{t+1,\mathrm{B}} \;=\; \mu_t + s_t\int u\phi_{t+1}(w|\mathrm{B})s_t\mathrm{d}u. \qquad \text{(B21)}$$

Using the formula for J_1 in (B17), we compute

$$\mu_{t+1,\mathrm{B}} \;=\; \mu_t + \sigma_t\,\frac{\phi(-a_t)+\phi(-b_t)}{\Phi(-a_t)+\Phi(-b_t)+\frac{1-\rho}{\rho}}. \qquad \text{(B22)}$$

Similarly, we compute the posterior mean conditional on a sell order:

$$\mu_{t+1,\mathrm{S}} \;=\; \mu_t - \sigma_t\,\frac{\phi(a_t)+\phi(b_t)}{\Phi(a_t)+\Phi(b_t)+\frac{1-\rho}{\rho}}, \qquad \text{(B23)}$$

Equation (7) implies that the ask is $A_t = \mu_{t+1,\mathrm{B}}$ and the bid is $B_t = \mu_{t+1,\mathrm{S}}$. If we normalize these equations, we have $a_t = \frac{\mu_{t+1,\mathrm{B}}-\mu_t}{\sigma_t}$ and $b_t = \frac{\mu_{t+1,\mathrm{S}}-\mu_t}{\sigma_t}$. We thus obtain the following equations for the normalized ask and bid:

$$a_t \;=\; \frac{\phi(-a_t)+\phi(-b_t)}{\Phi(-a_t)+\Phi(-b_t)+\frac{1-\rho}{\rho}}, \qquad b_t \;=\; -\frac{\phi(a_t)+\phi(b_t)}{\Phi(a_t)+\Phi(b_t)+\frac{1-\rho}{\rho}}, \tag{B24}$$

We now show that this system has a unique solution. We use the notation from the proof of Proposition 2, adapted to this particular case. For $(a,b)\in(0,\infty)\times(-\infty,0)$, define $F(a,b)=\frac{\phi(a)+\phi(b)}{a}-\Phi(-a)-\Phi(-b)-\frac{1-\rho}{\rho}$ and $G(a,b)=\frac{\phi(a)+\phi(b)}{-b}-\Phi(a)-\Phi(b)-\frac{1-\rho}{\rho}$. As in the proof of Proposition 2, for $b\in[-\infty,0]$ define $\alpha(b)$ as the unique solution of $F(\alpha(b),b)=0$; and for $a\in[0,\infty]$ define $\beta(a)$ as the unique solution of $G(a,\beta(a))=0$. For $a\in(0,\infty)$, define $f(a)=\alpha(\beta(a))$. We show that any solution a of the equation $a-f(a)=0$ must satisfy $a<1$. Let $b=\beta(a)$. Since $a=\alpha(b)$, by definition $F(a,b)=0$. As in the proof of Proposition 2, one shows that F is decreasing in a, and that $F(0,b)=+\infty>0$. As $b<0$, $F(1,b)=\phi(1)-\Phi(-1)+\phi(b)-\Phi(-b)-\frac{1-\rho}{\rho}<\phi(1)-\Phi(-1)+\phi(0)-\Phi(0)\approx-0.0177<0$. As $F(a,b)=0$ and F is decreasing in a (and a is positive), we have just proved that $a\in(0,1)$. A similar argument (adapted to the function G) shows that $b=\beta(a)\in(-1,0)$.

As in equation (B9), define $f(a)=\alpha\big(\beta(a)\big)$. We need to show that the equation $a-f(a)=0$ has a unique solution in $(0,\infty)$. By contradiction, suppose there are at least two solutions $a_1<a_2$, and suppose a_1 is the smallest such solution and a_2 the largest. As f is continuous and takes values in some compact interval $[\underline{b},\overline{b}]$ (see the proof of Proposition 2), a_1 and a_2 are well defined. Also, since f is increasing, f is a bijection of $[a_1,a_2]$. The argument above then shows that both a_1 and a_2 are in $(0,1)$. If we prove that $f'<1$ on $[a_1,a_2]$, it follows that $a-f(a)$ is increasing on $[a_1,a_2]$ and cannot therefore be equal to zero at both ends. This contradiction therefore proves uniqueness, as long as we show that indeed $f'<1$ on $(0,1)$. Let $a\in(0,1)$ and denote $b=\beta(a)$ and $a'=\alpha(b)$. Then by the chain rule $f'(a)=\alpha'(b)\beta'(a)$. Differentiating the equations $F(\alpha(b),b)=0$ and $G(a,\beta(a)=0$, we have $\alpha'(b)=\frac{\phi(b)(a'-b)a'}{\phi(a')+\phi(b)}$ and $\beta'(b)=\frac{\phi(a)(a-b)(-b)}{\phi(a)+\phi(b)}$. Both these derivatives are of the form $\frac{\phi(x_1)(x_1+x_2)x_1}{\phi(x_1)+\phi(x_2)}$ with $x_1,x_2\in(0,1)$. This function is increasing in x_2, hence it is smaller than $\frac{\phi(x_1)(x_1+1)x_1}{\phi(x_1)+\phi(1)}$, which is increasing in x_1, hence smaller than one, which is the value corresponding to $x_1=1$. Thus, $f'<1$ on $(0,1)$ and the uniqueness is proved.

We now search for the solution. Because of symmetry, we suspect that $a_t=-b_t$. If we impose this condition, we have $\Phi(a_t) + \Phi(b_t) = \Phi(-a_t) + \Phi(-b_t) = 1$. Therefore, we need to solve the equation $a_t = 2\rho\phi(a_t)$ for $a_t > 0$, or equivalently $g(a_t) = 2\rho$, where $g(x) = \frac{x}{\phi(x)}$. As the derivative of ϕ is $\phi'(x) = -x\phi(x)$, the derivative of g is $g'(x) = \frac{1+x^2}{\phi(x)} > 0$ for all x. Moreover, $g(0) = 0$ and $\lim_{x\to\infty} g(x) = \infty$, hence g is increasing and a one-to-one and mapping of $(0, \infty)$. Thus, if we define $\delta = g^{-1}(2\rho)$, which is the same formula as in (14), we have $g(\delta) = 2\rho$. The solution of (B24) is then

$$a_t \;=\; -b_t \;=\; \delta, \qquad \text{or} \qquad A_t \;=\; \mu_t + \delta\sigma_t, \quad B_t \;=\; \mu_t - \delta\sigma_t. \tag{B25}$$

Thus, the posterior mean satisfies

$$\mu_{t+1,\mathrm{B}} \;=\; \mu_t + \delta\sigma_t, \qquad \mu_{t+1,\mathrm{S}} \;=\; \mu_t - \delta\sigma_t, \tag{B26}$$

which proves the first part of equation (13). After the change of variables $w = \mu_t + s_t u$, the posterior variance satisfies

$$\begin{aligned} \sigma^2_{t+1,\mathcal{O}_t} \;&=\; \int (w - \mu_t - (\mu_{t+1,\mathcal{O}_t} - \mu_t))^2 \phi_{t+1}(w|\mathcal{O}_t)\mathrm{d}w \\ &=\; (\mu_{t+1,\mathcal{O}_t} - \mu_t)^2 - 2(\mu_{t+1,\mathcal{O}_t} - \mu_t)^2 + s_t^2 \int u^2 \phi_{t+1}(w|\mathcal{O}_t) s_t \mathrm{d}u. \end{aligned} \tag{B27}$$

Using the formula for J_2 in (B17), we compute $\int u^2 \phi_{t+1}(w|\mathcal{O}_t)s_t\mathrm{d}u = 1$ for $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$. From $s_t^2 = \sigma_t^2 + \sigma_v^2$, it follows that the posterior variance satisfies

$$\sigma^2_{t+1,\mathcal{O}_t} \;=\; \sigma_t^2 + \sigma_v^2 - (\mu_{t+1,\mathcal{O}_t} - \mu_t)^2. \tag{B28}$$

From equation (B26), we get $(\mu_{t+1,\mathcal{O}_t} - \mu_t)^2 = \delta^2\sigma_t^2$ for $\mathcal{O}_t \in \{\mathrm{B}, \mathrm{S}\}$. Therefore, the posterior variance satisfies

$$\sigma^2_{t+1,\mathrm{B}} \;=\; \sigma^2_{t+1,\mathrm{S}} \;=\; (1 - \delta^2)\sigma_t^2 + \sigma_v^2, \tag{B29}$$

which proves the second part of equation (13). □

Proof of Proposition 4. Recall that the function $g : [0, \infty) \to [0, \infty)$ is increasing and $\delta =$

$g^{-1}(2\rho)$, with $\rho \in (0,1)$. Hence, $\delta < g^{-1}(2) \approx 0.647$, and in particular $\delta < 1$. Equation (B29) implies that the efficient variance evolves according to $\sigma_t^2 = (1-\delta^2)\sigma_{t-1}^2 + \sigma_v^2$ for any $t \geq 0$ (by convention, $\sigma_{-1} = 0$). Iterating this equation, we obtain $\sigma_t^2 = (1-\delta^2)^t \sigma_0^2 + \frac{1-(1-\delta^2)^t}{\delta^2}\sigma_v^2$. Using $\sigma_* = \frac{\sigma_v}{\delta}$, we obtain $\sigma_t^2 = \sigma_*^2 + (1-\delta^2)^t(\sigma_0^2 - \sigma_*^2)$, which proves (16). As $\delta \in (0,1)$, it is clear that σ_t^2 converges monotonically to σ_*^2 for any initial value σ_0. The bid-ask spread satisfies $s_t = 2\sigma_t\delta$, hence it converges to $2\sigma_*\delta = 2\frac{\sigma_v}{\delta}\delta = 2\sigma_v = s_*$. □

Proof of Corollary 2. Following the proof of Proposition 4, recall that g is increasing on $(0,\infty)$. Its inverse g^{-1} is therefore also increasing, and $\sigma_* = \sigma_v / g^{-1}(2\rho)$ is decreasing in ρ. The dependence on σ_v is straightforward. □

Proof of Corollary 3. Conditional on the information at t, each order (buy or sell) is equally likely. Therefore, the change in the efficient mean $\mu_{t+1,\mathcal{O}_t} - \mu_t$ has a binary distribution with probability 1/2, which has standard deviation equal to σ_v, which is the fundamental volatility. □

Proof of Corollary 4. Equation (B29) shows that the efficient variance σ_t^2 evolves according to $\sigma_{t+1}^2 = (1-\delta^2)\sigma_t^2 + \sigma_v^2$. Taking the limit on both sides, we get $\sigma_*^2 = (1-\delta^2)\sigma_*^2 + \sigma_v^2$. Subtracting the two equations above, we get $\sigma_{t+1}^2 - \sigma_*^2 = (1-\delta^2)(\sigma_t^2 - \sigma_*^2)$, which proves the speed of convergence formula (21) for the efficient variance. As $\sigma_t^2 - \sigma_*^2 = (\sigma_t - \sigma_*)(\sigma_t + \sigma_*)$, the formula (21) is true for the efficient volatility as well. Finally, the bid-ask spread is $s_t = 2\delta\sigma_t$, which proves (21) for the bid-ask spread. □

Proof of Proposition 5. The only difference from the setup of Section 2 is that after trading at t (but before trading at $t+1$) the dealer receives a signal $\Delta s_{t+1} = \Delta v_{t+1} + \Delta\eta_{t+1}$. By notation, just before trading at t, v_t is distributed as $\mathcal{N}(\cdot, \mu_t, \sigma_t)$. We thus follow the proof of Propositions 3 and 4, and infer that after trading at t the dealer regards v_t to be distributed as $\mathcal{N}(\cdot, \mu_t', \sigma_t')$, where $\mu_t' = \mu_t \pm \delta\sigma_t$ and $\sigma_t'^2 = (1-\delta^2)\sigma_t^2$.[17] After observing $\Delta s_{t+1} = \Delta v_{t+1} + \Delta\eta_{t+1}$, the dealer computes $\mathsf{E}\big(\Delta v_{t+1} | \Delta s_{t+1}\big) = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2}\Delta s_{t+1}$ and $\mathsf{Var}\big(\Delta v_{t+1} | \Delta s_{t+1}\big) = \frac{\sigma_v^2\sigma_\eta^2}{\sigma_v^2 + \sigma_\eta^2} = \sigma_{v\eta}^2$. Hence, after observing the signal, the dealer regards v_{t+1} to be distributed as $\mathcal{N}(\cdot, \mu_{t+1}, \sigma_{t+1})$,

[17] The sign $\pm$ is plus if a buy order is submitted at t, and minus if a sell order is submitted at t.

with

$$\mu_{t+1} \;=\; \mu'_t + \frac{\sigma_v^2}{\sigma_v^2+\sigma_\eta^2}\Delta s_{t+1}, \qquad \sigma_{t+1}^2 \;=\; \sigma_t'^2 + \sigma_{v\eta}^2 \;=\; (1-\delta^2)\sigma_t + \sigma_{v\eta}^2. \tag{B30}$$

The recursive equation for σ_t is the same as (B29), except that instead of σ_v we now have $\sigma_{v\eta}$. Then, the same proof as in Propositions 3 and 4 can be used to derive all the desired results.

Note that equation (23) implies that the change in efficient mean is $\Delta\mu_{t+1} = \pm\delta\sigma_t + \frac{\sigma_v^2}{\sigma_v^2+\sigma_\eta^2}\Delta s_{t+1}$. Thus, in the stationary equilibrium, $\mathsf{Var}(\Delta\mu_{t+1}) = \delta^2\sigma_*^2 + \frac{\sigma_v^4}{(\sigma_v^2+\sigma_\eta^2)^2}(\sigma_v^2+\sigma_\eta^2) = \sigma_{v\eta}^2 + \frac{\sigma_v^4}{\sigma_v^2+\sigma_\eta^2} = \sigma_v^2 = \mathsf{Var}(\Delta v_{t+1})$. This verifies the result in Appendix C that in any stationary filtration problem the variance of the change in efficient mean must equal the fundamental variance. Moreover, the half spread is equal to $\delta\sigma_* = \sigma_{v\eta}$, which does not depend on the informed share ρ. □

Appendix C. Stationary Filtering

We show that in a filtration problem that is stationary (in a sense to be defined below) the variance of value changes is the same as the variance of the efficient mean changes. Let v_t be a discrete time random walk process with constant volatility σ_v. Suppose each period the market gets (public) information about v_t. Let $\mathcal{I}_t$ be the public information set available at time t. Denote by $\mu_t = \mathsf{E}(v_t|\mathcal{I}_t) = \mathsf{E}_t(v_t)$ the efficient mean at time t, i.e., the expected asset value given all public information. This filtration problem is called *stationary* if the efficient variance is constant over time:

$$\mathsf{Var}_t(v_t) = \mathsf{Var}_{t+1}(v_{t+1}). \tag{C1}$$

The next result gives a necessary and sufficient for the filtration problem to be stationary.

Proposition 6. *The filtration problem is stationary if and only if*

$$\mathsf{Var}(v_{t+1} - v_t) \;=\; \mathsf{Var}(\mu_{t+1} - \mu_t).$$

Proof. Since $\mu_t = \mathsf{E}_t(v_t)$, we have the decomposition $v_t = \mu_t + \eta_t$, where η_t is orthogonal on the information set $\mathcal{I}_t$. Moreover, $\mathsf{Var}(\eta_t) = \mathsf{Var}_t(v_t)$. Similarly, $v_{t+1} = \mu_{t+1} + \eta_{t+1}$, and

$\mathsf{Var}(\eta_{t+1}) = \mathsf{Var}_{t+1}(v_{t+1})$. Thus, the stationary condition reads $\mathsf{Var}(v_{t+1} - \mu_{t+1}) = \mathsf{Var}(v_t - \mu_t)$. We can decompose $v_{t+1} - \mu_t$ in two ways:

$$\begin{aligned} v_{t+1} - \mu_t &= (v_{t+1} - \mu_{t+1}) + (\mu_{t+1} - \mu_t) \\ &= (v_{t+1} - v_t) + (v_t - \mu_t). \end{aligned} \tag{C2}$$

We verify that these are orthogonal decompositions. The first condition is that $\mathsf{cov}(v_{t+1} - \mu_{t+1}, \mu_{t+1} - \mu_t) = 0$, i.e., that $\mathsf{cov}(\eta_{t+1}, \mu_{t+1} - \mu_t) = 0$. But η_{t+1} is orthogonal on $\mathcal{I}_{t+1}$, which contains μ_{t+1} and μ_t. The second condition is that $\mathsf{cov}(v_{t+1} - v_t, v_t - \mu_t) = 0$. But v_t has independent increments, so $v_{t+1} - v_t$ is independent of v_t and anything contained in the information set at time t. (This is true as long as the market does not get at t information about the asset value at a future time.)

The total variance of the two orthogonal decompositions in (C2) must be the same, hence $\mathsf{Var}(v_{t+1} - \mu_{t+1}) + \mathsf{Var}(\mu_{t+1} - \mu_t) = \mathsf{Var}(v_{t+1} - v_t) + \mathsf{Var}(v_t - \mu_t)$. But being stationary is equivalent to $\mathsf{Var}(v_{t+1} - \mu_{t+1}) = \mathsf{Var}(v_t - \mu_t)$, which is then equivalent to $\mathsf{Var}(v_{t+1} - v_t) = \mathsf{Var}(\mu_{t+1} - \mu_t)$.

□

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