The Impact of Credit Risk on Equity Options

A Preprint

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Abstract

The aim of this work is to understand and measure to what extent equity options price credit risk. With the exception of Toft and Prucyk (1997), which is a dated work based on the simplistic assumption of a reference company issuing perpetual debt, all the work in the literature which try to establish a link between the credit default swaps and equity options rely on reduced-form models (Carr and Linetsky 2006, Carr and Wu 2010, Carr et al. 2010). Thus, the connection between the firm's fundamentals and the pricing of these derivative contracts are ignored. With a novel structural model, which stems from Geske (1977, 1979), equity is priced as a compound call option written on the firm's asset. The proposed structural model is first able to reconcile some puzzling findings documented by Carr and Wu (2017) on increasing leverage due to increasing business risk. Moreover, as the price equity options is shown to depends on the probability of the option expiring in-the-money, conditional on the firm's survival, this decomposition allows to construct a new measure of impact of credit risk on options. It is then shown that put options, contrary to calls, are sensitive to changes in the default risk in the underlying company. In addition, this measure is able to forecast future changes of the negative skew of long-term maturity options written on the equity of the same company. Finally, I show that the implied volatilities estimated á la Black-Scholes tend to average out the effect of credit risk over the moneyness space, leading to potential biases when applied for risk management purposes.

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MSC classification: 91G20, 91G40, 91G50

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1 Introduction

Markets for both stock options and credit derivatives have experienced a significant growth in the last decades. Along with the rapid growth, academics have started investigating a possible link between stock option implied volatilities and credit default swaps (CDS) spreads. In fact, when a company experiences credit-rating downgrades, its equity inevitably drops by a sizable amount. As a result, the possibility of default induces negative skewness in the probability distribution of stock returns. This negative skewness is manifested in the relative pricing of stock options across different strikes: when the Black and Scholes (1973) implied volatility is plotted against moneyness at fixed maturities, the slope of the

graph is positively related to the risk-neutral skewness of the stock return distribution. This phenomenon is commonly referred in the literature as the leverage effect.

The aim of this work is to exploit this overlapping information on the market risk and the credit risk of a company to provide better identification of the dynamics of the stock return variance and default events, and how these impact equity option prices. Ultimately, this allows to measure the impact of credit risk on each equity option contract and shed light on the mechanism leading to future movements of the option skew due to credit-related events.

The attempts in the literature to extract information from debt-related instruments has mostly focus on bond prices. I instead choose to use CDS spread to infer the term structure of implied probabilities of default. This is motivated by the fact that CDS spreads constitute a more direct and clean signal for the underlying default risk. In fact, CDS spreads provide relatively pure pricing of the default event of the underlying entity as they are typically traded on standardised terms. In fact, unlike bonds, CDSs have a constant maturity, the underlying instrument is always par valued, they concentrate liquidity in one instrument, and are not affected by different taxation regimes; also, bond spreads are more likely to be affected by differences in contractual arrangements, such as differences in seniority, coupon rates, embedded options, and guarantees. Secondly, many corporate bonds are bought by investors who simply hold them to maturity, and the secondary market liquidity is therefore often poor. Furthermore, shorting bonds is even more difficult in the cash market as the repo market for corporate bond is often illiquid, and the tenor of the agreement is usually very short. CDS contracts instead allow investors to implement trading strategies to hedge credit risk over a longer period of time at a known cost. Finally, as shown by Blanco et al. (2005), CDS spreads tend to respond more quickly than bond spreads to changes in credit conditions in the short run.

The first works to point out a possible effect of leverage and credit risk on equity option are Black (1976) and Christie (1982). They both argue that the possibility by the company of defaulting on its obligations can induce negative skewness in the company's return distribution. This manifests when the implied volatility is plotted against a measure of moneyness and exhibits a decreasing pattern for increasing strike prices (the so-called negative volatility skew) rather than a flat line. The skew is often observed in the region where the implied volatility is estimated using out-of-the-money put options, being the latter intrinsically affected the most by credit risk. More recent empirical works, such as Collin-Dufresne et al. (2001), Elton et al. (2001), Cremers, Driessen and Meanhout (2008), Cremers, Driessen, Meanhout and Weinbaum (2008), and Cao et al. (2010), show that CDS and bond spreads are positively correlated with both stock option implied volatility levels and the skew of the implied volatility plotted against moneyness. Also, Campbell and Taksler (2003), and Ericsson et al. (2009) document a link between bond spreads and equity historical volatility. At the aggregate level, similar results hold for sovereign CDS spreads (Carr and Wu 2007), and credit default swap index (CDX) spreads and synthetic collateralised debt obligations (CDOs) on the same index (Collin-Dufresne et al. 2012).

The relative pricing of equity and debt related derivatives instruments has been mainly explored using reduced-form models of default. Carr and Linetsky (2006), Carr et al. (2010) and Carr and Wu (2010) develop joint frameworks of valuation for credit-sensitive derivatives contracts and equity options. Their estimations highlight the interaction between market risk (return variance) and credit risk (default arrival) in pricing stock options and credit default swaps. They also point out the need of developing future models that integrate both markets, rather than having separate valuation models. This work tries to bridge this gap providing a unique structural valuation framework where both the price of CDSs and options written on the same reference entity are driven by a unique state variable, namely the firm's asset value, which ultimately determines the default event properties.

On the other side of the spectrum, the use of structural model for jointly modelling credit and equity derivative contracts has not been exhaustively explored. With the exception of Toft and Prucyk (1997), which built on the Leland (1994) model to document the effect of leverage on the pricing of options, and Hull et al. (2004), which develop a new calibration methodology based on options to implement the Merton (1974) model, there have not been significant attempts to develop structural models of default able to transmit the company's credit risk to the pricing of equity options. Here, the firm is allowed to issue multiple bonds with different maturities, thus removing the restriction on

perpetual debt (as in Leland 1994) or a unique zero-coupon bond (as in Merton 1974). The work of Geske et al. (2016) moves towards this direction whilst investigating capital structure effects on the pricing of equity option. However, their work does not measure the extent to which leverage and credit risk impact the pricing of options which is assessed in this paper. Also, they focus on call options only, which intuitively should be affected by credit risk the least, whilst here the different impact of credit risk on both call and put options is extensively studied.

This work is also inspired by the findings in Carr and Wu (2011, 2017). Carr and Wu (2011) show that under suitable assumptions the price of equity put options struck deepest out-of-the-money (DOOM) is entirely driven by the default possibility of the reference entity. Also, they show that the contract values of credit-sensitive instruments and put options share similar magnitudes and show strong co-movements; furthermore, when the estimates from the two markets deviate from each other, the deviations predict future movements in both markets due to the future convergence.

On the other hand, Carr and Wu (2017) is a crucial piece of research in interpreting and assessing the extent to which results on the volatility skew obtained herein could be driven by factors other than the leverage effect. In their article, the authors identify three different channels able to generate the negative volatility skew documented in equity options. They show that the option skew can be driven, as expected, by increasing leverage but also by volatility feedback and self-exciting market disruptions. They also find that, contrary to conventional wisdom, financial leverage does not always decline with increased business risk: financial leverage can be positively correlated with business risk when the increase in risk is due to small, diffusive market movements. The proposed model, which accounts for stochastic diffusive movements only, is able to explain this apparently counter-intuitive finding.

In addition, this work serves as an investigation of the above-mentioned link between debt and equity markets, focusing on the relative pricing of the corresponding derivative contracts: credit default swaps for the debt and stock-options for the equity side. The theory employed for pricing equity options is based on Merton (1974), Geske (1977), and Geske (1979) where the equity is considered as a contingent claim on the firm's value. Here, having the firm issued more than one bond, equity turns into a n-fold compound option (where n is the number of bond outstanding). To the best of my knowledge, this is the first work that uses the theory of n-fold compound options for pricing vanilla equity options. In fact, the proposed structural model allows to derive pricing formulae for equity options, once a novel calibration technique is carried through using equity prices and model-free estimates of the risk-neutral survival probabilities extracted from the CDS spreads. It is worth mentioning that, with the exception of Vassalou and Xing (2004), Forte and Lovreta (2012) and Ericsson et al. (2015), none of the contributions on structural models of default use calibrations on stock prices, whilst opting for calibrations on bond prices/spreads or option-implied volatilities (which are usually model-dependent).

The rest of the paper is structured as follows: Section 2 introduces the value of the equity as a n-fold compound option and extend the pricing to equity options; Section 3 gives a description of the data and the calibration methodology; in Section 4 a new measure of impact of credit risk on equity options is introduced and the different impact of credit risk on calls and puts is investigated. Furthermore, it is shown that changes in the options' negative skew is driven by credit risk for long-term maturity options only. Some robustness test are also conducted, allowing to also investigate the integration of the CDS and option markets. Finally, Section 5 concludes.

2 Firm's Claims as Compound Options

The given structural model of default allows to price equity, debt and options written on equity and is inspired by Merton (1974) and Geske (1977). It extends the Merton model as it allows the firm to have issued a sequence of bonds, with different face values, maturities and coupon rates. It also borrows the intuition of Geske (1977) to construct and modify the definition of the default events. In Geske (1977), the equityholders are given the option to default on coupon payments as there is only one bond outstanding paying discrete coupons; in this work instead the optionality activates when the face value of the bonds outstanding are due and should be paid back by the company. In the model proposed

herein, shareholders can default on coupon payments only indirectly via a lower risk-neutral drift in the firm asset value process.

Thereafter, the pricing of equity is addressed first, and eventually the same mechanism is extended to the pricing of equity options.

2.1 Equity as a n – fold Compound Option on Asset Value

Consider a firm which has issued n bonds and equity, both receiving payments in the form of coupons and dividends. According to the indenture of the bonds: (1) the firm promises to repay each bond, with face value F_i, to the bondholders at known times $t_i \in (t_0, t_n]$, $i \in I := \{1, \dots, n\}$; (2) in case of default, the bondholders immediately take over the company and the shareholders receive nothing; (3) the firm cannot issue any senior or equivalent rank claims on the firm nor do share repurchases before t_n.

The firm refinances each bond payment with equity, and bankruptcy occurs when the firm fails to make the reimbursement payment because it is unable to issue new equity. If the whole outstanding debt is finally repaid, the firm is either liquidated, and the shareholders receive any remaining value as lump-sum liquidating dividend, or it refinances with another sequence of bonds. Usual assumptions in terms of transaction cost, taxes, bid/ask spreads, short-selling and indivisibility of assets apply.

For convenience of notation, set $t_0 := 0$ and denote the generic payoff at time t_i as $X_{t_i} := X_i$. Let V, S and D represent the firm's assets, equity and debt respectively. According to the structural approach and the Modigliani-Miller theorem, both equity and debt are function of the firm's assets and not vice versa (see Merton 1977). Also, I fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and assume no-arbitrage conditions in the economy. Under certain technical conditions, there exist a risk-neutral probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that the gain process associated with any admissible trading strategy deflated by the risk-free rate is a martingale[1]. Furthermore, the following notation for the (risk-free) discount factor

$$D(t_i, t_j) = \frac{B_i}{B_j} = \exp\left(-\int_{t_i}^{t_j} r_s \,\mathrm{d}s\right),$$

is used, being $B_t = \exp\left(\int_0^t r_s \,\mathrm{d}s\right)$ the value of the money-market account at time t and r_t a (possibly stochastic) positive function of time.

Within this framework, at each reimbursement time t_i, the shareholders decide on whether it is worthwhile to pay F_i (in the sense explained below). The optionality is embedded in the continuation value of the equity (i.e. the expected present value of future uncertain cash-flows in case of not defaulting): if at time t_i the latter is higher than F_i, the bond is paid off and the firm is kept alive; otherwise bondholders declare bankruptcy. In the case the bond is repaid, the same optionality occurs at the next payment date, t_{i+1}, and so on until the last payment date, t_n. This sequence of 'optional' decisions leads to a compound option model.

Hence, the default time is defined as

$$\tau := \inf_{i \in I} \{t_i : S_i^\star(V) < F_i\} \tag{1}$$

where $S_i^\star(V)$ is the continuation value of equity. That is, the default time is defined as the first time at which the value of the equity is lower than the face value of the bond; by definition, the firm can default only at discrete points in time.

[1]Without further assumptions, the market is incomplete as both the firm's equity and debt are contingent claims on the firm's asset which is a non-tradable asset (therefore, the replicating portfolio cannot be constructed). In order to implement the model, as described in Section 3, a calibration on the equity and options is in fact required (options are not redundant assets). However, as shown in Ericsson and Reneby (2012), if the interest rate market is complete and the firm's equity is traded, completeness can be achieved (thus leading to a unique measure $\mathbb{Q}$). In fact, considering a Black and Scholes (1973) economy, as the value of equity options can be replicated by trading the stock and a riskless bond, the converse il also true: the payoff of the stock can be replicated by trading the option and the bond. Similarly, as in the structural approach of default equity is an option on the firm value, the payoff of the latter can be replicated by trading the firm's equity and the risk-free bond.

Being the value of the equity a function of the firm value, the default times can be re-expressed as events in the asset value space: for each value of equity which triggers default corresponds one and one only value of the firm assets, namely $\bar{V}_i$ at time t_i, which implies (1), that is

$$\tau = \inf_{i \in I} \left\{ t_i : V_i < \bar{V}_i \right\}.$$

The default barrier $\left(\bar{V}_i\right)_{i \in I}$ can be interpreted as a latent sequence of thresholds embedded into the firm's capital structure and riskiness. This sequence of values is found recursively starting from the default threshold in t_n which coincides with F_n as in the Merton model. The other values of the barrier are calculated as the solution of an integral equation, where the dimension of the integrals increases alongside with the number of bond outstanding (i.e., given n bond outstanding, $n-1$ integral equation must be solved, being the last integral to be solved an $n-1$ – dimensional integral). For further details on the estimation of the default barrier, see Appendix E.

More specifically, both the present and the continuation value of the equity can be calculated as the risk-neutral expectation of their terminal payoffs. The terminal value of the equity can be expressed as

$$S_n(V) = V_n \mathbb{1}_{\tau > t_n} - \sum_{k=i+1}^{n} \frac{F_k}{D\left(t_k, t_n\right)} \mathbb{1}_{\tau > t_k}.$$

The interpretation of the payoff function is straightforward: equity holders receive the asset value in t_n (if the firm has been able to repay all its outstanding debt), net of all the future reimbursements (if the firm has survived at each default point).

The continuation value of the equity at time $t_i \in [0, t_n]$ is given by the present value of the expected payoff of the equity, before having checked for the potential default occurring at t_i, that is

$$S_i^{\star}(V) = \mathbb{E}_i^{\mathbb{Q}} \left[D\left(t_i, t_n\right) S_n(V) \right], \tag{2}$$

where the expectation is taken under the risk-neutral measure. As a consequence, the value of the equity is given by

$$S_i(V) = \max \left\{ S_i^{\star}(V) - F_i, 0 \right\}. \tag{3}$$

See Figure 1 for a visual representation of the continuation and actual value of the equity. Under (1), (2) can be further expressed in terms of events in equity space and, ultimately, in the asset value space. Therefore

$$\begin{aligned} S_i^{\star}(V) &= \mathbb{E}_i^{\mathbb{Q}} \left(D\left(t_i, t_n\right) V_n \mathbb{1}_{\bigcap_{h=i+1}^{n} \left\{ S_h^{\star}(V) \geq F_h \right\}} \right) - \sum_{k=i+1}^{n} F_k \mathbb{E}_i^{\mathbb{Q}} \left(D\left(t_i, t_k\right) \mathbb{1}_{\bigcap_{h=i+1}^{k} \left\{ S_h^{\star}(V) \geq F_h \right\}} \right) \\ &= \mathbb{E}_i^{\mathbb{Q}} \left(D\left(t_i, t_n\right) V_n \mathbb{1}_{\bigcap_{h=i+1}^{n} \left\{ V_h \geq \bar{V}_h \right\}} \right) - \sum_{k=i+1}^{n} F_k \mathbb{E}_i^{\mathbb{Q}} \left(D\left(t_i, t_k\right) \mathbb{1}_{\bigcap_{h=i+1}^{k} \left\{ V_h \geq \bar{V}_h \right\}} \right). \end{aligned} \tag{4}$$

Notice that (4) is the most general expression for the continuation value of the equity. So far, no distributional assumptions have been made on the process driving the asset value nor on the form of the discount factor. The asset value process could be a Lévy process, as well as a process with continuous paths and stochastic volatility; similarly, the discount factor could be assumed stochastic. However, Frey and Sommer (1998) showed that compound option problems such as in Geske (1977), Geske (1979) and Geman et al. (1995) cannot be solved in a semi-closed form under stochastic interest rates nor stochastic volatility. Consequently, in order to preserve analytical tractability, a positive constant continuously compounded risk-free rate r is assumed throughout. Also, a geometric Brownian motion is considered for the asset value process, that is

$$\mathrm{d}V_t = (r - \varpi) V_t \, \mathrm{d}t + \sigma_V V_t \, \mathrm{d}W_t^{\mathbb{Q}}, \tag{5}$$

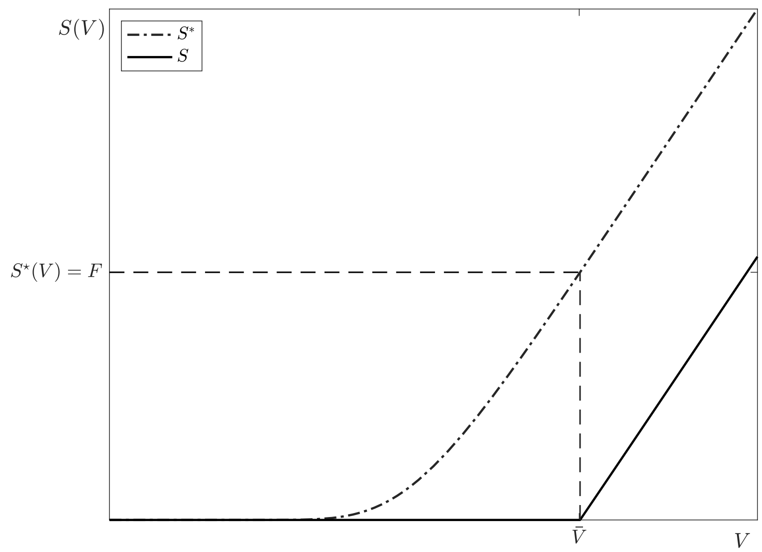


Figure 1: Continuation value (dashed and dotted line) and value (solid line) of the equity. Checking whether the continuation value of the equity $S^\star$ is greater than the face value of the bond F is equivalent to finding the value of the assets V greater than the default threshold $\bar{V}$.

where ϖ is the continuously compounded payout rate (reflecting both dividends and coupon payments), σ_V the instantaneous volatility of the assets, and $W_t^{\mathbb{Q}}$ a $\mathbb{Q}$-standard Brownian motion.

Defining the events $\mathcal{V}_{i,k} := \bigcap_{h=i+1}^{k} \left\{V_h \geq \bar{V}_h\right\}$, the t_i-continuation value of the equity can be written as

$$S_i^\star(V) = e^{-r(t_n - t_i)}\mathbb{E}_i^{\mathbb{Q}}\left(V_n \mathbb{1}_{\mathcal{V}_{i,n}}\right) - \sum_{k=i+1}^{n} e^{-r(t_k - t_i)} F_k \mathbb{E}_i^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{V}_{i,k}}\right), \tag{6}$$

and for $t_i = t_0$, it follows

$$S_0(V) = e^{-rt_n}\mathbb{E}^{\mathbb{Q}}\left(V_n \mathbb{1}_{\mathcal{V}_n}\right) - \sum_{k=1}^{n} e^{-rt_k} F_k \mathbb{Q}\left(\mathcal{V}_k\right), \tag{7}$$

where $\mathcal{V}_k \equiv \mathcal{V}_{0,k}$, for $k \in I$. Notice that the t_0-continuation value of the equity and the contemporaneous value of the equity coincide as no debt is due in t_0 (i.e. $F_0 = 0$). In order to derive an analytical solution, a change of measure as in Geman et al. (1995) is performed, such that $\mathbb{M} \sim \mathbb{Q}$, with

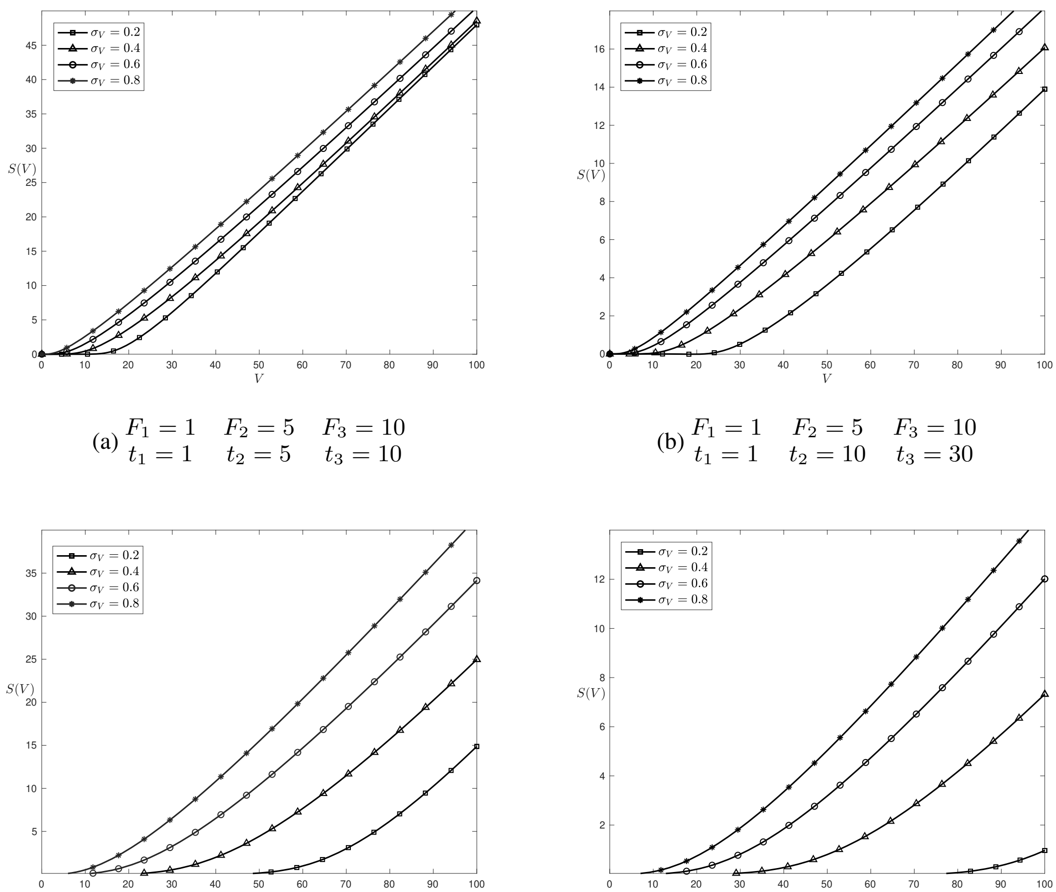
$$\left.\frac{\mathrm{d}\mathbb{M}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{V_t e^{\varpi t}}{V_0 B_t} = \exp\left(\sigma_V W_t^{\mathbb{Q}} - \frac{\sigma_V^2}{2} t\right).$$

The measure $\mathbb{M}$ is referred as the firm-value fund measure thereafter. Setting $t = t_n$, it follows

$$S_0(V) = e^{-\varpi t_n} V_0 \mathbb{M}\left(\mathcal{V}_n\right) - \sum_{k=1}^{n} e^{-rt_k} F_k \mathbb{Q}\left(\mathcal{V}_k\right).$$

In order to compute the probabilities under $\mathbb{M}$ and $\mathbb{Q}$, the result in Theorem C.1 in Appendix C is needed. Notice that the proposed change of measure is not strictly necessary to derive the value of the equity; it is only used as a useful mean to solve the model more easily[2]. Using those results, the two probabilities can expressed as the following

[2]Alternatively, the first expectation in (7) can be expressed in terms of truncated log-normal integrals using Theorem C.2 in Appendix C.



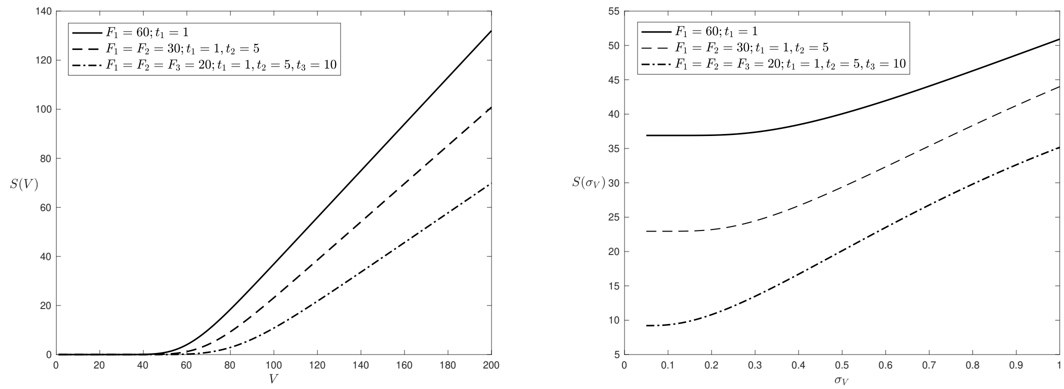
(a) $F_1 = 1 \quad F_2 = 5 \quad F_3 = 10$ / $t_1 = 1 \quad t_2 = 5 \quad t_3 = 10$

(b) $F_1 = 1 \quad F_2 = 5 \quad F_3 = 10$ / $t_1 = 1 \quad t_2 = 10 \quad t_3 = 30$

(c) $F_1 = 5 \quad F_2 = 10 \quad F_3 = 50$ / $t_1 = 1 \quad t_2 = 5 \quad t_3 = 10$

(d) $F_1 = 5 \quad F_2 = 10 \quad F_3 = 50$ / $t_1 = 1 \quad t_2 = 10 \quad t_3 = 30$

Figure 2: Equity as a function of the firm value for different sets of parameters ($r = 0.03$, $\varpi = 0.05$). F_i denotes the face value of debt due at time t_i, for $i = \{1, 2, 3\}$. Panels (a) to (d): impact of different levels of leverage and debt maturities on value of the equity (for a given σ_V, i.e. the riskiness of the firm). Safer firms (lower debt, shorter maturities: Panel (a)) are least affected by changes in σ_V. Progressively riskier firms (lower debt, longer maturities: Panel (b); larger debt, shorted maturities: Panel (c); larger debt, longer maturities: Panel (d)) display significant differences in the value of equity for different levels of riskiness. Firms in Panel (d) are the most sensitive to changes in σ_V.



(a) Asset value

(b) Asset volatility

Figure 3: The Figures show how clustering the whole firm debt at different maturities affect the value of equity (as a function of the asset value, as in Panel (a), and as a function of the asset volatility, as in Panel (b). Spreading the same debt level over longer maturities lowers the value of the equity given the greater uncertainty on the firm being able to repay the bonds at farther dates away in the future.

multivariate Gaussian integrals

$$S_0(V) = e^{-\varpi t_n} V_0 \Phi_n\left(\mathbf{d}^{\mathbb{M}}; \mathbf{\Gamma}_n\right) - \sum_{k=1}^{n} e^{-r t_k} F_k \Phi_k\left(\mathbf{d}_k^{\mathbb{Q}}; \mathbf{\Gamma}_k\right) \tag{8}$$

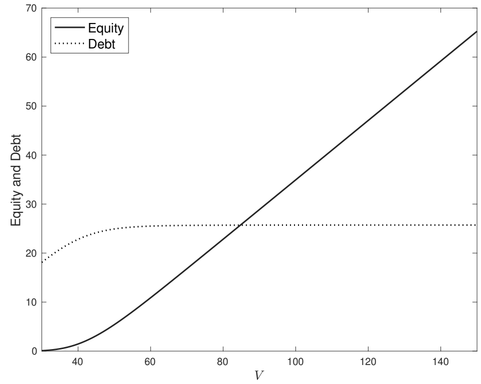
where $\mathbf{d}^{\mathbb{M}} := \left(d_i^{\mathbb{M}}\right)_{1\le i\le n}$ and $\mathbf{d}_k^{\mathbb{Q}} = \left(d_i^{\mathbb{M}} - \sigma_V \sqrt{t_i}\right)_{1\le i\le k}$ with

$$d_i^{\mathbb{M}} = \frac{\ln\left(V_0/\bar{V}_i\right) + \left(r - \varpi + \sigma_V^2/2\right) t_i}{\sigma_V \sqrt{t_i}}, \qquad \mathbf{\Gamma}_k = \begin{pmatrix} 1 & \sqrt{\frac{t_1}{t_2}} & \sqrt{\frac{t_1}{t_3}} & \dots & \sqrt{\frac{t_1}{t_k}} \\ & 1 & \sqrt{\frac{t_2}{t_3}} & \dots & \sqrt{\frac{t_2}{t_k}} \\ \dots & \dots & \dots & \dots & \dots \\ & & & 1 & \sqrt{\frac{t_{k-1}}{t_k}} \\ & & & & 1 \end{pmatrix}.$$

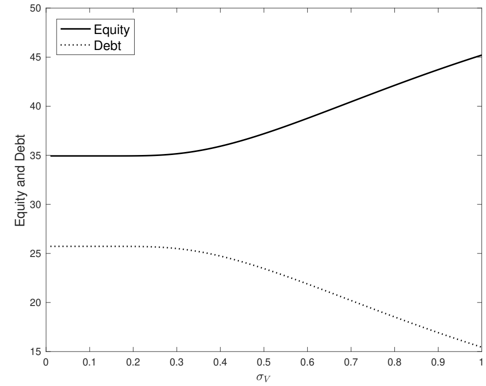
If $\varpi = 0$, then (8) is isomorphic to the results in Geske (1977). If also $n = 1$, the model coincides with the Merton model.

Figure 2 shows how the value of the equity changes for different sets of parameters, when equity is a compound option (the model is the 3–fold compound option which is also used throughout the empirical investigation). The motivations for selecting $n = 3$ are discussed in Section 5. Remarkably, equity is an increasing function of the asset volatility (as expected, being a compound call option). Firms with relatively lower debt and shorter maturities are the least affected by changes in the riskiness of the firms (see Panel(a)). Progressively riskier firms (lower debt, longer maturities: Panel (b); larger debt, shorted maturities: Panel (c); larger debt, longer maturities: Panel (d)) display significant differences in the value of equity for different levels of riskiness. Also, the effect of progressively larger asset volatility on equity is magnified for highly-levered firms with larger fraction of long-term debt (Panel (d)).

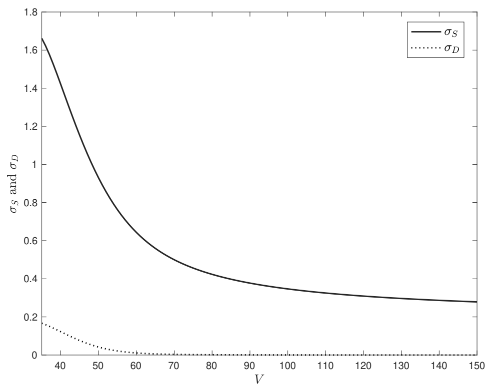
Furthermore, Figure 3 shows how the value of the equity is affected by the maturity of the firm's debt. Given the same total face value of the bonds outstanding, firms whose debt is concentrated at the first payment date are the ones whose equity is valued the most. This result is intuitive and is due to the fact that companies whose debt is spread further into the future are more exposed to uncertainty (i.e. they have more chances of defaulting). Therefore, the today-price of the equity incorporates this future risk.



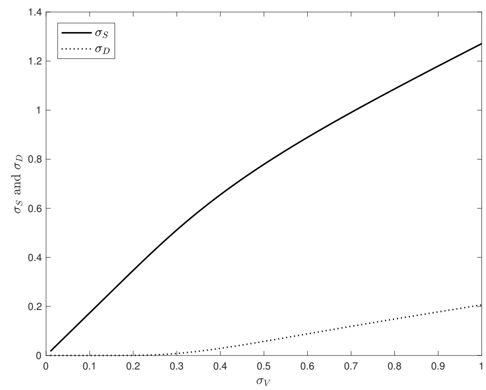
(a) S and D as functions of V



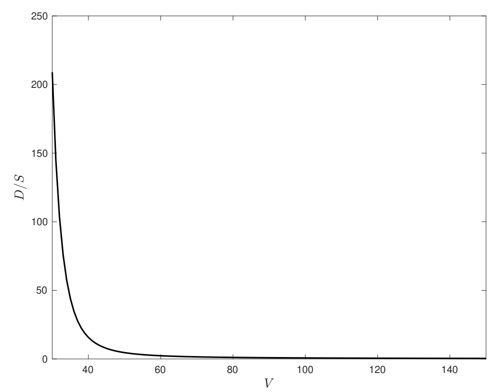
(b) S and D as functions of σ_V



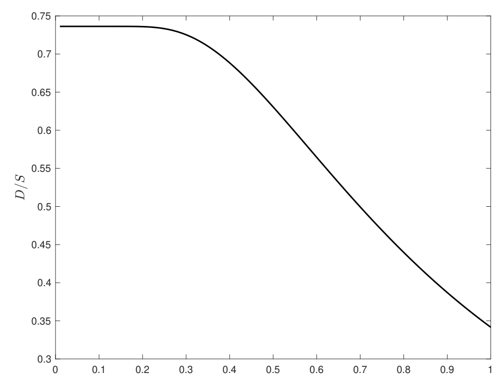
(c) σ_S and σ_D as functions of V



(d) σ_S and σ_D as functions of σ_V



(e) D/S as function of V



(f) D/S as function of σ_V

Figure 4: $F_1 = F_2 = F_3 = 10$; $t_1 = 1$, $t_2 = 5$, $t_3 = 10$, $r = 0.03$, $\varpi = 0.05$. When leverage is fixed (Panels (a), (c), (e)) the firm's riskiness is set to $\sigma_V = 0.2$; when the firm's riskiness is fixed (Panels (b), (d), (f)), the value of the asset is set to $V = 100$. Panel (a) shows that both equity and debt are increasing functions of V, with equity being convex whilst debt being concave. Panel (b) shows both equity and debt as functions of the firm's riskiness, with equity being an increasing and convex function, whilst debt being decreasing and concave. Panel (c) displays the volatility of equity and debt as function of leverage, showing σ_D far less sensitive to leverage than σ_S, with the latter also always larger. Panel (d) analyses the same volatilities with respect to the asset volatility: both are increasing in the volatility of the assets, despite being σ_V more affected. Panel (e) shows how leverage decreases as the value of the assets increases and panel (f) shows that leverage tend to decrease with larger level of asset volatility as the the numerator of D/S tends to decrease while the denominator increases.

To conclude, the stochastic properties of the process driving the value of the equity are discussed. By the virtue of Itô's Lemma, the value of the equity does not follow a geometric Brownian motion but instead a process that I refer as Stochastic Elasticity of Variance (SEV). As a matter of fact, it can be shown that

$$\mathrm{d}S_t = \alpha_S^{\mathbb{Q}}(V_t,t)S_t\,\mathrm{d}t + \sigma_V S_t^{\beta(V_t,t)}\,\mathrm{d}W_t^{\mathbb{Q}}, \tag{9}$$

with

$$\alpha_S^{\mathbb{Q}}(V_t,t) := \frac{1}{S_t}\left(\frac{\partial S}{\partial t} + \frac{\partial S}{\partial V}(r-\varpi)V_t + \frac{1}{2}\frac{\partial^2 S}{\partial V^2}\sigma_V^2 V_t^2\right) \quad \text{and} \quad \beta\left(V_t,t\right) := 1 + \frac{\ln \mathrm{El}_V(S_t)}{\ln S_t}$$

where $\mathrm{El}_V(S_t) := \frac{\partial S}{S_t}/\frac{\partial V}{V_t}$ is the elasticity of the firm's equity with respect to the asset value. This model closely resembles the Cox (1996) Constant Elasticity of Variance (CEV). Moreover, the volatility of equity is stochastic and given by

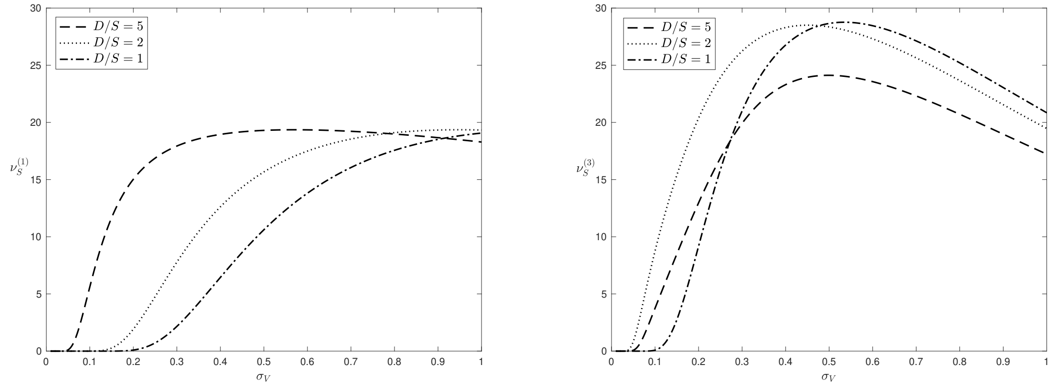
$$\sigma_{S,t} = \sigma_V \Delta_S^{(n)} \frac{V_t}{S_t} \tag{10}$$

where $\Delta_S^{(n)}$ is the sensitivity of the equity with respect to changes of the asset value (as equity is an option, it is the 'Delta' of the equity). Analytical expressions of $\Delta_S^{(n)}$ – which depends on the number of bond outstanding n – are available in Appendix F. It is worth highlighting that the process does not only have stochastic volatility, but it is also a model of local volatility in the sense of Dupire (1994) as it depends on the current level of the equity. Therefore, the model driving equity returns is a local-stochastic volatility model (for further details on this class of models, see Henry-Labordère 2009).

To some extent the process in (9) produces similar trajectories to the reduced-form default-extended CEV process in Carr and Linetsky (2006). The main difference between the two approaches is that their process is only able to statistically reproduce the patterns caused by the leverage effect, while here the statistical properties of the SEV are directly attributable to the firm's capital structure.

On a different note, a compound option default model is also able to explain some of the 'puzzling' findings in Carr and Wu (2017). The authors document that, contrary to conventional wisdom, financial leverage does not always decline with increased business risk. Instead, financial leverage can increase if the risk increase is due to small, diffusive market movements. Also Cremers, Driessen, Meanhout and Weinbaum (2008) document a negative correlation between market volatilities and credit spread (and, therefore, default probabilities) performing panel regressions. Given some suitable and reasonable assumptions, these findings can be reconciled with the predictions of my model, and are discussed below.

If the volatility of the company increases, every structural model à la Merton would predict that the value of its equity increases too as displayed in Figure 4 (b). Also, in line with common sense, Figure 4 (a), (c) and (e) shows that the firm equity and debt are increasing in the asset value (but they display opposite concavity), and that the volatility of equity and debt both tend to decline as the firm becomes safer due to a larger asset value. The same applies for the debt to equity ratio. On the other hand, panels (d) and (f) can help shedding some light on the counter-intuitive findings in Carr and Wu (2017). If the choice of rebalancing the capital structure to hit a targeted leverage ratio is not modelled (i.e. the capital structure is insensitive to changes in the business risk), the company's financial leverage drops (panel (f)). Instead, in a model where the firm can react to changes in its business risk and adjust its capital structure accordingly, as the riskiness of the firm increases, it would be optimal to take on more debt given the rise in equity so that the leverage ratio remains constant. Also, it would make sense to take in a larger fraction of debt than the percentage increase of the value of equity, being debt less volatile than equity (panel (d)). Herein, the company is not allowed to rebalance its capital structure targeting a leverage ratio though, as the main focus is to study the impact of credit risk on equity options, not to explain leverage ratios in response to changes in business risk. Thus, the findings in Carr and Wu (2017) of increasing leverage for increasing diffusive volatility can be easily reconciled with a model in which shareholders maximise the firm value, based on a targeted leverage ratio, and where equity is seen as a compound option.



(a) Merton model

(b) Compound option model

Figure 5: Sensitivity of equity with respect to asset volatility under the Merton (panel (a)) and a compound option model (panel (b)) ($r = 0.03$, $\varpi = 0.05$). In panel (a): $F_1 = 50$, $t_1 = 1$; in panel (b): $F_1 = F_2 = F_3 = 50/3$; $t_1 = 1$, $t_2 = 5$, $t_3 = 10$. Panel (a) shows how $\nu_S^{(1)}$ reacts to changes in the riskiness of the firm: it is optima for equity holders to increase leverage, as this makes equity more sensitive to further changes in volatility. Panel (b) shows how $\nu_S^{(3)}$ reacts to increasing business risk: pushing leverage is optimal as long as the sensitivity of equity starts decreasing. The sensitivity of a highly-levered (dashed) company falls between the sensitivities of the medium-levered (dotted) and low-levered (dotted-dashed) companies for plausible value of $\sigma_V \in (0, 0.3)$ p.a..

Furthermore, in order to understand how equity, and thus leverage, reacts to increase in business risk (that is, σ_V), the sensitivity of the equity with respect to asset volatility, $\nu_S^{(n)}$ (i.e. the 'Vega' of the equity), can be looked at. Analytical expressions for $\nu_S^{(n)}$ are available in Appendix G. Figure 5 shows how equity reacts to volatility changes over different capital structures and aggregation schemes. It also provides a valid motivation for preferring a compound option model with respect to the Merton model. Panel (a) shows the sensitivity of the equity under the Merton model (where the whole firm's debt is clustered at unique date in the future); panel (b) shows how the same sensitivity displays a different behaviour by having allowed for a more realistic aggregation scheme of the company's capital structure (here, as in the following, the firm's debt is clustered at three future dates).

Comparing the two panels, it is evident that the effect of leverage is exacerbated in a compound option model: the equity of moderately-levered firms changes more severely in panel (b). For reasonable combinations of asset volatility and leverage, there is an incentive by the shareholders to increase the leverage. Considering plausible values of the asset volatility (say between 5% and 40% p.a.), panel (b) in Figure 5 shows that equityholders of the 'dashed company' would be better off increasing leverage to become the 'dotted company'. However, increasing leverage further, that is turning into the 'dotted-dashed company', would reflect into a reduction on the sensitivity of the equity. Notice that the Merton model is not able to reproduce this pattern: it is always optimal for shareholders to increase leverage further.

2.2 Call and Put Equity Options as a $(n + 1)$ – fold Compound Options on Asset Value

Intuitively, as equity is an n–fold compound option, vanilla options are $(n + 1)$–fold compound options on the firm's assets. Consider an European option with maturity $T \in (t_i, t_{i+1})$, with $0 \leq t_i < t_{i+1} \leq t_n$, and strike price K written on the firm's equity. If the company is allowed to default at any t_i, the generic terminal payoff of the option is

$$P_{T,\xi} = \xi \left(S_T(V) - K \right) \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{\{\xi S_T(V) \geq \xi K\}}$$

where ξ is a binary variable taking values $+1$ (respectively -1) in the case in which the option is a call (respectively a put), $\mathbb{1}_{\{\xi S_T(V) \geq \xi K\}}$ determines the condition for the option to expire in-the-money, and $\mathbb{1}_{\{\tau > T\}} \equiv \mathbb{1}_{\{\tau > t_i\}}$ is the condition to ensure that the firm has not defaulted before the maturity of the option. Notice that the possibility for the company to default after the maturity of the option is already accounted in the value of $S_T(V)$. Under risk-neutral

valuation, the present value of the option is given by

$$P_{0,\xi} = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left(\gamma_{T,\xi}\right).$$

In the same fashion as Section 2.1, the event of the option expiring in-the-money, $\{\xi S_T(V) \geq \xi K\}$, can be redefined as the event $\left\{\xi V_T \geq \xi \bar{V}_K\right\}$, where $\bar{V}_K$ is nothing but the value of the assets corresponding to the equity value that makes the option expire in-the-money. Letting $\mathcal{V}_T^{\xi} := \left\{\xi V_T \geq \xi \bar{V}_K\right\}$ and rearranging, the value of the option can be written as

$$P_{0,\xi} = \xi e^{-rT}\left[\mathbb{E}^{\mathbb{Q}}\left(S_T(V)\mathbb{1}_{\mathcal{V}_i \cap \mathcal{V}_T^{\xi}}\right) - K\mathbb{Q}\left(\mathcal{V}_i \cap \mathcal{V}_T^{\xi}\right)\right],$$

where, using (6), the value of the equity at the maturity of the option is given by

$$S_T(V) = e^{-r(t_n - T)}\mathbb{E}_T^{\mathbb{Q}}\left(V_n \mathbb{1}_{\mathcal{V}_{i,n}}\right) - \sum_{k=i+1}^{n} e^{-r(t_k - T)} F_k \mathbb{E}_T^{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{V}_{i,k}}\right).$$

Being $\mathbb{1}_{\mathcal{V}_i \cap \mathcal{V}_T^{\xi}}$ $\mathcal{F}_T$-measurable, and applying the law of iterated expectations, it follows

$$P_{0,\xi} = \xi\left[e^{-rt_n}\mathbb{E}^{\mathbb{Q}}\left(V_n \mathbb{1}_{\mathcal{V}_n \cap \mathcal{V}_T^{\xi}}\right) - \sum_{k=i+1}^{n} e^{-rt_k} F_k \mathbb{Q}\left(\mathcal{V}_k \cap \mathcal{V}_T^{\xi}\right) - e^{rT} K\mathbb{Q}\left(\mathcal{V}_i \cap \mathcal{V}_T^{\xi}\right)\right].$$

Operating the same change of measure of Section 2.1, the price of the option can be expressed as

$$P_{0,\xi} = \xi\left[e^{-\varpi t_n} V_0 \mathbb{M}\left(\mathcal{V}_n \cap \mathcal{V}_T^{\xi}\right) - \sum_{k=i+1}^{n} e^{-rt_k} F_k \mathbb{Q}\left(\mathcal{V}_k \cap \mathcal{V}_T^{\xi}\right) - e^{-rT} K\mathbb{Q}\left(\mathcal{V}_i \cap \mathcal{V}_T^{\xi}\right)\right].$$

Furthermore, the probabilities under $\mathbb{M}$ and $\mathbb{Q}$ can be computed using the result in Theorem C.1 in Appendix C, and the price can be expressed in terms of multivariate Gaussian integral as

$$P_{0,\xi} = \xi\left[e^{-\varpi t_n} V_0 \Phi_{n+1}\left(\mathbf{d}_{\xi}^{\mathbb{M}}; \boldsymbol{\Gamma}_{n+1,\xi}\right) - \sum_{k=i+1}^{n} e^{-rt_k} F_k \Phi_{k+1}\left(\mathbf{d}_{\xi,k+1}^{\mathbb{Q}}; \boldsymbol{\Gamma}_{k+1,\xi}\right) - e^{-rT} K \Phi_{i+1}\left(\mathbf{d}_{\xi,T}^{\mathbb{Q}}; \boldsymbol{\Gamma}_{T,\xi}\right)\right] \quad (11)$$

with $\mathbf{d}_{\xi}^{\mathbb{M}} = \left(\left(d_i^{\mathbb{M}}\right)_{i=1}^{k}, \xi d_T^{\mathbb{M}}, \left(d_i^{\mathbb{M}}\right)_{i=k+1}^{n}\right)$, $\mathbf{d}_{\xi,k+1}^{\mathbb{Q}} = \left(d_{\xi,i}^{\mathbb{M}} - \sigma_V\sqrt{t_i}\right)_{1 \leq i \leq k+1}$, where $d_i^{\mathbb{M}/\mathbb{Q}}$ are defined as in Section 2.1 and

$$d_T^{\mathbb{M}} = \frac{\ln\left(V_0/\bar{V}_K\right) + \left(r - \varpi + \sigma_V^2/2\right)T}{\sigma_V\sqrt{T}}, \qquad \boldsymbol{\Gamma}_{k+1,\xi} = \begin{pmatrix} 1 & \sqrt{\frac{t_1}{t_2}} & \dots & \xi\sqrt{\frac{t_1}{T}} & \dots & \sqrt{\frac{t_1}{t_k}} \\ & 1 & \dots & \xi\sqrt{\frac{t_2}{T}} & \dots & \sqrt{\frac{t_2}{t_k}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & 1 & \sqrt{\frac{t_{k-1}}{t_k}} \\ & & & & & 1 \end{pmatrix}.$$

For $\varpi = 0$, $n = 1$ and $t_1 > T$, the pricing formula coincides with that in Geske (1979). The probabilities which appear in (7) and (11) are used in Section 4 in order to construct a novel measure of impact of credit risk based on option prices.

3 Data and Estimation Methodology

The dataset under investigation is composed by US companies, constituents of the S&P100 during the period January 2013 – December 2017. Companies with either preferred equity or subject to merges or acquisitions are excluded. Also, only companies for which both CDS spreads and option quotes available are included. The final sample is formed by 66 companies. Previous studies investigating the relative pricing of options and CDSs rely on much smaller samples (see

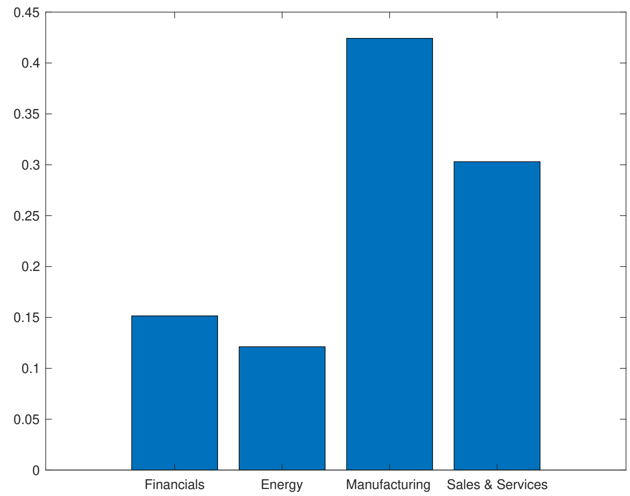


Figure 6: Industry/sector composition of the whole sample: (a) Financial companies (15%); (b) Mining, Energy and Utilities companies (12%); (c) Manufacturing (42%); (d) Retail, Wholesales and Services (30%).

for instance Hull et al. 2004, Carr and Wu 2010, and Carr and Wu 2017). Table B.1 displays the complete name list, alongside the SIC code of the companies.

In the same spirit of Carr and Wu (2017), the sample is further divided into four categories based on the industry/type or business: (a) Financial companies; (b) Mining, Energy and Utilities companies; (c) Manufacturing; (d) Retail, Wholesale and Services. See Figure 6 for the relative frequencies of the different industries/sectors. A sub-sample analysis based on these groups is carried out in Section 4.2.

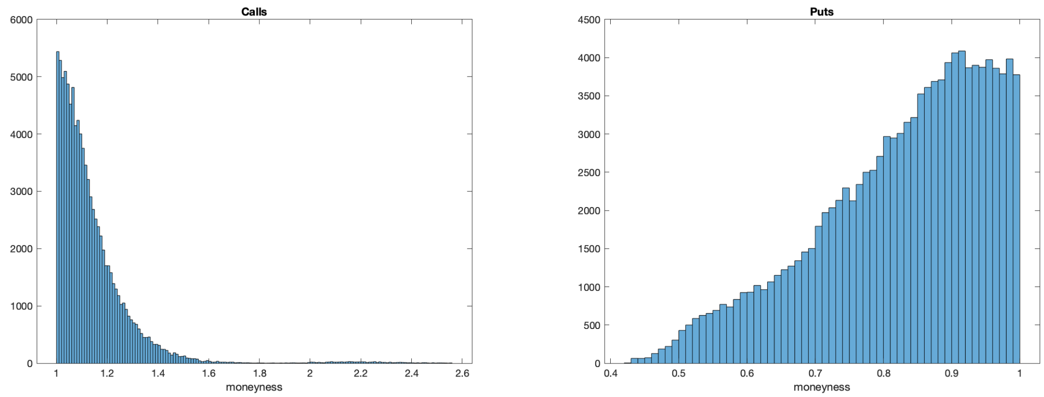
Data on stock prices, number of shares outstanding, dividends and the risk-free yield curve are obtained from Bloomberg. Option quotes are collected from Optionmetrics. CDS spreads are from Thompson Reuters Datastream. Information relative to the firms' capital structures and cost of debt is gathered from Compustat and the 10-K documents.

Option quotes, CDS spreads and equity prices are observed at daily frequency. The data from Compustat on the firms' debt is available only at quarterly frequency though. Therefore, it is assumed that the capital structure remains fixed within quarters, having only adjusted the time to maturity of the firm's debt due to the passage of time. It appears a reasonable assumption given the empirical evidence on how often US firms decide to rebalance their capital structures (see Strebulaev and Whited 2012).

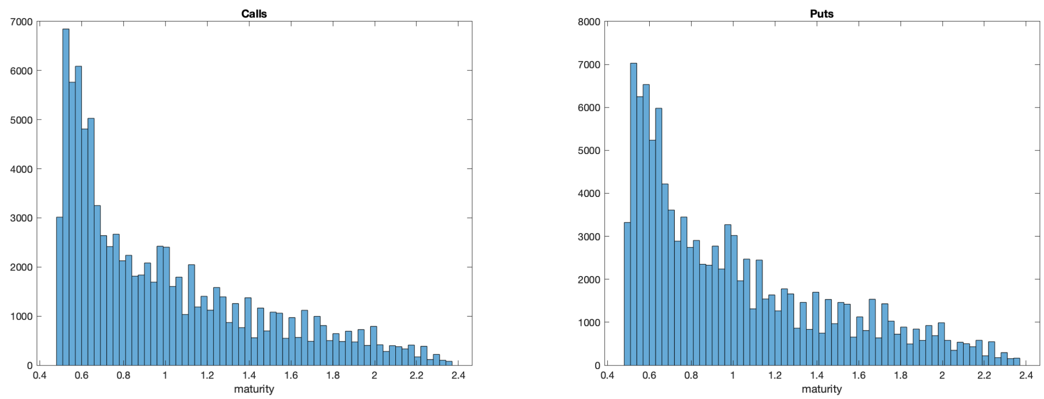
Given the large amount of option data, only the most liquid OTM call and put options traded every Wednesdays with time-to-maturity greater than six months are taken into consideration. To determine the most liquid traded options, those prices whose moneyness is outside the 5th to 95th percentile range are first removed. Secondly, only those options with volume above their annual median are kept.

As one of the aims of this work is to study the interplay between market and credit risk, and how this is reflected into the relative pricing of derivatives contracts (options and CDSs) written on the same company, I opt for focusing on those option for which the impact of credit risk is presumably not negligible. In fact, as the database is composed by firms members of the S&P100 – which should be considered as 'safe' companies in the short-term – it seems very unlikely that options with maturities lesser than six months would price any credit risk. This intuition is also confirmed by Cremers, Driessen, Meanhout and Weinbaum (2008).

The price of the option is defined as the average of the bid and ask price when both are available; the observation is removed otherwise. Finally, options with zero trading volume and negative bid-ask spread are also excluded. The final



(a) OTM Calls' moneyness (b) OTM Puts' moneyness



(c) OTM Calls' maturity (d) OTM Puts' maturity

Figure 7: Empirical distributions of moneyness (K/S) and maturity (T) of option data. Panel (a) shows that most of the valid call option observations have moneyness in $(1, 1.8)$, whilst for put options, displayed in panel (b), the moneyness is in $(0.6, 1)$. The distribution of put options is less peaked than the one of calls. Panels (c) and (d) takes into account the maturities of the options which range from 0.5 to 2.4 years form both call and put quotes, being the distributions very similar.

sample counts 92,879 valid call and 112,347 put options observations over 259 weeks. Figure 7 shows the options' distribution in terms of moneyness and maturity.

One of the main disadvantage of working with equity options is that they are usually American-style. This is the case in the analysed dataset and, in order to test and implement the model, European quotes should be used. Hence, the de-Americanization procedure introduced by Carr and Wu (2010) and further tested in Burkovska et al. (2018) is applied. The aim of the de-Americanization is to find the corresponding European price (the so-called pseudo-European price) for a given American price. That is, the price ought to be observed if the contract would not allow to exercise the option before maturity. In a nutshell, a binomial tree is used to price the American option. The volatility parameter such that the squared difference between the market price and the price generated by the tree is minimised is set as the option implied volatility. Once estimated, the pseudo-European price is found by applying the Black-Scholes formula for European options.

In order to estimate the impact of credit risk on equity options, and verify whether the option market participants price credit risk consistently with the swap market, a calibration based on equity prices, CDS spreads and option quotes is carried out. In addition, the term-structure of the firm's debt must be known or approximated somehow. I opt for clustering the firm's debt at three fixed point, $t_i = \{1, 5, 10\}$ years, $i = 1, 2, 3$. The face values of the bond due in

$t_1 = 1$ represents the company's short-term debt and is computed as the Compustat variable `DD1Q` (Long-Term Debt Due in One Year). The remaining two bonds clustered at $t_2 = 5$ and $t_3 = 10$ are obtained as half of `DLTQ` (Long-Term Debt Total) each.

The choice of setting $n = 3$ appears optimal as it is the smallest number of maturity dates needed in order to match both the level, slope and curvature of the term structure of the survival probabilities extracted from the CDSs. As a matter of fact, an effective calibration of the model should aim at reproducing the aforementioned term structure as accurately as possible. For a more in-depth description of the construction of F_i and the payout rate see Appendix B. Furthermore, as shown in the previous Section, a 3–fold compound option model displays some desirable properties in terms of optimal leverage.

The unknown parameters of the model are the value of the company's asset, V_0, and its volatility, σ_V. In order to estimate their values, (8) and (11) are used (the two pricing equations constitute a non-linear system of equations in two unknowns). In the same spirit of estimating the implied volatility à la Black-Scholes, the value of the asset volatility, $\sigma_V^\star$, and corresponding asset value, $V_0^\star$, are found such that both the market price of equity and the option quote are matched.

Notice that the calibration can be implemented without using the risk-neutral probability of survival estimated from the CDS spreads (to solve for two unknowns, only two equations are needed, namely (8) and (11)). However the risk-neutral probabilities in (8), $\mathbb{Q}(\tau > t_i)$, can be replaced by those estimated from the CDS spreads. Nonetheless, the price of the firm's equity is still represented by an equation unknown in σ_V and V_0 as the probability under the firm-value fund measure, $\mathbb{M}(\tau > t_n)$, cannot be determined from CDS quotes. Thus, comparing the model estimates with and without the inclusion of CDS data should constitute a test of relative pricing between the option and CDS markets. If the default probabilities generated by the model – based on the calibration on the option and stock prices only – is able to reproduce those estimated form the CDS spreads, it could be inferred that the two markets are well integrated. For a further discussion on this topic and on how compute the implied default barrier (which determines the model-implied probabilities of survival), see Appendix E.

Furthermore, the calibration on the equity and option prices allows to construct the asset volatility surface. Moreover, using (10), the equivalent of the Black-Scholes volatility smile/smirk for the equity volatility can be computed and compared: according to the model, the contemporaneous volatility of the equity should depend on the volatility of the asset, the sensitivity of the equity with respect to the asset value and the leverage ratio. As, after the calibration, the optimal value of the asset is known, the model-implied market value of the debt can be used in order to estimate the leverage ratio. Finally, the sensitivity of the firm's equity to changes in asset value (see Appendix F for analytical expressions) can also be computed, thus allowing to estimate the volatility of the equity. The ability of the model to price option based on the previous asset volatility surface is tested also in Section 4.4.

4 Empirical Results

In this section, a new measure of impact of credit risk on options is first introduced. Using this measure I later test whether call and put options display different price impacts in terms of credit risk, and I further look at differences among industries in terms of credit risk pricing. Finally, it is shown that this measure (calculated on put options only) is able to forecast future movements of the options' negative skew. In the robustness section, evidence for the integration of the option and CDS markets is also discussed.

4.1 Information Content Ratios as Measure of Impact of Credit Risk

After having estimated the asset volatility and value such that (8) and (11) are met, a measure of impact of credit risk on option prices can be constructed based on the pricing equation (11). In the model, the option price is obtained by

	Calls		Puts	
	K	T	K	T
Correlation (w/o CDS)	-0.0963	0.2209	-0.2573	0.2902
p-value	0.000***	0.000***	0.000***	0.000***
Correlation (with CDS)	-0.1078	0.2199	-0.2611	0.2876
p-value	0.000***	0.000***	0.000***	0.000***

Table 1: Average sample correlation of $AICR$ (calls and puts, with and without having used the CDS to calibrate) with strike price (K) and maturity (T) of the options. As expected, for both calls and puts, the $AICR$ is negatively correlated with the strike price, as lower strikes proxies for default thresholds, and positively correlated with the time-to-maturity of the option, as the probability of defaulting is increasing with the time horizon. The estimates are carried out both having calculated the $AICR$s with and without the extra calibration of the risk-neutral probabilities extracted from CDSs (and are statistically equivalent).

multiplying the asset value and the discounted face values of the bonds by

$$\mu\left(\tau > t_i \cap \xi S_T > \xi K\right) \tag{12}$$

with $i \in I$ and μ intended as the risk-neutral measure (when it multiplies the discounted face values) and the firm-value fund measure (when it multiplies the asset value). These are the probabilities of the event for which the firm survives (up to t_i) and the option expires in-the-money. Most importantly, they can be factorised as

$$\mu\left(\tau > t_i\right)\mu\left(\xi S_T > \xi K | \tau > t_i\right). \tag{13}$$

The first factor is the probability of the firm surviving until t_i, whilst the second is the probability of the option expiring in-the-money conditional on the firm having survived. Therefore, this decomposition rigorously disentangles the source of credit and market risk and how these reflect into option prices. Notice that the estimation of $\sigma_V^\star$ and $V_0^\star$ as in Section 3 allows to compute the probabilities in (12) as well as the risk neutral probabilities of survival in (13). Hence, the probability of the option expiring in-the-money conditional on surviving can be computed as well.

Alternatively, the risk-neutral probabilities of survival can be directly estimated in a model-free fashion using the CDSs spreads (see Appendix D). The comparison of the latter with those produced by the model allows to investigate the its ability to replicate the observed term-structure of default probabilities and, indirectly, to test for the integration of the two markets. In fact, if the model is calibrated on option quotes only and the generated probabilities are close those estimated via the CDSs, this would point towards a similar and consistent pricing of the two derivatives contracts by the economic agents trading in the two markets.

In order to turn the multiplicative link into an additive one, instead of looking at raw probabilities, their information content[3] is instead considered. Thus, the Information Content Ratio (ICR) for each probability is defined as

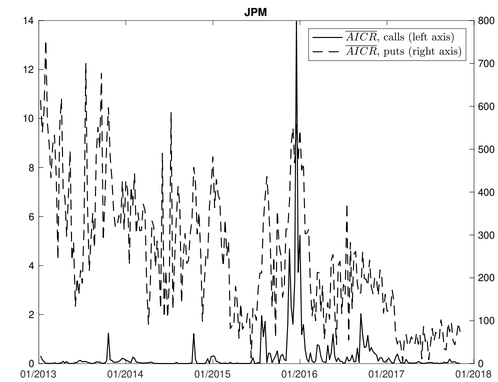
$$ICR_{i,\mu,\xi}(K,T) = \frac{\log\mu\left(\tau > t_i\right)}{\log\mu\left(\tau > t_i \cap \xi S_T > \xi K\right)}. \tag{14}$$

This represents the percentage of credit risk over the whole event of the firm surviving and the option expiring in-the-money, expressed in terms of the information content of the two events.

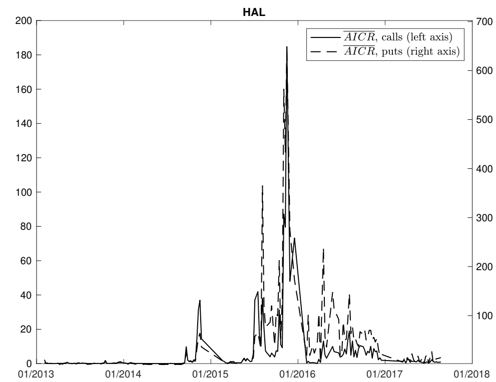
Once these ratios are computed for all the probabilities contributing to the price of the option, they can be then aggregated in order to measure the impact of credit risk on each option contract. The Average Information Content Ratio ($AICR$) is thus defined as a weighted average of the information content ratios, that is

$$AICR_\xi(K,T) = \frac{\sum_{i,\mu} w_{i,\mu} \cdot ICR_{i,\mu,\xi}(K,T)}{\sum_{i,\mu} w_{i,\mu}}. \tag{15}$$

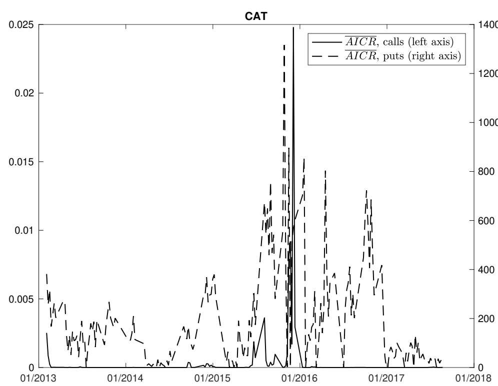
[3] In Information Theory, information content (or surprisal) of a signal is the amount of information gained when it is sampled. It is defined as minus the log-probability of the event: the less likely the event, the greater is the "surprise" associated if it happens. See Cover and Thomas (2006) for further details.



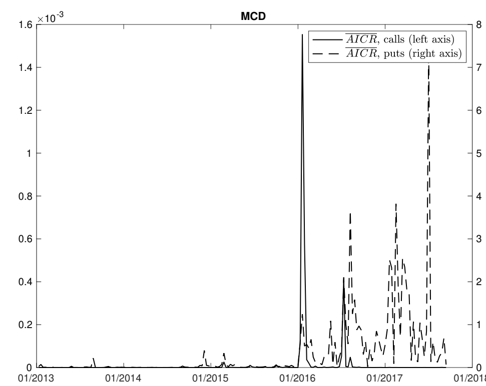
(a) Financials



(b) Mining, Energy and Utilities



(c) Manufacturing



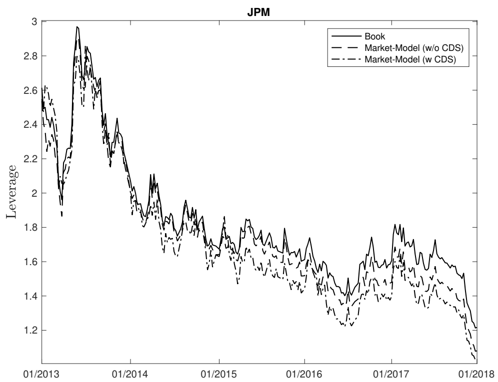
(d) Retail, Wholesale and Services

Figure 8: Comparison of daily average $\overline{AICR}$ calculated over call and put options for four companies representing the four selected industries/sectors. As expected, $\overline{AICR}$ calculated using call (solid lines) options is much smaller than those obtained from put (dashed lines) options. $\overline{AICR}$s are expressed in basis points.

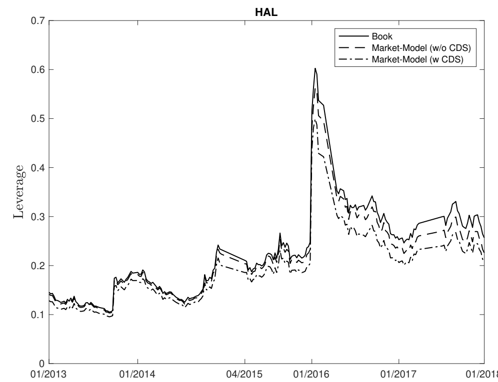
The weights $w_{i,\mu}$ are the net asset value, $V_0 e^{-\varpi t_n}$, for $\mathbb{M}(\tau > t_n)$, and the discounted face values of the bonds, $F_i e^{-rt_i}$, for $\mathbb{Q}(\tau > t_i)$ (and the present value of the strike, Ke^{-rT}, for $\mathbb{Q}(\tau > T)$). Notice that if $\tau > t_i$ a.s. for all $i \in I$, then $AICR_\xi(K, T) = 0$ (i.e. no impact of credit risk). For each option contract $AICR$ is computed. Given that every day several contracts are traded, the daily average $AICR$ is further calculated over all valid option prices. This allows us to construct a time series of a measure of the company's credit risk which is based on option quotes.

As expected, Figure 8 shows that the daily average $AICR$ calculated over calls is substantially smaller than those computed using put quotes. In addition, it appears that financial companies (panel (a)) display the greatest variability (both in terms of calls and puts) in the average daily $AICR$, thus suggesting a prompt reaction in option prices to changes in the company's credit risk and leverage.

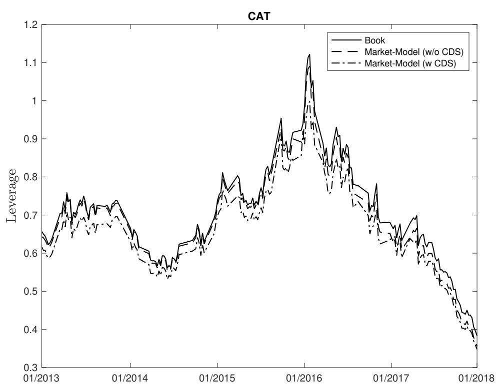
Also, a correlation analysis shows that $AICR$s display a statistically significant negative correlation with the strike price: the lower the strike price, the higher the price of credit risk embedded in the option. In fact, lower strikes 'belong' to the left side of the distribution of the firm's equity, which is the one affected by credit events the most. In addition, $AICR$s is positively correlated with the maturity of the option: given the probability of survival being a decreasing function of the time horizon, long-maturity options display a larger impact of credit risk. Table 1 reports these estimates. Furthermore, these sample correlations highlight a much stronger link between credit risk and put options.



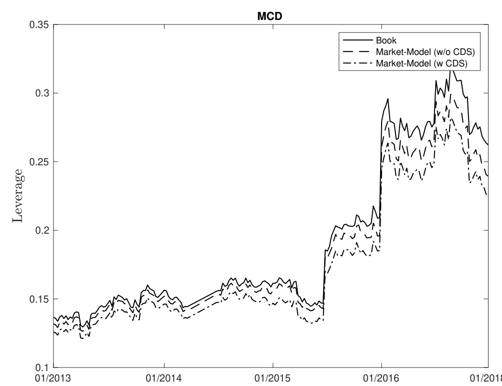
(a) Financials



(b) Mining, Energy and Utilities



(c) Manufacturing



(d) Retail, Wholesale and Services

Figure 9: Measures of leverage for four companies representing the four selected industries/sectors. The solid line is a measure of leverage calculated as book value of debt over market value of equity. The dashed line is a measure of leverage calculated as model-implied market value of debt (without having used the risk-neutral probabilities of default from CDS spreads) over the market value of equity. The dashed-dotted line is a measure of leverage calculated as model-implied market value of debt (having used the risk-neutral probabilities of default from CDS spreads) over the market value of equity.

The same comparison can be done by computing the $AICR$ of each option contract using either the model-implied probabilities or those extracted from the CDSs. As $AICR$ is a linear functions of information contents in (14), the discrepancies between the $AICR$s computed using the two different sets of probabilities can shed light on the level of integration of the two markets. for a good level of integration between the option and CDS market. The next section addresses the link between the two constructions as well as the difference behaviour of put and call options in term of pricing of credit risk. A more formal test of integration is conducted in Section 4.4.

4.2 Cross Sectional Differences in Calls and Puts

Economic and financial intuition suggests that a different pricing behaviour of credit risk should be observed between calls and puts. As default is an event which is priced in the left side of the equity distribution, put options, which constitute insurances against price falling, should be affected more by credit-related events more than calls. As a matter of fact, Carr and Wu (2011) show that, under a general class of stock price dynamics, a portfolio of two deep out-of-the-money American-style equity put options can replicate a pure credit insurance contract that pays off if and only if the company defaults prior to the option expiry.

A preliminary test is conducted on the relationship between the two measures of impact of credit risk, either constructed using option quotes only ($AICR$) or based on the option prices and CDS spreads ($AICR'$). As these measures are option-specific, the daily average for the whole set of available call and put options is calculated each week ($\overline{AICR}$ and $\overline{AICR}'$ respectively). In order to estimate the risk-neutral probabilities from the CDSs, a 50% loss given default is assumed. A 50% recovery rate is consistent with the median value for senior unsecured bonds reported in Duffie and Singleton (1999). Other values of loss given default are investigated as a robustness test in the next section.

The following set of unbalanced panel pooled regressions are estimated

$$\overline{AICR}'_{j,t,\xi} = \alpha + \beta\overline{AICR}_{j,t,\xi} + \eta_{j,t,\xi}, \tag{16}$$

with $j \in J$, being J the set of 66 US companies, t the weekly observation, and ξ the binary variable that takes value $\xi = 1$ for calls and $\xi = -1$ for puts. Firstly, these regressions serve as a sanity check of the co-movements of the two measures of impact of credit risk. Secondly, the residuals $\eta_{j,t,\xi}$ summarise the pricing information carried by CDS in determining the ICRs which is not obtainable from option prices. Results are reported in Table 2. As expected, the two measures strongly co-move and the loading coefficients have the predicted sign.

The next step is assessing the different impact of credit risk on call and put options. Different measures of leverage (book leverage, market model-implied leverage obtained from the calibration on options only, market model-implied leverage using both options and CDSs) can be used as proxies of credit risk. These measures of leverage are then regressed onto $\overline{AICR}$ and $\overline{AICR}'$ of calls and puts for each set of weekly observations. By common sense and the model predictions, a higher leverage should induce a larger impact of credit risk on the company's securities. Put options are expected to show statistically significant positive loadings. More specifically, panel regressions are implemented with year- and industry-fixed effects. These are

$$\begin{aligned}
\overline{AICR}_{j,t,\xi} &= \alpha_\xi + \mu_{i,\xi} + \theta_{y,\xi} + \beta LEV_{j,t} + \varepsilon_{j,t,i,y,\xi} \\
\overline{AICR}'_{j,t,\xi} &= \alpha'_\xi + \mu'_{i,\xi} + \theta'_{y,\xi} + \beta' LEV_{j,t} + \varepsilon'_{j,t,i,y,\xi} \\
&.
\end{aligned} \tag{17}$$

Year-fixed effects ($\theta_{y,\xi}$ and $\theta'_{y,\xi}$, with $y = \{2013, \ldots, 2017\}$) should capture the time-series variation in the volatility of the market, whilst the industry-fixed effects ($\mu_{i,\xi}$ and $\mu'_{i,\xi}$, with $i =${Financials; Mining, Energy and Utilities; Manufacturing; Retail, Wholesale and Services}) account for possible cross-sectional heterogeneity due to the sector the company operates.

For the variable LEV, the market leverage defined as model-implied value of debt over market value of equity (dotted-dashed line in Figure 9) is used. More specifically, after estimating the asset volatility surface via a joint calibration on options and CDSs, the average value $\bar{\sigma}_V$ is used to compute the model implied asset value, that is the value of the firm such that (8) holds. Subsequently, the model implied market value of debt is obtained. Unreported results, available upon request, show that these findings are robust across other measures of leverage.

The results are reported in Table 3. Remarkably, despite the two measures of credit risk strongly co-move, in the case of call options, only the $\overline{AICR}$ obtained by adding the information carried by CDSs is able to capture the credit risk of the company. Since the main effect of default on equity is to reduce its value by a sizable amount, it is understandable that the credit risk factor has its major impacts on put options. Therefore, the joint calibration on options and CDS carries extra information that, especially for call options, is relevant to capture default risk dynamics.

To stress this point further, the following regressions

$$\eta_{j,t,\xi} = \alpha_\xi + \mu_{i,\xi} + \theta_{y,\xi} + \beta LEV_{j,t} + \epsilon_{j,t,i,y,\xi}, \tag{18}$$

are estimated. Here, the left-hand side is constituted by the residuals obtained from regression (16). These residuals indeed capture the extra information provided by the calibration on CDSs which is not accounted for by options. As

Regressand						Adj–R^2: 0.8804
$\overline{AICR}'_1$						
Regressors	**Coefficient**	**Robust Standard Error**	**t–stat**	**p–value**		
$\overline{AICR}_1$	0.7813466	0.0053647	145.65	0.000	***	
α_1	0.0015068	0.0000465	32.39	0.000	***	

(a): $\overline{AICR}'$ regressed onto $\overline{AICR}$ (both constructed on calls).

Regressand						Adj–R^2: 0.5359
$\overline{AICR}'_{-1}$						
Regressors	**Coefficient**	**Robust Standard Error**	**t–stat**	**p–value**		
$\overline{AICR}_{-1}$	0.5577679	0.0362096	15.40	0.000	***	
α_{-1}	0.0003161	0.0001010	3.13	0.002	***	

(b): $\overline{AICR}'$ regressed onto $\overline{AICR}$ (both constructed on puts).

Table 2: Estimation of regression (16).
(a): Estimates of the pooled panel regression of average $AICR$ obtained from call options ($\xi = 1$) and CDS regressed onto average $AICR$ obtained from call options only. Number of observations: 15,470. F–stat: 21,212.88 (p–value: 0.0000).
(b): Estimates of the pooled panel regression of average $AICR$ obtained from put options ($\xi = -1$) and CDS regressed onto average $AICR$ obtained from put options only. Number of observations: 15,027. F–stat: 237.28 (p–value: 0.0000).
A sandwich estimator for panel data is used to obtain robust standard errors. Significance levels: 10% (*), 5% (**), 1% (***).

shown in Table 4, the residuals obtained from regression (17) when call options are used are still strongly correlated with leverage, even after having controlled for year-fixed and industry-fixed effects. Furthermore, the relatively high adjusted R^2 signals an undoubted explanatory power of leverage on those residuals. Therefore, I conclude that CDS quotes carry extra information regarding the credit risk of the companies when compared with pricing information embedded in call options. Conversely, I conclude that call options do not price credit risk and, if the $AICR$ measure is used to asses the default risk of a company, a joint calibration with CDS spreads on the same reference entity is needed. The same conclusion does not hold for equity put options which share most of the pricing information in terms of default with the CDS written on the same reference entity.

In order to investigate further the link between options and credit risk, the same regressions in (17) are re-estimated over the four sub-samples based on industry (therefore, the industry-fixed effect is removed, whilst the year-fixed effect is kept). Based on the findings in Carr and Wu (2017), a cross-sectional diversity should emerge. Only put options are taken into consideration as the impact of default risk on call options was shown to be insignificant. Results are reported in Table 5.

As manufacturing companies usually invest in long-term assets, they tend to have the same debt for a long period of time, without actively rebalancing their capital structure. Therefore, their financial leverage varies passively with the stock price fluctuations. Also, in addition to having a relatively small average $\overline{AICR}'_{-1}$ (see Table 6), they also have the smallest mean leverage across the categories. This is reflected into the results of the regression: default does not play a major role in affecting the price of put options, displaying a relatively low adjusted R^2. The same conclusion applies to companies operating in sales and services: given an average leverage comparable to manufacturing companies, they also show the smallest $\overline{AICR}'_{-1}$.

On the other hand, financial firms tend to actively manage their capital structures according to changes in market conditions and, for banks, to satisfy regulatory requirements. The results indeed capture the impact of credit risk to be driven by financial leverage (highest adjusted R^2 and significance). Remarkably, the same result is found for utility

Regressand						Adj–R^2: 0.0336
$\overline{AICR}_1$						
Regressors	**Coefficient**	**Robust Standard Error**	**t–stat**	**p–value**		
LEV	0.0030977	0.0041647	0.74	0.511		
α_1	-0.000620	0.0024973	-0.25	0.820		
Industry-FE	✓					
Year-FE	✓					

(a): $\overline{AICR}$ (constructed on calls) regressed onto LEV.

Regressand						Adj–R^2: 0.1113
$\overline{AICR}'_1$						
Regressors	**Coefficient**	**Robust Standard Error**	**t–stat**	**p–value**		
LEV	0.0128021	0.0027542	4.65	0.019	**	
α_1	-0.0020046	0.0014777	-1.36	0.268		
Industry-FE	✓					
Year-FE	✓					

(b): $\overline{AICR}'$ (constructed on calls and CDSs) regressed onto LEV.

Regressand						Adj–R^2: 0.4586
$\overline{AICR}_{-1}$						
Regressors	**Coefficient**	**Robust Standard Error**	**t–stat**	**p–value**		
LEV	0.0336547	0.0037234	9.04	0.003	***	
α_{-1}	-0.0083151	0.0032709	-2.54	0.085	*	
Industry-FE	✓					
Year-FE	✓					

(c): $\overline{AICR}$ (constructed on puts) regressed onto LEV.

Regressand						Adj–R^2: 0.5707
$\overline{AICR}'_{-1}$						
Regressors	**Coefficient**	**Robust Standard Error**	**t–stat**	**p–value**		
LEV	0.0290758	0.0095415	3.05	0.056	*	
α_{-1}	-0.0085324	0.0078328	-1.09	0.356		
Industry-FE	✓					
Year-FE	✓					

(d): $\overline{AICR}'$ (constructed on puts and CDSs) regressed onto LEV.

Table 3: Estimation of regression (17)
(a): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto average $AICR$ calculated over call options only. Number of observations: 15,470.
(b): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto average $AICR$ calculated over call options and CDSs. Number of observations: 15,470.
(c): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options only. Number of observations: 15,027.
(d): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto average $AICR$ calculated based on put options and CDSs. Number of observations: 15,027.
Standard errors are adjusted for four clusters based on industry. Significance levels: 10% (*), 5% (**), 1% (***).

Regressand					Adj–R^2:	0.6001
η_1						
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
LEV	0.0103818	0.0021359	4.86	0.017	**	
α_1	-0.003027	0.0015659	-1.93	0.149		
Industry-FE	✓					
Year-FE	✓					

(a): Extra-information provided by CDSs when calls are used to infer credit risk.

Regressand					Adj–R^2:	0.1503
η_{-1}						
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
LEV	0.0103043	0.0076949	1.34	0.273		
α_{-1}	-0.0042105	0.0063554	-0.66	0.555		
Industry-FE	✓					
Year-FE	✓					

(b): Extra-information provided by CDSs when puts are used to infer credit risk.

Table 4: Estimation of regression (18)
(a): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto the residuals obtain from regression (16) (calls). Number of observations: 15,470.
(b): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto the residuals obtain from regression (16) (puts). Number of observations: 15,027.
Standard errors are adjusted for four clusters based on industry. Significance levels: 10% (*), 5% (**), 1% (***).

and energy companies, despite the regression adjusted R^2 is just a third of that obtained from financial companies. Operating in regulated businesses and being strongly influenced by systemic factors as the state of the economy, perhaps the regression fit could be enhanced accounting for macroeconomic factors. In addition, unreported results available upon request show that these findings across the four sub-samples still hold after having accounted for a firm-fixed effect.

4.3 Explaining the Skew

Many attempts have been made in the literature in order to explain the shapes of the implied volatilities obtained from options. It is well-known that inverting the Black-Scholes formula to determine the value of the volatility which matches the observed price produces the so called volatility smile or smirk, instead of a flat line (which would be expected if the Black-Scholes model were correct). It is also well-documented that equity volatilities display more often a smirk, that is the volatility is a decreasing and convex function of the moneyness of the option. The negative slope of the implied volatility function is referred to as negative skew.

The first works attempting to give a explanation for the observed skew are Black (1976) and Christie (1982). Both attribute the negative slope to the possibility of the underlying to default, the so-called leverage effect: if the underlying of the option can default, the left tail of its distribution should be more sensitive to credit-related events which notoriously make the value of the firm's equity significantly drop. This increased probability of the underlying falling due to default would be then reflected in the pricing of options. Also, as put options protects the buyer against price falling, they should price both market and credit related events. As shown in Section 4.2, put options indeed price credit risk (and doubtlessly price market risk). However, some more recent works shed light on the drivers of the volatility skew.

Regressand					Adj–R^2: 0.6608
$\overline{AICR}'_{-1}$					

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0391051	0.0012387	31.57	0.000	***
α_{-1}	-0.0145449	0.0011321	-12.85	0.000	***
Year-FE	✓				

(a): Financials

Regressand					Adj–R^2: 0.2128
$\overline{AICR}'_{-1}$					

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0069186	0.0004261	16.24	0.000	***
α_{-1}	-0.0007640	0.0001489	-5.13	0.000	***
Year-FE	✓				

(b): Mining, Energy and Utilities

Regressand					Adj–R^2: 0.1173
$\overline{AICR}'_{-1}$					

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0014455	0.0000861	16.78	0.000	***
α_{-1}	-0.0000211	0.0000163	-1.29	0.197	
Year-FE	✓				

(c): Manufacturing

Regressand					Adj–R^2: 0.2450
$\overline{AICR}'_{-1}$					

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0075047	0.0009004	8.34	0.000	***
α_{-1}	-0.0007342	0.0001427	-5.14	0.000	***
Year-FE	✓				

(d): Retail, Wholesale and Services

Table 5: Estimation of regression (17) over the four sub-samples.
(a): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Financials. Number of observations: 1,938. F–stat = 199.90 (p–value = 0.000).
(b): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Mining, Energy and Utilities. Number of observations: 1,916. F–stat = 73.60 (p–value = 0.000).
(c): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Manufacturing. Number of observations: 6,515. F–stat = 80.21 (p–value = 0.000).
(d): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Retail, Wholesale and Services. Number of observations: 4,658. F–stat = 20.63 (p–value = 0.000).
A sandwich estimator for panel data is used to obtain robust standard errors. Significance levels: 10% (*), 5% (**), 1% (***).

	Financials	Energy and Utilities	Manufacturing	Sales and Services
$\overline{LEV}$	0.9090	0.3791	0.1853	0.1905
$\overline{\overline{AICR}}'_{-1}$	0.0148	0.0023	0.0003	0.0007

Table 6: Sub-sample averages for leverage and $\overline{AICR}$ obtained from put options and CDSs.

Carr and Wu (2017) show that the leverage effect can be generated by other sources than leverage. As a matter of fact, the skew is also displayed by other assets, such as commodities or indexes, which cannot default. In their work, they individuate three possible channels influencing volatility: (1) return volatility increases with financial leverage; (2) positive shocks to systematic risk generate a negative correlation between the market's return and its volatility, regardless of the magnitude of financial leverage; (3) large negative market disruptions show self-exciting behaviours.

Their estimations show that the volatility feedback effect (2) reveals itself mainly in the variations of short-term options, the self-exciting behaviour (3) affects both short-term and long-term option variations, and the financial leverage variation (1) has its largest impact on long-dated options. Finally, when the model of Carr and Wu (2017) is applied to individual companies, the three economic channels show up differently and to different degrees according to the company's specific business and capital structure. Therefore, their work constitutes a valuable ruler in order to asses my model's predictions and results.

From a different perspective, the shape of the volatility smile/smirk provides an insight on how the Black-Scholes model prices risk under the presence of both market and credit risk, and how the estimation of the implied volatility is affected by them. Figure 10 shows in fact how the Black-Scholes implied volatility is an average of the implied volatility estimated with a compound option model (which, instead, allows to account for the relative impact of credit and market risk separately).

Evidently, the Black-Scholes averages across the surface the impact of credit risk. Analytically, the probabilities involved into the calculation of the option price à la Black-Scholes are such as

$$\mu\left(\xi S_T > \xi K\right)$$

whilst those of the compound option model are of the type

$$\mu\left(\xi S_T > \xi K | \tau > t_i\right).$$

By the law of total probability

$$\mu\left(\xi S_T > \xi K\right) = \mathbb{E}^{\mu}\left[\mu\left(\xi S_T > \xi K | \tau > t_i\right)\right],$$

thus showing that the probabilities involved in the Black-Scholes model are an average of the probability of the option expiring in-the-money conditional on the firm surviving at the reimbursement dates. Therefore, the volatility smirk produced by Black-Scholes should lie within the implied volatilities produced by the model. Figure 10 confirms it.

Inspecting Figure 10 , this averaging effect impacts mostly long-maturity options as the underlying probability of defaulting increases with the time horizon. Interestingly, for financial companies, despite displaying a negative slope for both short and long-term maturity options, the skew is more likely to be associated with the leverage effect for long-maturity options only (panel (b)). As a matter of fact, the compound option implied volatilities of the short-term maturity options of financial companies (panel (a)) replicate almost perfectly the Black-Scholes implied volatility which does not accommodate for default risk explicitly.

More generally, the closer the two equity volatility estimates are, the lower the probability of default and therefore the impact of credit risk (i.e. leverage effect) on the pricing of the option. Notice also that, if $\sum_i F_i \downarrow 0$, the compound option model coincides with Black and Scholes (1973). Therefore, the larger the firm's financial leverage the farther the Black-Scholes implied volatilities surface should be with respect to those estimated via the compound option

model. However, if the two models reproduce very similar volatility skews when the company is highly levered (as for financials), it can be argued that 'apparent' leverage effect is not driven by leverage and, therefore, financial leverage is not a good proxy for default risk, at least in the short-term. This is exactly what Carr and Wu (2017) document.

Moreover, having defined the equity as a compound call options allows to further motivate why the skew is observed over the region constructed using put options. It is well-known that the price of a call option is an increasing function of the volatility of its underlying; similarly, it can be shown that the price of a compound call on call option is also an increasing function of the volatility. By induction, it can be shown that also the price of a n-fold compound call option is increasing of the underlying volatility.

Under Black and Scholes (1973), the Vega of a put option is an increasing function of the volatility of the underlying equity. However, as equity is itself a compound call option, an increase in the asset volatility causes an increase in the value of the equity which ultimately makes the put options less likely to expire in-the-money. Therefore, the Vega of in-the-money put options under the compound option model is negative. Figure 11 illustrates this aspect[4].

So far the volatility skew was only described as the negative slope displayed by the graph of the implied volatility plotted against moneyness. Following Carr and Wu (2017), the volatility skew is formally defined as

$$\text{Skew} = \frac{25\Delta \text{ Put } \sigma_{IV} - 25\Delta \text{ Call } \sigma_{IV}}{50\Delta \text{ Put } \sigma_{IV}}, \tag{19}$$

where $x\Delta$ Put σ_{IV} is the implied volatility of the put option whose delta is $-x\%$, and $y\Delta$ Call σ_{IV} is the implied volatility of the call with delta equal to $y\%$. The proposed measure of skewness is more positive when the risk-neutral return distribution is more negatively skewed (if the implied volatility is downward sloping, the numerator of (19) is positive).

In order to investigate to what extent the displayed skew is driven by leverage, the following panel regression are carried out

$$\Delta\text{Skew}_{j,t,T} = \alpha_T + \phi_{j,T} + \theta_{y,T} + \beta\overline{AICR}_{j,t-1,-1,T} + \varepsilon_{j,t,y,T}, \tag{20}$$

where $\overline{AICR}_{j,t-1,T,-1}$ is the average $AICR$ of put options having time to maturity equal to T observed in the previous week. Rather than using industry-fixed effect, a firm-fixed effect (ϕ_j) is estimated as, ultimately, I want to show the ability of this measure to forecast future changes in the skew observed at the company level. If the skew is caused by the leverage effect, a positive and statistically significant effect should be observed.

The variable ΔSkew is calculated based on (19). Call options are excluded, given the results in Section 4.2. Also, ΔSkew is used rather than the contemporaneous level as the time-series component of the latter is non-stationary for some companies (see Figure 12).

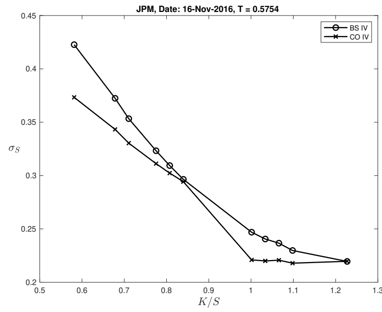
In particular, based on the previous findings, two sets of regression are estimated. As for every day t, multiple maturities are observed, the skew of the shortest ($\min_T : T < 1$ year) and longest ($\max_T : T > 1$ year) maturity options is used in two separate sets of regressions. Based on the different behaviour of the model for short and long term maturity options, the skew should be driven by credit risk (here proxied as the $\overline{ICR}$ of out-of-the-money put options) mostly for $T > 1$ rather than for $T < 1$.

The estimates are presented in Table 7. Consistently with economic intuition and the empirical evidence in Carr and Wu (2011) and Carr and Wu (2017), only the changes in the skew of long-maturity options are driven by the credit risk of the company. Also, the average $AICR$ of put options, is able, to some extent, to predict the future changes in the skew for those options. The same variable is not able to explain the movements of the skew for options with shorter

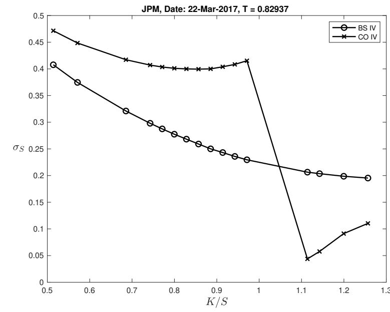
[4] Figure 11 clearly shows $\partial\gamma_{-1}^{CO}/\partial\sigma_V < 0$, for $S_t < K$. Therefore

$$\frac{\partial\gamma_{-1}^{CO}}{\partial\sigma_V} = \frac{\partial\gamma_{-1}^{CO}}{\partial\sigma_S}\frac{\partial\sigma_S}{\partial\sigma_V}$$

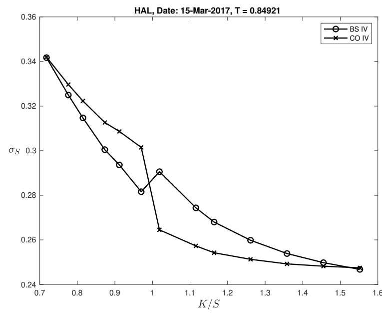
implies $\partial\gamma_{-1}^{CO}/\partial\sigma_S < 0$, as $\partial\sigma_S/\partial\sigma_V > 0$ (see Figure 4 panel(d)).



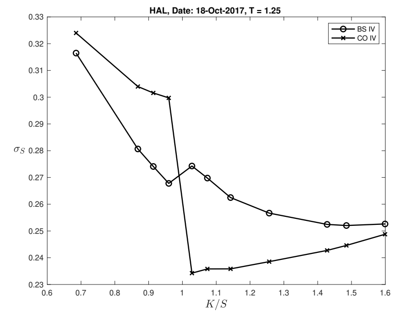
(a) $T < 1$



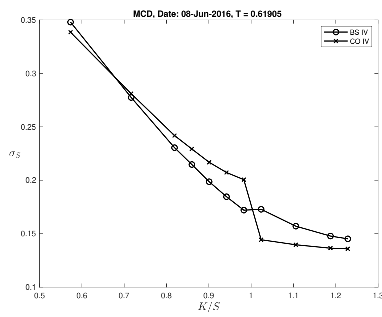
(b) $T > 1$



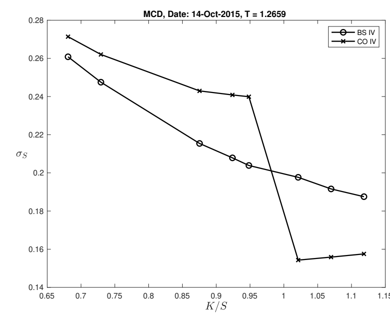
(c) $T < 1$



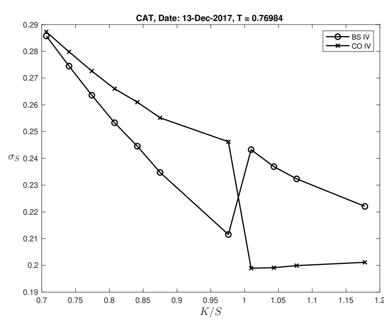
(d) $T > 1$



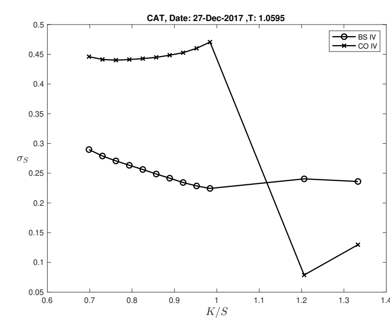
(e) $T < 1$



(f) $T > 1$



(g) $T < 1$



(h) $T > 1$

Figure 10: Volatility skews for different option maturities. The lines marked with circles are the implied volatility estimates produced by the Black-Scholes model, whilst those marked with crosses are the estimates produced by the compound option model. It is evident that the Black-Scholes estimates of the implied volatility lie within the estimates produced by the model, thus suggesting the aforementioned averaging effect. This effect is more pronounced for long-term maturity options (panels (b), (d), (f), (h)) than for options with shorter maturities (panels (a), (c), (e), (g)), as the distance between the two lines is larger for $T > 1$.

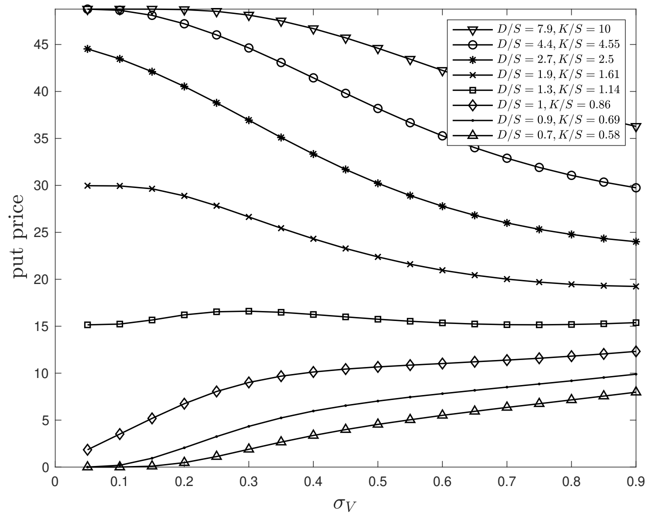
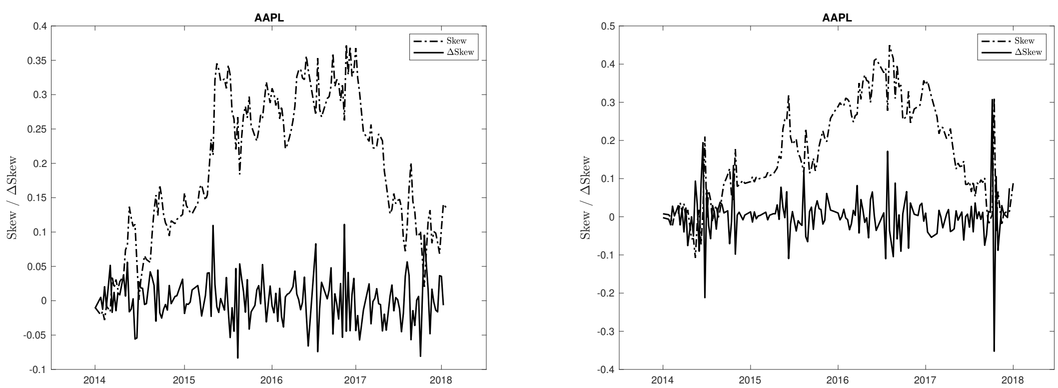


Figure 11: Put price as function of asset volatility. Conversely to Black-Scholes put price, the price of compound put option on a call can be a decreasing function of the volatility. This is observed when the put option is deep into-the-money, that is when price of equity is approaching zero (therefore, default becomes more likely). Here $K = 50$, $T = 0.5$, $F_1 = F_2 = F_3 = 30$, $t_1 = 1$, $t_2 = 5$, $t_3 = 10$, $r = 0.03$ and $\varpi = 0.05$.



(a) $T < 1$ (b) $T > 1$

Figure 12: Time-series of Skew extracted from short-term maturity options ($T < 1$, panel (a)) and from long-term maturity options ($T > 1$, panel (b)) for AAPL. Both panels show that the time series of the levels of Skew (dotted-dashed line) is non-stationary whilst the increments (solid line) are stationary.

maturities. Therefore, the high significance of the average $AICR$ obtained from put options, as well as the correct sign of its loading, points towards a connections between the future changes of the negative skew and the today's credit risk of the company. However, the low fit can be attributed either to the presence of other factors driving the skew or to the highly non-linear link between the movements of the skew and the impact of credit risk (or both).

In the same spirit of the tests conducted in the previous section, the sample is further split accordingly to industry classification and the regressions in (20) are re-estimated. As there is no effect for short-maturity options, only long dated options are taken into consideration. Results are reported in Table 8.

The four sets of regressions show industry differences in the ability of the measure of credit risk to predict future changes in the option skew. Mirroring the results of previous sections, a larger predictive power is associated with companies

Regressand						Adj–R^2: 0.0007
ΔSkew$_{T<1}$						
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
$\overline{AICR}'_{-1,T<1}$	0.9772085	0.714567	1.37	0.176		
$\alpha_{T<1}$	-0.0037777	0.0009448	-4.00	0.000	***	
firm-FE	✓					
year-FE	✓					

(a): Predictive regression for short-term skew.

Regressand						Adj–R^2: 0.0005
ΔSkew$_{T>1}$						
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
$\overline{AICR}'_{-1,T>1}$	0.353236	0.0784575	4.50	0.000	***	
$\alpha_{T>1}$	-0.0114802	0.0017093	-6.72	0.000	***	
firm-FE	✓					
year-FE	✓					

(b): Predictive regression for long-term skew.

Table 7: Estimation of regression (20).
(a): Predictive regression for short-term skew based on the average ICR calculated over short-term put options and CDSs. Number of observations: 7,656. F–stat = 10.35 (p–value = 0.000).
(b): Predictive regression for long-term skew based on the average ICR calculated over long-term put options and CDSs. Number of observations: 6,818. F–stat = 11.87 (p–value = 0.000).
A sandwich estimator for panel data is used to obtain robust standard errors. Significance levels: 10% (*), 5% (**), 1% (***).

which are sensibly levered (see Table 8, panels (a) and (b)). Despite the average lower leverage, also the changes in the skew of retail, wholesale and services companies is partially attributable to the credit risk of the companies. On the other hand, the future changes in the negative skew of manufacturing companies (which are the least levered in the sample, see Table 6) are not explained by the level of credit risk and is likely to be driven by other factors. Finally, the weaker predictive power of this measure for financial companies could be explained by their relatively large probability of survival (despite being highly levered). This would mirror the graphical evidence of the implied volatility of the compound option model being very close to the implied volatility of the Black-Scholes model.

4.4 Robustness Tests

I first focus on how the compound option pricing model (CO) performs out-of-sample compared to the Black-Scholes model (BS). Given the option dataset, the volatility surface of equity (for BS) and asset (for CO) is estimated every week (in-sample). These estimates are used to reprice the options traded the following week (out-of-sample). Then, the pricing error is assessed as the absolute percentage error with respect to the market price, that is $|\gamma^{market} - \gamma^{model}|/\gamma^{market}$. The compound option model is implemented both using stock and option price only, as well as with the triangulation stock-option-CDS, having assumed LGD = $\{50, 60, 80\}\%$. In addition to 50% assumed so far in the analysis, the loss given default parameter is also set to 60% and 80%, being these values explicitly suggested by the ISDA for the pricing of CDSs[5]. Results are reported in Tables 9 and 10.

[5]Visit `https://www.cdsmodel.com/cdsmodel/assets/cds-model/docs` for the document `ISDA Standard CDS Contract Converter Specification - Sept 4, 2009.pdf`. A market-wide loss given default of 60% is also used by Collin-Dufresne et al. (2010).

Regressand						
$\Delta\text{Skew}_{T>1}$				Adj–R^2:	0.0022	
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
$\overline{AICR}'_{-1,T>1}$	0.3942216	0.1693359	2.33	0.048	**	
$\alpha_{T>1}$	-0.0140584	0.0116132	-1.21	0.268		
firm-FE	✓					
year-FE	✓					

(a): Financials.

Regressand						
$\Delta\text{Skew}_{T>1}$				Adj–R^2:	0.0017	
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
$\overline{AICR}'_{-1,T>1}$	0.2832847	0.0658556	4.30	0.004	***	
$\alpha_{T>1}$	-0.0194797	0.002694	-7.23	0.000	***	
firm-FE	✓					
year-FE	✓					

(b): Mining, Energy and Utilities.

Regressand						
$\Delta\text{Skew}_{T>1}$				Adj–R^2:	0.0006	
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
$\overline{AICR}'_{-1,T>1}$	-2.931313	2.452305	-1.20	0.247		
$\alpha_{T>1}$	-0.0078513	0.0021041	-3.73	0.001	***	
firm-FE	✓					
year-FE	✓					

(c): Manufacturing.

Regressand						
$\Delta\text{Skew}_{T>1}$				Adj–R^2:	0.0007	
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
$\overline{AICR}'_{-1,T>1}$	0.4410598	0.0564674	7.81	0.000	***	
$\alpha_{T>1}$	-0.0105209	0.0026711	-3.94	0.001	***	
firm-FE	✓					
year-FE	✓					

(d): Retail, Wholesale and Services.

Table 8: Estimation of regression (20) over the four sub-samples.
(a): Predictive regression for short-term skew based on the average ICR calculated over long-term put options and CDSs of Financials. Number of observations: 810. F–stat = 5.18 (p–value = 0.021).
(b): Predictive regression for short-term skew based on the average ICR calculated over long-term put options and CDSs of Mining, Energy and Utilities. Number of observations: 791. F–stat = 32.75 (p–value = 0.000).
(c): Table 5b: Predictive regression for short-term skew based on the average ICR calculated over long-term put options and CDSs of Manufacturing. Number of observations: 2,305 F–stat = 4.02 (p–value = 0.012).
(d): Predictive regression for short-term skew based on the average ICR calculated over long-term put options and CDSs of Retail, Wholesale and Services. Number of observations: 2,912 F–stat = 38.71 (p–value = 0.000).
A sandwich estimator for panel data is used to obtain robust standard errors. Significance levels: 10% (*), 5% (**), 1% (***).

$LEV \in (0, 0.5]$

T	K/S	BS	CO	$CO50$	$CO60$	$CO80$	N	m_C
$[0.5, 1)$	$(1, m_C)$	0.06	0.08	0.15	0.14	0.12	43,598	1.09
	$[m_C, \infty)$	0.11	0.12	0.21	0.19	0.17		
$[1, 1.5)$	$(1, m_C)$	0.05	0.10	0.13	0.13	0.11	16,634	1.12
	$[m_C, \infty)$	0.09	0.12	0.15	0.15	0.12		
$[1.5, 2)$	$(1, m_C)$	0.06	0.12	0.15	0.15	0.13	8,963	1.12
	$[m_C, \infty)$	0.10	0.12	0.15	0.15	0.13		
$[2, \infty)$	$(1, m_C)$	0.06	0.11	0.13	0.12	0.11	2,529	1.12
	$[m_C, \infty)$	0.10	0.13	0.12	0.13	0.11		

$LEV \in (0.5, 1]$

T	K/S	BS	CO	$CO50$	$CO60$	$CO80$	N	m_C
$[0.5, 1)$	$(1, m_C)$	0.08	0.14	0.69	0.57	0.49	4,239	1.08
	$[m_C, \infty)$	0.14	0.22	1.00	0.76	0.63		
$[1, 1.5)$	$(1, m_C)$	0.06	0.19	0.80	0.71	0.62	1,061	1.11
	$[m_C, \infty)$	0.14	0.32	0.94	0.78	0.65		
$[1.5, 2)$	$(1, m_C)$	0.08	0.31	0.79	0.72	0.64	653	1.11
	$[m_C, \infty)$	0.16	0.31	0.79	0.67	0.56		
$[2, \infty)$	$(1, m_C)$	0.10	0.24	0.67	0.61	0.52	225	1.09
	$[m_C, \infty)$	0.18	0.32	0.66	0.64	0.48		

$LEV \in (1, 2]$

T	K/S	BS	CO	$CO50$	$CO60$	$CO80$	N	m_C
$[0.5, 1)$	$(1, m_C)$	0.06	0.24	0.93	0.58	0.68	4,017	1.09
	$[m_C, \infty)$	0.13	0.49	1.81	0.82	1.18		
$[1, 1.5)$	$(1, m_C)$	0.06	0.33	1.06	0.95	0.84	226	1.16
	$[m_C, \infty)$	0.19	0.72	1.89	1.68	1.25		
$[1.5, 2)$	$(1, m_C)$	0.10	0.32	0.92	0.87	0.71	119	1.16
	$[m_C, \infty)$	0.20	0.55	1.38	1.01	0.97		
$[2, \infty)$	$(1, m_C)$	0.03	0.28	0.80	0.74	0.61	53	1.12
	$[m_C, \infty)$	0.12	0.35	0.85	0.78	0.63		

$LEV \in (2, \infty)$

T	K/S	BS	CO	$CO50$	$CO60$	$CO80$	N	m_C
$[0.5, 1)$	$(1, m_C)$	0.06	0.18	1.29	0.93	0.98	1,038	1.12
	$[m_C, \infty)$	0.15	0.47	2.47	1.21	1.70		
$[1, 1.5)$	$(1, m_C)$	0.34	8.38	7.32	12.76	15.58	14	1.36
	$[m_C, \infty)$	0.55	19.88	34.61	37.47	35.34		
$[1.5, 2)$	$(1, m_C)$	0.16	1.57	3.78	3.69	3.54	7	1.37
	$[m_C, \infty)$	0.40	5.98	8.93	9.05	9.05		
$[2, \infty)$	$(1, m_C)$	-	-	-	-	-	1	1.35
	$[m_C, \infty)$	0.59	7.39	16.26	17.04	17.43		

Table 9: Out-of-sample average absolute percentage pricing error using the Black-Scholes model versus a compound option model for call options. The compound option model is either implemented via the joint calibration stock and option price or via the triangulation stock-option-CDS spreads (having assumed LGD = $\{50, 60, 80\}\%$.) The errors are clustered based on progressively more leveraged firms ($LEV \equiv D/S$). Then, the errors are analysed based on the time-to-maturity (T) and the moneyness (K/S). The moneyness dimension is further split into OTM, $K/S \in (1, m_C)$, and deep OTM, $K/S \in [m_C, \infty)$, where m_C is the median moneyness in each cluster. N is the number of prices in each bucket. Both the Black-Scholes and the compound option pricing errors increases with T and as the option becomes more and more OTM. The pricing error of the simple compound option (CO) generally increases with leverage.

$LEV \in (0, 0.5]$

T	K/S	BS	CO	$CO50$	$CO60$	$CO80$	N	m_P
$[0.5, 1)$	$[m_P, 1)$	0.05	0.08	0.11	0.10	0.09	51,519	0.86
	$(0, m_P)$	0.10	0.12	0.16	0.15	0.13		
$[1, 1.5)$	$[m_P, 1)$	0.04	0.09	0.11	0.11	0.09	20,469	0.83
	$(0, m_P)$	0.08	0.11	0.13	0.13	0.11		
$[1.5, 2)$	$[m_P, 1)$	0.04	0.11	0.12	0.12	0.10	12,042	0.81
	$(0, m_P)$	0.10	0.13	0.13	0.14	0.11		
$[2, \infty)$	$[m_P, 1)$	0.06	0.14	0.13	0.14	0.13	3,554	0.79
	$(0, m_P)$	0.15	0.18	0.18	0.19	0.17		

$LEV \in (0.5, 1]$

T	K/S	BS	CO	$CO50$	$CO60$	$CO80$	N	m_P
$[0.5, 1)$	$[m_P, 1)$	0.05	0.12	0.41	0.41	0.28	5,373	0.88
	$(0, m_P)$	0.10	0.20	0.49	0.49	0.34		
$[1, 1.5)$	$[m_P, 1)$	0.04	0.36	0.32	0.42	0.23	1,928	0.85
	$(0, m_P)$	0.10	0.39	0.37	0.51	0.27		
$[1.5, 2)$	$[m_P, 1)$	0.05	0.36	0.33	0.42	0.26	1,280	0.83
	$(0, m_P)$	0.10	0.36	0.34	0.49	0.26		
$[2, \infty)$	$[m_P, 1)$	0.06	0.45	0.29	0.39	0.23	438	0.82
	$(0, m_P)$	0.13	0.80	0.36	0.56	0.29		

$LEV \in (1, 2]$

T	K/S	BS	CO	$CO50$	$CO60$	$CO80$	N	m_P
$[0.5, 1)$	$[m_P, 1)$	0.06	0.18	0.49	0.57	0.38	2,532	0.81
	$(0, m_P)$	0.20	0.44	0.68	0.73	0.60		
$[1, 1.5)$	$[m_P, 1)$	0.04	0.09	0.35	0.32	0.27	1,010	0.83
	$(0, m_P)$	0.08	0.13	0.44	0.39	0.37		
$[1.5, 2)$	$[m_P, 1)$	0.05	0.09	0.42	0.44	0.43	846	0.80
	$(0, m_P)$	0.11	0.16	0.41	0.39	0.38		
$[2, \infty)$	$[m_P, 1)$	0.05	0.43	0.29	0.56	0.21	213	0.80
	$(0, m_P)$	0.15	0.74	0.34	0.66	0.36		

$LEV \in (2, \infty]$

T	K/S	BS	CO	$CO50$	$CO60$	$CO80$	N	m_P
$[0.5, 1)$	$[m_P, 1)$	0.05	0.08	0.85	0.94	0.95	117	0.68
	$(0, m_P)$	0.45	0.57	2.34	1.48	1.63		
$[1, 1.5)$	$[m_P, 1)$	0.04	0.06	0.56	0.56	0.60	221	0.83
	$(0, m_P)$	0.07	0.09	0.55	0.61	0.68		
$[1.5, 2)$	$[m_P, 1)$	0.05	0.05	0.41	0.42	0.54	188	0.83
	$(0, m_P)$	0.08	0.09	0.43	0.42	0.54		
$[2, \infty)$	$[m_P, 1)$	0.06	0.50	0.87	0.88	0.99	44	0.81
	$(0, m_P)$	0.14	0.57	1.05	1.01	1.15		

Table 10: Out-of-sample average absolute percentage pricing error using the Black-Scholes model versus a compound option model for put options. The compound option model is either implemented via the joint calibration stock and option price or via the triangulation stock-option-CDS spreads (having assumed LGD = $\{50, 60, 80\}\%$.) The errors are clustered based on progressively more leveraged firms ($LEV \equiv D/S$). Then, the errors are analysed based on the time-to-maturity (T) and the moneyness (K/S). The moneyness dimension is further split into OTM, $K/S \in [m_P, 1)$, and deep OTM, $K/S \in (0, m_P)$, where m_P is the median moneyness in each cluster. N is the number of prices in each bucket. Both the Black-Scholes and the compound option pricing errors increases with T and as the option becomes more and more OTM. The pricing error of the simple compound option (CO) generally increases with leverage.

I find that, in general, the Black-Scholes performs better out-of-sample than any compound option model, both for call and put options. The average pricing error for put options is though substantially lower than for call options, again pointing towards negligible sensitivity of the latter with respect to credit events.

More specifically, the pricing error are analysed across the leverage, time-to-maturity and moneyness dimension. The errors produced by the CO models are comparable in size with those of BS for moderately levered firms ($D/S \in (0, 0.5]$); however these tend to increase with leverage. Only for highly levered firms ($D/S \in (2, \infty)$), the pricing errors consistently reduce for both call and put options. Both the BS and the CO models show increasing pricing errors for options with longer maturities as well as for deep out-of-the-money options. Also, it is worth highlighting that when the CO model is implemented by making use of the CDS spreads, the smallest pricing error is observed for LGD = 80%.

The only available reference to compare the out-of-sample analysis with is Geske et al. (2016). There, the authors implement the Geske (1979) model and compare how a compound option model performs with respect to the Black-Scholes model. My results are in sharp contrast with their findings. On average, I obtain smaller pricing errors for the Black-Scholes than in the case of the compound option model (for those instances in which the samples are comparable). They document the opposite. Even more surprisingly, my errors (both for the BS and the CO) are significantly smaller than what they report. However, several differences in the approach proposed here are worth mentioning.

Firstly, their sample is formed by American call options only, with at most one year to maturity. I instead focus on both American call and put options with maturity at least equal to six months. Further, they model equity as a 1–fold compound option as they assume that the firms have issued a zero-coupon bond with maturity equal to the duration of the firm's debt. As shown in Section 2.1, Figure 5, when debt is clustered at one future date only, it would be optimal for shareholders to take on as much debt as possible as the positive sensitivity of equity with respect to business risk increases with leverage. This does not happens in a 3–fold compound option model for equity. Also, their definition of firm debt is much broader than ours (as instance, they also include accrued expense and deferred income as well as deferred federal tax in their definition of debt).

Secondly, their calibration methodology is different as they estimate a term structure of asset volatility (embedded in their model) using the three most at-the-money call options and the stock price. In fact, they opt for having two different parameters for the volatility of the assets, one for the duration of the firm's debt and one for the maturity of the option. In order to make a more appropriate comparison with the Black-Scholes model, I instead estimate the asset volatility surface by 'inverting' the compound option pricing model in order to match both the option and stock price (that is, in the same spirit the equity implied volatility is extracted à la Black-Scholes). Also, my estimates are based on both out-of-the-money call and put options.

Finally, they need to have four equations to calibrate the compound option model as they opt for embedding a term structure of the asset volatility within their model. If, instead, the volatility is assumed constant as in a geometric Brownian motion, the implied default barrier (and the implied strike price in the asset space) are implicit functions of the (unique) asset volatility (see Appendix E), which can be retrieved using two equations only, namely (8) and (11). Also, they do not obtain pseudo-European option prices but consider only those options which, retrospectively, did not pay dividends between the valuation day and the maturity of the option. This information, however, is not usually available when the market participants trade options. Therefore, the proposed approach seems more parsimonious and makes the comparison with the Black-Scholes model more straightforward.

A second robustness test is based on the comparison of the estimated option-implied asset volatility obtained with and without the additional calibration on the risk-neutral probabilities of default extracted from CDS spreads. Then, the absolute percentage errors $|\sigma_V - \sigma'_V|/\sigma_V$ are computed, where σ'_V and σ_V are the implied asset volatility obtained with and without the CDS calibration for the values of LGD = $\{50, 60, 80\}\%$. This allows to infer the market-implied loss given default consistent with option prices. Table 11 reports the results.

The smallest absolute percentage error is observed for LGD $= 60\%$, both for calls and puts, consistently with Collin-Dufresne et al. (2010). The errors are also smaller for put than call options, and they appear to be increasing with leverage.

The robustness section is concluded by re-running the regressions in Sections 4.2 and 4.3 for alternative values of loss given default (again, 60% and 80%). The results are reported in the tables in Appendix H. The overall conclusions in terms of insensitivity of call options to credit events and the ability of the measure of impact of credit risk to predict the future changes in the negative skew of equity options remain unchanged.

5 Conclusions

In this paper I investigate the effect of credit related events on the pricing of equity options. Given a firm which has issued $n \geq 1$ defaultable coupon-bearing bonds, I generalise the results in Merton (1974) and Geske (1977), and price the firm's equity as an n–fold compound option call option on the asset value struck at the face values of the bonds outstanding. Further, I extend the pricing formula in Geske (1979) and show that European vanilla options on the firm's equity are $(n + 1)$–fold compound options written on the value of the firm's assets. This frameworks constitutes the most natural environment to study the impact of credit risk on equity options consistently with the structural approach to default.

I further explore the predictions of a compound option model on a sample of 66 US firms from January, 2013 to December, 2017. These are the constituents of the S&P100 which neither issued preferred equity nor engaged into extraordinary financial operations (such as M&As) during the selected sample period.

Given the probabilistic implications of the model, a new measure of impact of credit risk on options is constructed, thus allowing to rank the latter based on how much they are exposed to credit-related events. Consistently with the economic intuition and the results in Carr and Wu (2011) (who, instead, opt for a reduced-form approach to default), call and put option prices account for the possibility of the company to default very differently. More specifically, call options do not price credit risk, whilst the price of put options does embed it. To the best of my knowledge, this is the first work which explores and rigorously assesses this phenomenon using a large sample of options (both in the cross-section and the time-series dimension).

I finally attempt at predicting the future changes in the negative skew displayed by equity options. I show that the novel measure of credit risk constructed on put prices is able to forecast future movements of the skew for long-maturity equity options. To the best of my knowledge, this is the first work which tries to capture and predict the changes in the option skew with a measure of credit risk. Further work is however required to capture these movements more precisely. Factors based on the channels described in Carr and Wu (2017) could be constructed in order to improve the fit of the proposed regressions.

The implications of this study are multifaceted. Given the importance of default risk for those assets which are more sensitive to events occurring in the left tail of equity distribution (e.g. put options), risk-management implications for those instruments are relevant, especially when the underlying is the equity of either highly-levered or financially-distressed companies.

For example, hedge funds often take highly levered positions in corporate bonds while hedging away interest rate risk by shorting treasuries. As a consequence, their portfolios become extremely sensitive to changes in credit spreads rather than changes in bond yields. If there is a nonnegligible probability of large negative jumps in firm value, then the appropriate hedging tool for corporate debt may not be the firm's equity, but rather deep out-of-the-money puts on the firm's equity. In turn, the writer of these options will need to hedge its short position.

It is trivial to show that the Delta-hedge under the Black-Scholes model, which ignores credit risk, is different than the hedge prescribed by a compound option model. Ignoring dividends for simplicity, the Delta-hedges under the Black-Scholes and a compound option model differ as such

		LGD					
		Call			Put		
	LEV	50%	60%	80%	50%	60%	80%
AAPL	0.09	0.035	0.010	0.031	0.018	0.007	0.017
ABT	0.11	0.124	0.042	0.106	0.077	0.021	0.075
ACN	0.00	0.000	0.000	0.000	0.000	0.000	0.000
ALL	0.23	0.097	0.089	0.062	0.048	0.037	0.033
AMGN	0.29	0.103	0.091	0.062	0.078	0.045	0.055
AMZN	0.05	0.081	0.068	0.062	0.070	0.044	0.057
BA	0.10	0.025	0.025	0.013	0.027	0.012	0.021
BAC	1.79	1.172	0.570	0.943	0.212	0.272	0.149
BMY	0.08	0.030	0.015	0.023	0.016	0.009	0.012
C	1.63	1.562	0.615	1.284	0.125	0.281	0.311
CAT	0.60	0.390	0.193	0.281	0.261	0.060	0.221
CL	0.10	0.043	0.027	0.032	0.028	0.016	0.020
CMCSA	0.40	0.115	0.102	0.109	0.103	0.064	0.072
COF	1.02	0.499	0.361	0.358	0.380	0.234	0.249
COP	0.37	0.178	0.158	0.113	0.114	0.078	0.072
COST	0.08	0.021	0.020	0.013	0.024	0.010	0.019
CSCO	0.16	0.072	0.045	0.050	0.044	0.023	0.034
CVS	0.21	0.105	0.067	0.075	0.060	0.035	0.043
CVX	0.14	0.130	0.061	0.104	0.048	0.024	0.038
DD	0.21	0.092	0.066	0.063	0.057	0.035	0.042
DIS	0.11	0.081	0.019	0.072	0.054	0.010	0.051
EMR	0.11	0.126	0.049	0.105	0.073	0.020	0.071
EXC	0.81	0.550	0.416	0.389	0.272	0.185	0.165
F	1.99	1.205	0.742	0.926	0.577	0.120	0.483
FDX	0.17	0.085	0.072	0.054	0.053	0.042	0.034
GD	0.08	0.015	0.017	0.008	0.023	0.010	0.018
GE	1.00	0.509	0.293	0.364	0.294	0.108	0.233
HAL	0.22	0.077	0.066	0.048	0.057	0.046	0.035
HD	0.13	0.040	0.034	0.025	0.027	0.017	0.019
IBM	0.24	0.182	0.078	0.146	0.103	0.029	0.087
INTC	0.12	0.047	0.037	0.030	0.035	0.021	0.026
JNJ	0.06	0.022	0.018	0.014	0.009	0.008	0.006
JPM	1.51	0.751	0.658	0.527	0.176	0.141	0.122
KO	0.14	0.063	0.053	0.040	0.029	0.025	0.019
LLY	0.09	0.034	0.029	0.022	0.018	0.015	0.012
LOW	0.21	0.064	0.054	0.041	0.031	0.026	0.019
MCD	0.18	0.076	0.064	0.049	0.039	0.033	0.024
MDT	0.26	0.111	0.093	0.071	0.052	0.043	0.032
MMM	0.09	0.029	0.024	0.018	0.012	0.010	0.007
MO	0.14	0.082	0.069	0.053	0.033	0.028	0.021
MON	0.14	0.068	0.058	0.044	0.033	0.028	0.021
MRK	0.15	0.020	0.017	0.013	0.009	0.008	0.006
MS	2.71	0.786	0.699	0.577	0.554	0.615	0.715
MSFT	0.09	0.027	0.023	0.017	0.015	0.013	0.010
ORCL	0.21	0.070	0.059	0.045	0.041	0.034	0.026
OXY	0.12	0.065	0.055	0.042	0.046	0.039	0.030
PEP	0.22	0.137	0.115	0.088	0.051	0.043	0.032
PFE	0.17	0.065	0.055	0.041	0.033	0.027	0.021
PG	0.10	0.044	0.037	0.028	0.016	0.014	0.010
PM	0.19	0.109	0.092	0.070	0.043	0.036	0.027
RTN	0.14	0.046	0.038	0.029	0.025	0.021	0.016
SLB	0.13	0.050	0.042	0.032	0.028	0.023	0.017
SO	0.68	0.632	0.537	0.417	0.209	0.172	0.127
SPG	0.47	0.451	0.388	0.304	0.207	0.173	0.130
T	0.51	0.506	0.432	0.335	0.208	0.173	0.129
TGT	0.35	0.150	0.127	0.097	0.072	0.060	0.045
TWX	0.36	0.171	0.145	0.111	0.105	0.087	0.065
TXN	0.08	0.033	0.028	0.021	0.015	0.013	0.010
UNH	0.20	0.078	0.066	0.050	0.037	0.031	0.023
UNP	0.16	0.036	0.031	0.025	0.023	0.019	0.014
USB	0.47	0.253	0.214	0.164	0.084	0.069	0.051
UTX	0.23	0.103	0.087	0.066	0.044	0.037	0.028
VZ	0.55	0.476	0.406	0.314	0.211	0.175	0.130
WFC	0.85	0.492	0.422	0.327	0.172	0.138	0.096
WMT	0.20	0.089	0.075	0.057	0.043	0.036	0.027
XOM	0.06	0.020	0.017	0.013	0.008	0.007	0.005
Mean		0.212	0.148	0.153	0.092	0.066	0.073

Table 11: Absolute percentage errors between the option-implied asset volatility obtained with and without the CDS spreads calibration for LGD = $\{50, 60, 80\}\%$. The average errors are reported at the company level (where the time-average leverage is also reported). The last row display the overall average, thus suggesting a market implied loss given default of 60%.

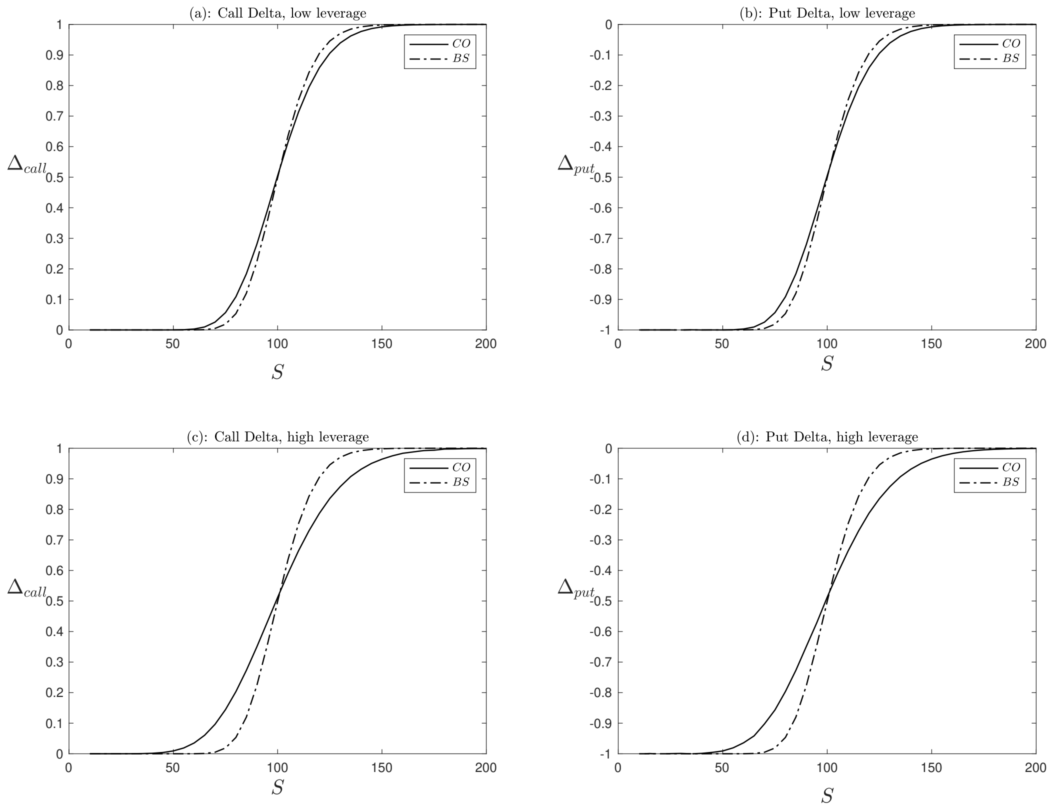


Figure 13: Comparison of the delta of call and put options under the Black-Scholes model and a 3–fold compound option model for the underlying. Pictures (a) and (b) show the effect of ignoring the possibility of default for a low levered firm ($F_1 = F_2 = F_3 = 10$), whilst pictures (c) and (d) show the same comparison for a company which is more levered ($F_1 = F_2 = F_3 = 30$). The optimal hedge ratio is obviously more distorted for the case of highly-levered firms as they are naturally more exposed to credit-related events. Also here $K = 100$, $T = 0.5$, $t_1 = 1$, $t_2 = 5$, $t_3 = 10$, $r = 0.03$ and $\varpi = 0.05$.

$$\frac{\partial \gamma_\xi^{CO}}{\partial S} = \frac{\partial \gamma_\xi^{CO}/\partial V}{\partial S/\partial V} = \frac{\Delta_{\gamma,\xi}^{(n)}}{\Delta_S^{(n)}} \neq \xi \Phi\left(\xi d_1^{\mathbb{M}}\right) = \frac{\partial \gamma_\xi^{BS}}{\partial S}.$$

Figure 13 shows that the Black-Scholes hedge underestimates the number of units of the underlying required to hedge the short option position when the option is out-of-the-money. Conversely, it overestimates the delta-hedged position in the equity when the option is in-the-money. This bias would apply to any other hedging strategy based on the Greeks of the option when credit risk is not modelled. Also, this distortion becomes more and more severe for increasingly levered firms. A more in-depth analysis of this imperfect hedging when default risk is not taken into account is though left for future research.

Furthermore, the countintuitive findings in Carr and Wu (2017) of increasing leverage for increasing diffusive volatility can be easily reconciled with a model in which shareholders maximise the firm value, based on a targeted leverage ratio, and equity is modelled as a compound option. Optimal capital structure policies, however, are not explored in this work and left for future research as well.

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Appendix A Notation and Abbreviation

If not differently specified, sets as well as univariate random variables are indicated as upper case letters (X), scalars as lower case letters (x), vectors as lower case bold letters ($\mathbf{x}$), matrices as well as multivariate random vectors as upper case bold letters ($\mathbf{X}$).

$\mathbb{N} = \{1, 2, \dots\}$	set of natural number
$\mathbb{R} = (-\infty, +\infty)$	set of real number
$\mathbb{R}_+ = [0, +\infty)$	set of non-negative real number
$\mathbb{R}^n$	Euclidean n-dimensional space
$\mathcal{M}^n_+$	space of positive definite matrices
$\circ$	composition of function (operator)
$f \in o(g)$ as $x \to a$	$\lim_{x\to a} f/g = 0$
$f \in O(g)$ as $x \to a$	$\limsup_{x\to a} \lvert f/g \rvert < \infty$
$f \sim g$ as $x \to a$	$\lim_{x\to a} f/g = 1$
$\mathbb{1}_A : X \to [0, 1]$	indicator function of a subset A of a set X
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$	filtered probability space, with $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$
$\mathcal{S}(\mathbb{F}, \mathbb{P})$	vector space of semimartingales on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$
$\mathbb{E}^{\mathbb{P}}$	expectation under $\mathbb{P}$
$\mathbb{E}^{\mathbb{P}}_t$ (or $\mathbb{E}^{\mathbb{P}}_k$)	conditional expectation under $\mathbb{P}$ given $\mathcal{F}_t$ (or $\mathcal{F}_{t_k}$)
$\mathbb{V}$	variance
$\mathbb{C}$ov	covariance
$\mathbb{C}$orr	correlation
$\mathbb{Q}^n$	t_n-forward measure
$\mathbb{Q}$	risk-neutral measure
$\mathbb{Q} \sim \mathbb{P}$	$\mathbb{Q}$ is equivalent with respect to $\mathbb{P}$
PDF	probability density function
CDF	cumulative distribution function
$X \sim \dots$	X is distributed as $\dots$
$n(\cdot; \mu, \sigma)$	PDF of $X \sim \mathcal{N}(\mu, \sigma)$
$n_m(\cdot; \boldsymbol{\mu}, \boldsymbol{\Sigma})$	PDF of $\mathbf{X} \sim \mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
$N(\cdot; \mu, \sigma)$	CDF of $X \sim \mathcal{N}(\mu, \sigma)$
$N_m(\cdot; \boldsymbol{\mu}, \boldsymbol{\Sigma})$	CDF of $\mathbf{X} \sim \mathcal{N}_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
$\phi(\cdot)$	PDF of $X \sim \mathcal{N}(0, 1)$
$\phi_m(\cdot; \boldsymbol{\Gamma})$	PDF of $\mathbf{X} \sim \mathcal{N}_m(\mathbf{0}, \boldsymbol{\Gamma})$ with $\gamma_{ii} = 1, \forall i \le m$
$\Phi(\cdot)$	CDF of $X \sim \mathcal{N}(0, 1)$
$\Phi_n(\cdot; \boldsymbol{\Gamma})$	CDF of $\mathbf{X} \sim \mathcal{N}_m(\mathbf{0}, \boldsymbol{\Gamma})$ with $\gamma_{ii} = 1, \forall i \le m$
$\Phi_n(\cdot)$	CDF of $\mathbf{X} \sim \mathcal{N}_m(\mathbf{0}, \mathbf{I})$ with $\mathbf{I}$ the $m \times m$ identity matrix
X_{t-} (or X_{i-})	$\lim_{s\uparrow t} X_s$ (or $\lim_{s\uparrow t_i} X_s$)
r_t	instantaneous spot rate at time t
$B_t = \exp\left(\int_0^t r_s \, \mathrm{d}s\right)$	value of a bank account at time t
$D(t, T) = \frac{B_t}{B_T}$	discount factor between time t and T
τ	default time
LGD	loss given default
V_t	market value of the firm at time t
S_t	market value of the firm's equity at time t
$D_{t,T}$	market value of the firm's debt with maturity T at time t
$\mathrm{El}_x(y)$	elasticity of y with respect to x

Appendix B Further Description of the Data and Construction of the Variables

In order implement the calibration described in Section 3, one of the most crucial aspects is to effectively and efficiently represent the firms' capital structures. In particular, every firm can actually have $n \geq 0$ bonds outstanding; however, it is virtually impossible to solve for the unobservable (V_0, σ_V) if n is very large, as the calibration involves the solution of an n-dimentional integral equation. Moreover, the estimation of the asset value and its volatility involes the inversion of the multivariate normal CDF, which is implemented in Matlab via the function `mvncdf`. For bivariate and trivariate distributions, `mvncdf` uses adaptive quadrature on a transformation of the t density, based on methods developed in Genz (2004). For four or more dimensions, `mvncdf` uses a quasi-Monte Carlo integration algorithm based on methods developed by Genz and Bretz (1999) and Genz and Bretz (2002). As a matter of fact, for $n \geq 4$, the algorithm becomes much slower due to the quasi-Monte Carlo integration.

In addition, as each company have a different n, it is extremely impractical not to have a 'standard framework' of valuation. Also, for US firms detailed data about individual bonds outstanding is available only at yearly frequency in the 10-K document (and the face value of the bonds are to be collected manually). On the other hand, in order to homogenise different capital structures and construct a standard framework for the implementation, the number of bond outstanding is set such that $n \leq 3$ as described below.

From Compustat, the variables `DLTTQ` (Long-Term Debt – Total) and `DD1Q` (Long-Term Debt Due in One Year) are downloaded every quarter. `DLTTQ` represents debt obligations due more than one year from the company's Balance Sheet date or due after the current operating cycle, whilst `DD1Q` represents the current portion of long-term debt. They proxy as long-term and short-term debt respectively. Finally, the three synthetic bonds are defined as

$$F_1 = \frac{\texttt{DD1Q}}{\texttt{CSHO}}, \qquad F_2 = F_3 = 0.5 \cdot \frac{\texttt{DLTTQ}}{\texttt{CSHO}},$$

with $\{t_1, t_2, t_3\} = \{1, 5, 10\}$ and `CSHO` the number of common share outstanding. Those are reset at each quarter, and the time to maturity is adjusted accordingly for the effect of passage of time when the estimates of the implied volatility of the assets are carried out.

The choice of $t_1 = 1$ is trivial and given by the definition of the variables; $t_2 = 5$ is chosen as it is well-documented that the most actively traded CDS is the 5-year contract (which is used for the calibration). Also t_3 is set at 10 years in order not to rely on CDS with very long maturities (such as 20 or 30-years contracts) as they could be very illiquid.

Unfortunately, the variable `DD1Q` can either be not available at quarterly frequency or is not reported at all (the latter is usually observed for banks and energy companies). Under these cases, the variable is estimated using the quarterly variable Debt in Current Liabilities (`DLCQ`) which is always available. Notice that `DD1Q` (when available) is a fraction of `DLCQ`. Therefore, in case of missing observation, the last available `DD1Q`/`DLCQ` ratio is used to determine the contemporaneous `DD1Q`. Finally, if `DD1Q` is never reported, the average `DD1Q`/`DLCQ` of comparable companies based on Division/Sub-division (see Table B.1) is estimated and `DD1Q` is projected accordingly.

Moreover, I believe that the choice of setting $n = 3$[6] constitutes the optimal number of bonds such that both *level*, *slope* and *curvature* of the term structure of the survival probabilities extracted from CDS are matched by the model. As a matter of fact, the calibration procedure to be effective relies on the ability of the structural model of default to reproduce the aforementioned term structure.

For completeness, I report the composition of `DD1Q` and `DLTTQ`.

`DD1Q` includes:

1. Current portion of any item defined as long-term debt (for example, the current portion of a long-term lease obligation);

[6] There are instances where $n < 3$ as the company does not have either short-term or long-term debt; however the most frequent capital structures have $n = 3$.

2. Instalments on a loan;
3. Sinking fund payments.

This item excludes:

1. Current portions of debt that do not reflect discounts on long term debt;
2. Debt that includes interest payments due;
3. Demand notes;
4. Debt in default if there is no associated long term debt reported as part of the long term liabilities;
5. Estimated claims and other liabilities under Chapter XI or other bankruptcy proceedings;
6. Interest on capitalized lease obligations.

`DLTTQ` includes:

1. Advances to finance construction;
2. Bonds, mortgages, and similar debt;
3. ESOP loan guarantees;
4. Extractive industries' advances for exploration and development;
5. Forestry and paper companies' timber contracts;
6. Gold and bullion loans;
7. Guaranteed Preferred Beneficial Interests in Corporation's Junior Subordinated Deferred Interest Debentures;
8. Indebtedness to affiliates;
9. Industrial revenue bonds;
10. Instalment Obligations – nonrecourse;
11. Line of credit, when reclassified as a non-current liability;
12. Loans;
13. Loans on insurance policies;
14. Long-term lease obligations (capitalized lease obligations);
15. Mandatorily Redeemable Capital Securities of Subsidiary Trust;
16. Notes payable, due within one year to be refunded by long-term debt when carried as noncurrent liability;
17. Obligations called"note" or"deb" whether or not they are interest-bearing;
18. Obligations requiring interest payment that are not specified by type;
19. Production payments and advances for exploration and development;
20. Publishing companies' royalty contracts payable;
21. Purchase obligations and payments to officers (when listed as long-term liabilities);
22. Unamortized debt discount.

This item excludes:

1. Accounts payable due after one year (included in Liabilities – Other);
2. Accrued interest on long-term debt (included in Liabilities – Other);

3. Chapter XI bankruptcy terms;
4. Current portion of long-term debt (included in Current Liabilities);
5. Customers' deposits on bottles, kegs, and cases (included in Liabilities – Other);
6. Deferred compensation;
7. Subsidiary preferred stock (included in Minority Interest).

Finally, the payout ratio is calculated as the weighted average cost of capital for the company. The cost of equity (i.e. dividend yield, q) is estimated as the average dividend yield over the previous year. These data are downloaded from Bloomberg. The cost of debt is calculated as

$$c = \min\left\{\frac{\sum_{i=1}^{n} c_i F_i}{F}, \ln\left(1 + \frac{\mathtt{XINT}}{F}\right)\right\},$$

where $F = \sum_{i=1}^{n} F_i$ and c_i the continuously compounded debt payout rate. `XINT` is the Compustat variable Interest and Related Expense – Total. The individual rates c_i are observed at yearly frequency and manually collected from the 10-K documents. Eventually, the payout rate is estimated every year as

$$\varpi = \frac{cF + qS}{F + S},$$

where S is the value of the equity at the beginning of the year and q the dividend yield estimated as described above.

Table B.1: List of the selected companies and their SIC code.

Ticker	SIC	Division
AAPL	3663	Manufacturing
ABT	2834	Manufacturing
ACN	8742	Services
ALL	6331	Finance, Insurance and Real Estate
AMGN	2836	Manufacturing
AMZN	5961	Wholesale Trade
BA	3721	Manufacturing
BAC	6020	Finance, Insurance and Real Estate
BMY	2834	Manufacturing
C	6199	Finance, Insurance and Real Estate
CAT	3531	Manufacturing
CL	2844	Manufacturing
CMCSA	4841	Transportation, Communications, Electric, Gas and Sanitary service
COF	6141	Finance, Insurance and Real Estate
COP	1311	Mining
COST	5399	Wholesale Trade
CSCO	3576	Manufacturing
CVS	5912	Retail Trade
CVX	2911	Manufacturing
DD	2821	Manufacturing
DIS	4888	Transportation, Communications, Electric, Gas and Sanitary service
EMR	3823	Manufacturing
EXC	4911	Transportation, Communications, Electric, Gas and Sanitary service
F	3711	Manufacturing
FDX	4513	Transportation, Communications, Electric, Gas and Sanitary service
GD	3721	Manufacturing
GE	4911	Transportation, Communications, Electric, Gas and Sanitary service
HAL	1389	Mining
HD	5211	Wholesale Trade
IBM	7370	Services
INTC	3674	Manufacturing
JNJ	2834	Manufacturing
JPM	6020	Finance, Insurance and Real Estate
KO	2086	Manufacturing
LLY	2834	Manufacturing
LOW	5211	Wholesale Trade
MCD	5812	Retail Trade
MDT	3845	Manufacturing
MMM	2670	Manufacturing
MO	2111	Manufacturing
MON	5169	Retail Trade
MRK	2834	Manufacturing
MS	6211	Finance, Insurance and Real Estate
MSFT	7372	Services
ORCL	7370	Services
OXY	1311	Mining
PEP	2080	Manufacturing
PFE	2834	Manufacturing
PG	2840	Manufacturing
PM	2111	Manufacturing
RTN	3812	Manufacturing
SLB	1389	Mining
SO	4911	Transportation, Communications, Electric, Gas and Sanitary service
SPG	6798	Finance, Insurance and Real Estate
T	4812	Transportation, Communications, Electric, Gas and Sanitary service
TGT	5331	Wholesale Trade
TWX	8748	Services
TXN	3674	Manufacturing
UNH	6324	Finance, Insurance and Real Estate
UNP	4011	Transportation, Communications, Electric, Gas and Sanitary service
USB	6020	Finance, Insurance and Real Estate
UTX	3724	Manufacturing
VZ	4812	Transportation, Communications, Electric, Gas and Sanitary service
WFC	6020	Finance, Insurance and Real Estate
WMT	5331	Retail Trade
XOM	1311	Mining

Appendix C Gaussian Integrals

Theorem C.1. *Given a binary variables ξ taking values ± 1, $a \in \mathbb{R}$, and $\mathbf{b}, \mathbf{c} \in \mathbb{R}^m$ then*

$$\xi \int_a^{\xi\infty} n\left(x;\mu,\sigma\right) \Phi_m \left(\mathbf{b}x + \mathbf{c}; \tilde{\mathbf{\Gamma}}\right) \mathrm{d}x = \Phi_{m+1}\left(\xi \frac{\mu - a}{\sigma}, \mathbf{d}; \mathbf{\Gamma}_\xi\right), \tag{C.1}$$

for $\mathbf{d} \in \mathbb{R}^m : d_i = \frac{b_i \mu + c_i}{\sqrt{1 + b_i^2 \sigma^2}}$,

$$\mathbf{\Gamma}_\xi = \begin{pmatrix} \tilde{\mathbf{\Gamma}} & \boldsymbol{\gamma}_\xi \\ \boldsymbol{\gamma}_\xi^\top & 1 \end{pmatrix},$$

and $\boldsymbol{\gamma}_\xi \in \mathbb{R}^m : \gamma_{i,\xi} = \xi \frac{b_i \sigma}{\sqrt{1 + b_i^2 \sigma^2}}$.

Proof. For notational convenience, define

$$I_\xi := \xi \int_a^{\xi\infty} n\left(x;\mu,\sigma\right) \Phi_m \left(\mathbf{b}x + \mathbf{c}; \tilde{\mathbf{\Gamma}}\right) \mathrm{d}x.$$

I distinguish the two cases $\xi = 1$ and $\xi = -1$ (for convenience of notation the dependence on the distributions' parameters such as μ, σ and $\tilde{\mathbf{\Gamma}}$ is omitted).

1. Consider first $\xi = 1$, that is

$$\begin{aligned} I_1 &= \int_a^\infty n\left(x\right) \left[\int_{-\infty}^{b_1 x + c_1} \cdots \int_{-\infty}^{b_m x + c_m} \frac{1}{\sqrt{(2\pi)^m \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{\mathbf{y}^\top \tilde{\mathbf{\Gamma}}^{-1} \mathbf{y}}{2}\right) \mathrm{d}^m\, \mathbf{y}\right] \mathrm{d}x \\ &= \int_{D_0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \frac{1}{\sqrt{(2\pi)^m \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{\mathbf{y}^\top \tilde{\mathbf{\Gamma}}^{-1} \mathbf{y}}{2}\right) \mathrm{d}^m\, \mathbf{y}\, \mathrm{d}x \end{aligned}$$

   where $D_0 := \{x \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^m : x \geq a, \bigcap_{i=1}^m \{y_i \leq b_i x + c_i\}\}$.

   Consider the transformation $\mathbf{T_1} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$

$$\mathbf{T_1} := \begin{cases} x = \mu - \sigma w \\ \mathbf{y} = \mathbf{y} \end{cases}$$

   with Jacobian

$$\mathbf{J_1} = \begin{pmatrix} -\sigma & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} -\sigma & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

   where $\mathbf{0}$ is the null column vector in $\mathbb{R}^m$ and $\mathbf{I}$ the $m \times m$ identity matrix.

Furthermore, $D_1 := D_0(\mathbf{T_1}) = \left\{w \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^m : w \leq \frac{\mu - a}{\sigma}, \bigcap_{i=1}^m \{y_i \leq -b_i\sigma w + b_i\mu + c_i\}\right\}$. Hence the integral becomes

$$
\begin{aligned}
I_1 &= \int_{D_1} n\left(\mu - \sigma w\right) \frac{1}{\sqrt{(2\pi)^m \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{\mathbf{y}^\top \tilde{\mathbf{\Gamma}}^{-1}\mathbf{y}}{2}\right) |\det \mathbf{J_1}|\, \mathrm{d}^m\mathbf{y}\, \mathrm{d}w \\
&= \int_{D_1} \phi\left(w\right) \frac{1}{\sqrt{(2\pi)^m \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{\mathbf{y}^\top \tilde{\mathbf{\Gamma}}^{-1}\mathbf{y}}{2}\right) \mathrm{d}^m\mathbf{y}\, \mathrm{d}w.
\end{aligned}
$$

Consider another transformation $\mathbf{T_2} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$

$$
\mathbf{T_2} := \begin{cases} w = w \\ \mathbf{y} = \mathbf{\Lambda}\left(\mathbf{z} - \boldsymbol{\gamma} w\right) \end{cases}
$$

with

$$
\mathbf{\Lambda} := \begin{pmatrix} \frac{1}{\sqrt{1-\gamma_1^2}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{1-\gamma_2^2}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\sqrt{1-\gamma_m^2}} \end{pmatrix}, \qquad \boldsymbol{\gamma} := \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \dots \\ \gamma_m \end{pmatrix},
$$

where $\gamma_i \in (-1, 1)$, for all $i \leq m$, are parameters that will be determined later. The corresponding Jacobian is

$$
\mathbf{J_2} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -\frac{\gamma_1}{\sqrt{1-\gamma_1^2}} & \frac{1}{\sqrt{1-\gamma_1^2}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -\frac{\gamma_m}{\sqrt{1-\gamma_m^2}} & 0 & \dots & \frac{1}{\sqrt{1-\gamma_m^2}} \end{pmatrix}.
$$

Furthermore,

$$
D_2 := D_1(\mathbf{T_2}) = \left\{ w \in \mathbb{R}, \mathbf{z} \in \mathbb{R}^m : w \leq \frac{\mu - a}{\sigma}, \bigcap_{i=1}^m \left\{ z_i \leq \left(\gamma_i - b_i\sigma\sqrt{1-\gamma_i^2}\right) w + \left(b_i\mu + c_i\right)\sqrt{1-\gamma_i^2} \right\} \right\}.
$$

Hence

$$
\begin{aligned}
I_1 &= \int_{D_2} \phi\left(w\right) \frac{1}{\sqrt{(2\pi)^m \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{\left(\mathbf{\Lambda}\left(\mathbf{z} - \boldsymbol{\gamma} w\right)\right)^\top \tilde{\mathbf{\Gamma}}^{-1}\left(\mathbf{\Lambda}\left(\mathbf{z} - \boldsymbol{\gamma} w\right)\right)}{2}\right) |\det \mathbf{J_2}|\, \mathrm{d}^m\mathbf{z}\, \mathrm{d}w \\
&= \int_{D_2} \phi\left(w\right) \frac{1}{\sqrt{(2\pi)^m \prod_{i=1}^m \left(1-\gamma_i^2\right) \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{\left(\mathbf{z} - \boldsymbol{\gamma} w\right)^\top \mathbf{\Lambda}^\top \tilde{\mathbf{\Gamma}}^{-1} \mathbf{\Lambda}\left(\mathbf{z} - \boldsymbol{\gamma} w\right)}{2}\right) \mathrm{d}^m\mathbf{z}\, \mathrm{d}w.
\end{aligned}
$$

Noticing that $\prod_{i=1}^m \left(1-\gamma_i^2\right) = \det\left(\mathbf{\Lambda}^{-1}\right)^2$, then

$$
\begin{aligned}
I_1 &= \int_{D_2} \phi\left(w\right) \frac{1}{\sqrt{(2\pi)^m \det \hat{\mathbf{\Gamma}}}} \exp\left(-\frac{\left(\mathbf{z} - \boldsymbol{\gamma} w\right)^\top \hat{\mathbf{\Gamma}}^{-1}\left(\mathbf{z} - \boldsymbol{\gamma} w\right)}{2}\right) \mathrm{d}^m\mathbf{z}\, \mathrm{d}w \\
&= \int_{D_2} \phi\left(w\right) n_m(\mathbf{z}|W)\, \mathrm{d}^m\mathbf{z}\, \mathrm{d}w
\end{aligned}
$$

with $\hat{\boldsymbol{\Gamma}}^{-1} := \boldsymbol{\Lambda}^\top \tilde{\boldsymbol{\Gamma}}^{-1} \boldsymbol{\Lambda}$, and $\mathbf{Z}|W \sim \mathcal{N}_m \left(\gamma W, \hat{\boldsymbol{\Gamma}}\right)$. This means that ij-element of the covariance matrix $\hat{\boldsymbol{\Gamma}}$ is given by

$$\hat{\gamma}_{ij} = \tilde{\gamma}_{ij} \sqrt{\left(1-\gamma_i^2\right)\left(1-\gamma_j^2\right)}$$

with $\tilde{\gamma}_{ii} = 1$.

Furthermore, by setting $\gamma_i - b_i \sigma \sqrt{1-\gamma_i^2} = 0$, that corresponds to $\gamma_i = \frac{b_i \sigma}{\sqrt{1+b_i^2\sigma^2}}$, the region of integration reduces to $D_2 = \left\{ w \in \mathbb{R}, \mathbf{z} \in \mathbb{R}^m : w \leq \frac{\mu - a}{\sigma}, \bigcap_{i=1}^m \left\{ z_i \leq \frac{b_i \mu + c_i}{\sqrt{1+b_i^2\sigma^2}} \right\} \right\}$, and the expression for the generic element of $\hat{\boldsymbol{\Gamma}}$ reduces to

$$\hat{\gamma}_{ij} = \frac{\tilde{\gamma}_{ij}}{\sqrt{\left(1+b_i^2\sigma^2\right)\left(1+b_j^2\sigma^2\right)}}.$$

Finally, using the result in Greene (2008) (pag. 1013), I_1 can be written as

$$I_1 = \int_{D_2} n\left(\tilde{\mathbf{z}}\right) \mathrm{d}^{m+1}\tilde{\mathbf{z}} = \Phi_{m+1}\left(\frac{\mu - a}{\sigma}, \frac{b_1\mu + c_1}{\sqrt{1+b_1^2\sigma^2}}, \ldots, \frac{b_m\mu + c_m}{\sqrt{1+b_m^2\sigma^2}}; \boldsymbol{\Gamma}_1\right)$$

with $\tilde{\mathbf{Z}}^\top := \begin{pmatrix} \mathbf{Z} & W \end{pmatrix} \sim \mathcal{N}_{m+1}\left(\mathbf{0}, \boldsymbol{\Gamma}_1\right)$; the correlation matrix $\boldsymbol{\Gamma}_1$ after proving the case $\xi = -1$.

2. Consider $\xi = -1$, that is

$$\begin{aligned} I_{-1} &= \int_{-\infty}^{a} n(x) \left[\int_{-\infty}^{b_1 x + c_1} \cdots \int_{-\infty}^{b_m x + c_m} \frac{1}{\sqrt{(2\pi)^m \det \tilde{\boldsymbol{\Gamma}}}} \exp\left(-\frac{\mathbf{y}^\top \tilde{\boldsymbol{\Gamma}}^{-1} \mathbf{y}}{2} \right) \mathrm{d}^m \mathbf{y} \right] \mathrm{d}x \\ &= \int_{D_0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 \right) \frac{1}{\sqrt{(2\pi)^m \det \tilde{\boldsymbol{\Gamma}}}} \exp\left(-\frac{\mathbf{y}^\top \tilde{\boldsymbol{\Gamma}}^{-1} \mathbf{y}}{2} \right) \mathrm{d}^m \mathbf{y}\, \mathrm{d}x \end{aligned}$$

where $D_0 := \{x \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^m : x \leq a, \bigcap_{i=1}^m \{y_i \leq b_i x + c_i\}\}$. The derivation follows the same steps as in the previous case, with the only difference that in the transformation $\mathbf{T_1}$ the change of variable is $x = \mu + \sigma w$. Consequently

$$I_{-1} = \int_{D_2} n\left(\tilde{\mathbf{z}}\right) \mathrm{d}^{m+1}\, \tilde{\mathbf{z}} = \Phi_{m+1}\left(\frac{a - \mu}{\sigma}, \frac{b_1\mu + c_1}{\sqrt{1+b_1^2\sigma^2}}, \ldots, \frac{b_m\mu + c_m}{\sqrt{1+b_m^2\sigma^2}}; \boldsymbol{\Gamma}_{-1}\right)$$

where $D_2 = \left\{ w \in \mathbb{R}, \mathbf{z} \in \mathbb{R}^m : w \leq \frac{a-\mu}{\sigma}, \bigcap_{i=1}^m \left\{ z_i \leq \frac{b_i \mu + c_i}{\sqrt{1+b_i^2\sigma^2}} \right\} \right\}$, and $\tilde{\mathbf{Z}}^\top := \begin{pmatrix} \mathbf{Z} & W \end{pmatrix} \sim \mathcal{N}_{m+1}\left(\mathbf{0}, \boldsymbol{\Gamma}_{-1}\right)$.

Finally the general solution of the integral can be written in compact way as

$$\xi \int_a^{\xi\infty} n(x)\, \Phi_m\left(\mathbf{b}x + \mathbf{c}; \tilde{\boldsymbol{\Gamma}}\right) \mathrm{d}x = \Phi_{m+1}\left(\xi \frac{\mu - a}{\sigma}, \mathbf{d}; \boldsymbol{\Gamma}_\xi\right).$$

where $\mathbf{d} \in \mathbb{R}^m : d_i = \frac{b_i\mu + c_i}{\sqrt{1+b_i^2\sigma^2}}$.

In order to determine the covariance matrix of $\tilde{\mathbf{Z}}$, the relationship between $\hat{\boldsymbol{\Gamma}}$ and $\tilde{\boldsymbol{\Gamma}}$ provided in Greene (2008) (pag. 1013) can be used, that is

$$\hat{\boldsymbol{\Gamma}} = \tilde{\boldsymbol{\Gamma}} - \boldsymbol{\gamma}_\xi \boldsymbol{\gamma}_\xi^\top, \tag{C.2}$$

where $\boldsymbol{\gamma}_\xi \in \mathbb{R}^m$ is a vector which needs to be determined. Having constructed the correlation matrix $\boldsymbol{\Gamma}_\xi$ as

$$\boldsymbol{\Gamma}_\xi = \begin{pmatrix} \tilde{\boldsymbol{\Gamma}} & \boldsymbol{\gamma}_\xi \\ \boldsymbol{\gamma}_\xi^\top & 1 \end{pmatrix},$$

(C.2) can be solved as

$$\frac{\tilde{\gamma}_{ij}}{\sqrt{\left(1+b_i^2\sigma^2\right)\left(1+b_j^2\sigma^2\right)}} = \tilde{\gamma}_{ij} - \gamma_{i,\xi}\gamma_{j,\xi},$$

$$\gamma_{i,\xi}\gamma_{j,\xi} = \tilde{\gamma}_{ij}\left(1 - \frac{1}{\sqrt{\left(1+b_i^2\sigma^2\right)\left(1+b_j^2\sigma^2\right)}}\right),$$

if $i = j$, then

$$\gamma_{i,\xi} = \xi\frac{b_i\sigma}{\sqrt{1+b_i^2\sigma^2}}.$$

$\square$

Theorem C.2. *Given a binary variables ξ taking values ± 1, $a \in \mathbb{R}$, and $\mathbf{b}, \mathbf{c} \in \mathbb{R}^m$ then*

$$\xi\int_a^{\xi\infty} e^x n\left(x;\mu,\sigma\right)\Phi_m\left(\mathbf{b}x+\mathbf{c};\tilde{\boldsymbol{\Gamma}}\right)\mathrm{d}x = e^{\mu+\frac{\sigma^2}{2}}\Phi_{m+1}\left(\xi\frac{\mu+\sigma^2-a}{\sigma},\mathbf{f};\boldsymbol{\Gamma}_\xi\right), \tag{C.3}$$

for $\mathbf{f} \in \mathbb{R}^m : f_i = \frac{b_i\left(\mu+\sigma^2\right)+c_i}{\sqrt{1+b_i^2\sigma^2}}$,

$$\boldsymbol{\Gamma}_\xi = \begin{pmatrix} \tilde{\boldsymbol{\Gamma}} & \boldsymbol{\gamma}_\xi \\ \boldsymbol{\gamma}_\xi^\top & 1 \end{pmatrix},$$

and $\boldsymbol{\gamma}_\xi \in \mathbb{R}^m : \gamma_{i,\xi} = \xi\frac{b_i\sigma}{\sqrt{1+b_i^2\sigma^2}}$.

Proof. For notational convenience, define

$$J_\xi := \xi\int_a^{\xi\infty} e^x n\left(x;\mu,\sigma\right)\Phi_m\left(\mathbf{b}x+\mathbf{c};\tilde{\boldsymbol{\Gamma}}\right)\mathrm{d}x.$$

I distinguish the two cases $\xi = 1$ and $\xi = -1$ (for convenience of notation the dependence on the distributions' parameters such as μ, σ and $\tilde{\boldsymbol{\Gamma}}$ is omitted).

1. Consider first $\xi = 1$, that is

$$\begin{aligned} J_1 &= \int_a^\infty e^x n\left(x\right)\left[\int_{-\infty}^{b_1x+c_1}\cdots\int_{-\infty}^{b_mx+c_m}\frac{1}{\sqrt{(2\pi)^m\det\tilde{\boldsymbol{\Gamma}}}}\exp\left(-\frac{\mathbf{y}^\top\tilde{\boldsymbol{\Gamma}}^{-1}\mathbf{y}}{2}\right)\mathrm{d}^m\mathbf{y}\right]\mathrm{d}x \\ &= \int_{D_0} e^x\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)\frac{1}{\sqrt{(2\pi)^m\det\tilde{\boldsymbol{\Gamma}}}}\exp\left(-\frac{\mathbf{y}^\top\tilde{\boldsymbol{\Gamma}}^{-1}\mathbf{y}}{2}\right)\mathrm{d}^m\mathbf{y}\,\mathrm{d}x \end{aligned}$$

where $D_0 := \{x \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^m : x \geq a, \bigcap_{i=1}^m \{y_i \leq b_ix+c_i\}\}$.

Consider the transformation $\mathbf{T_1} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$

$$\mathbf{T_1} := \begin{cases} x = \mu + \sigma w \\ \mathbf{y} = \mathbf{y} \end{cases}$$

with Jacobian

$$\mathbf{J_1} = \begin{pmatrix} \sigma & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} = \begin{pmatrix} \sigma & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

where $\mathbf{0}$ is the null column vector in $\mathbb{R}^m$ and $\mathbf{I}$ the $m \times m$ identity matrix. Furthermore, $D_1 := D_0(\mathbf{T_1}) = \left\{w \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^m : w \geq \frac{a-\mu}{\sigma}, \bigcap_{i=1}^m \{y_i \leq -b_i\sigma w + b_i\mu + c_i\}\right\}$. Hence the integral becomes

$$\begin{aligned} J_1 &= \int_{D_1} e^{\mu+\sigma w} n\left(\mu + \sigma w\right) \frac{1}{\sqrt{(2\pi)^m \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{\mathbf{y}^\top \tilde{\mathbf{\Gamma}}^{-1}\mathbf{y}}{2}\right) |\det \mathbf{J_1}| \, \mathrm{d}^m\mathbf{y} \, \mathrm{d}w \\ &= e^{\mu+\frac{\sigma^2}{2}} \int_{D_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{(w-\sigma)^2}{2}} \frac{1}{\sqrt{(2\pi)^m \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{\mathbf{y}^\top \tilde{\mathbf{\Gamma}}^{-1}\mathbf{y}}{2}\right) \mathrm{d}^m\mathbf{y} \, \mathrm{d}w. \end{aligned}$$

Consider another transformation $\mathbf{T_2} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$

$$\mathbf{T_2} := \begin{cases} w = -v + \sigma \\ \mathbf{y} = \mathbf{\Lambda}\left(\mathbf{z} - \boldsymbol{\gamma} v\right) \end{cases}$$

with

$$\mathbf{\Lambda} := \begin{pmatrix} \frac{1}{\sqrt{1-\gamma_1^2}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{1-\gamma_2^2}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\sqrt{1-\gamma_m^2}} \end{pmatrix}, \qquad \boldsymbol{\gamma} := \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \dots \\ \gamma_m \end{pmatrix},$$

where $\gamma_i \in (-1, 1)$, for all $i \leq m$, are parameters that will be determined later. The corresponding Jacobian is

$$\mathbf{J_2} = \begin{pmatrix} -1 & 0 & \dots & 0 \\ -\frac{\gamma_1}{\sqrt{1-\gamma_1^2}} & \frac{1}{\sqrt{1-\gamma_1^2}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -\frac{\gamma_m}{\sqrt{1-\gamma_m^2}} & 0 & \dots & \frac{1}{\sqrt{1-\gamma_m^2}} \end{pmatrix}.$$

and

$$\begin{aligned} D_2 :&= D_1\left(\mathbf{T_2}\right) \\ &= \left\{v \in \mathbb{R}, \mathbf{z} \in \mathbb{R}^m : v \leq \frac{\mu + \sigma^2 - a}{\sigma}, \bigcap_{i=1}^m \left\{z_i \leq \left(\gamma_i - b_i\sigma\sqrt{1-\gamma_i^2}\right) v + \left[b_i\left(\mu + \sigma^2\right) + c_i\right]\sqrt{1-\gamma_i^2}\right\}\right\}. \end{aligned}$$

Hence

$$J_1 = e^{\mu+\frac{\sigma^2}{2}} \int_{D_2} \phi(v) \frac{1}{\sqrt{(2\pi)^m \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{(\mathbf{\Lambda}(\mathbf{z}-\boldsymbol{\gamma} v))^\top \tilde{\mathbf{\Gamma}}^{-1} (\mathbf{\Lambda}(\mathbf{z}-\boldsymbol{\gamma} v))}{2}\right) |\det \mathbf{J_2}| \, \mathrm{d}^m \mathbf{z} \, \mathrm{d}v$$

$$= e^{\mu+\frac{\sigma^2}{2}} \int_{D_2} \phi(v) \frac{1}{\sqrt{(2\pi)^m \prod_{i=1}^m (1-\gamma_i^2) \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{(\mathbf{z}-\boldsymbol{\gamma} v)^\top \mathbf{\Lambda}^\top \tilde{\mathbf{\Gamma}}^{-1} \mathbf{\Lambda} (\mathbf{z}-\boldsymbol{\gamma} v)}{2}\right) \mathrm{d}^m \mathbf{z} \, \mathrm{d}v.$$

Noticing that $\prod_{i=1}^m \left(1-\gamma_i^2\right) = \det\left(\mathbf{\Lambda}^{-1}\right)^2$, then

$$J_1 = e^{\mu+\frac{\sigma^2}{2}} \int_{D_2} n(v) \frac{1}{\sqrt{(2\pi)^m \det \hat{\mathbf{\Gamma}}}} \exp\left(-\frac{(\mathbf{z}-\boldsymbol{\gamma} v)^\top \hat{\mathbf{\Gamma}}^{-1} (\mathbf{z}-\boldsymbol{\gamma} v)}{2}\right) \mathrm{d}^m \mathbf{z} \, \mathrm{d}v$$

$$= e^{\mu+\frac{\sigma^2}{2}} \int_{D_2} n(v) \, n_m(\mathbf{z}|V) \, \mathrm{d}^m \mathbf{z} \, \mathrm{d}v$$

with $\hat{\mathbf{\Gamma}}^{-1} := \mathbf{\Lambda}^\top \tilde{\mathbf{\Gamma}}^{-1} \mathbf{\Lambda}$, and $\mathbf{Z}|V \sim \mathcal{N}_m\left(\boldsymbol{\gamma} V, \hat{\mathbf{\Gamma}}\right)$. This means that the ij-element of the covariance matrix $\hat{\mathbf{\Gamma}}$ is given by

$$\hat{\gamma}_{ij} = \tilde{\gamma}_{ij} \sqrt{\left(1-\gamma_i^2\right)\left(1-\gamma_j^2\right)}$$

with $\tilde{\gamma}_{ii} = 1$.

Furthermore, by setting $\gamma_i - b_i \sigma \sqrt{1-\gamma_i^2} = 0$, that corresponds to $\gamma_i = \frac{b_i \sigma}{\sqrt{1+b_i^2 \sigma^2}}$, the region of integration reduces to $D_2 = \left\{ v \in \mathbb{R}, \mathbf{z} \in \mathbb{R}^m : v \leq \frac{\mu+\sigma^2-a}{\sigma}, \bigcap_{i=1}^m \left\{ z_i \leq \frac{b_i\left(\mu+\sigma^2\right)+c_i}{\sqrt{1+b_i^2\sigma^2}} \right\} \right\}$, and the expression for the generic element of $\hat{\mathbf{\Gamma}}$ reduces to

$$\hat{\gamma}_{ij} = \frac{\tilde{\gamma}}{\sqrt{\left(1+b_i^2\sigma^2\right)\left(1+b_j^2\sigma^2\right)}}.$$

Finally, using the result in Greene (2008) (pag. 1013), J_1 can be written as

$$J_1 = e^{\mu+\frac{\sigma^2}{2}} \int_{D_2} n(\tilde{\mathbf{z}}) \, \mathrm{d}^{m+1} \tilde{\mathbf{z}}$$

$$= e^{\mu+\frac{\sigma^2}{2}} \Phi_{m+1}\left(\frac{\mu+\sigma^2-a}{\sigma}, \frac{b_1\left(\mu+\sigma^2\right)+c_1}{\sqrt{1+b_1^2\sigma^2}}, \ldots, \frac{b_m\left(\mu+\sigma^2\right)+c_m}{\sqrt{1+b_m^2\sigma^2}}; \mathbf{\Gamma}_1\right)$$

with $\tilde{\mathbf{Z}}^\top := \begin{pmatrix} \mathbf{Z} & W \end{pmatrix} \sim \mathcal{N}_{m+1}(\mathbf{0}, \mathbf{\Gamma}_1)$; the correlation matrix $\mathbf{\Gamma}_1$ after proving the case $\xi = -1$.

2. Consider $\xi = -1$, that is

$$J_{-1} = \int_{-\infty}^{a} e^x n(x) \left[\int_{-\infty}^{b_1 x + c_1} \cdots \int_{-\infty}^{b_m x + c_m} \frac{1}{\sqrt{(2\pi)^m \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{\mathbf{y}^\top \tilde{\mathbf{\Gamma}}^{-1} \mathbf{y}}{2}\right) \mathrm{d}^m \mathbf{y} \right] \mathrm{d}x$$

$$= \int_{D_0} e^x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \frac{1}{\sqrt{(2\pi)^m \det \tilde{\mathbf{\Gamma}}}} \exp\left(-\frac{\mathbf{y}^\top \tilde{\mathbf{\Gamma}}^{-1} \mathbf{y}}{2}\right) \mathrm{d}^m \mathbf{y} \, \mathrm{d}x$$

where $D_0 := \{x \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^m : x \leq a, \bigcap_{i=1}^m \{y_i \leq b_i x + c_i\}\}$. The derivation follows the same steps as in the previous case, with the only difference that in the transformation $\mathbf{T_1}$ the change of variable is $x = \mu + \sigma w$.

Consequently

$$J_{-1} = e^{\mu + \frac{\sigma^2}{2}} \Phi_{m+1} \left(\frac{a - \mu - \sigma^2}{\sigma}, \frac{b_1 \left(\mu + \sigma^2\right) + c_1}{\sqrt{1 + b_1^2 \sigma^2}}, \ldots, \frac{b_m \left(\mu + \sigma^2\right) + c_m}{\sqrt{1 + b_m^2 \sigma^2}}; \mathbf{\Gamma}_{-1} \right)$$

where $D_2 = \left\{ v \in \mathbb{R}, \mathbf{z} \in \mathbb{R}^m : v \leq \frac{a-\mu-\sigma^2}{\sigma}, \bigcap_{i=1}^m \left\{ z_i \leq \frac{b_i\left(\mu+\sigma^2\right)+c_i}{\sqrt{1+b_i^2\sigma^2}} \right\} \right\}$, and $\tilde{\mathbf{Z}}^\top := \begin{pmatrix} \mathbf{Z} & W \end{pmatrix} \sim \mathcal{N}_{m+1}\left(\mathbf{0}, \mathbf{\Gamma}_{-1}\right)$.

Thus the general solution of the integral can be written in compact way as

$$\xi \int_a^{\xi\infty} e^x n\left(x\right) \Phi_m \left(\mathbf{b}x + \mathbf{c}; \tilde{\mathbf{\Gamma}} \right) \mathrm{d}x = e^{\mu + \frac{\sigma^2}{2}} \Phi_{m+1} \left(\xi \frac{\mu + \sigma^2 - a}{\sigma}, \mathbf{f}; \mathbf{\Gamma}_\xi \right),$$

where $\mathbf{f} \in \mathbb{R}^m : f_i = \frac{b_i\left(\mu+\sigma^2\right)+c_i}{\sqrt{1+b_i^2\sigma^2}}$.

The correlation coefficients of $\mathbf{\Gamma}_\xi$ are the same of Theorem C.1. $\square$

Appendix D Estimating the Model-Free Risk Neutral Probability of Survival

The t-payoff of a CDS initiated at $t_0 = 0$ with maturity t_j and intermediate premium payments at $(t_i)_{i=1}^{j}$, $j \in \mathbb{N}$, and notional set to one is given by

$$\Pi_j(t) = DF(t,\tau)\,(\tau - \bar{t})\,s\mathbb{1}_{\{0<\tau\leq t_j\}} + s\sum_{i=1}^{j} DF(t,t_i)\alpha_i\mathbb{1}_{\{\tau\geq t_i\}} - DF(t,\tau)\text{LGD}\mathbb{1}_{\{0<\tau\leq t_j\}}$$

with $0 \leq t < t_j$, $\bar{t}$ the last payments date before t, that is $\bar{t} := \sup_{1\leq i\leq j}\{t_i \leq \tau\}$, α_i the year fraction between t_{i-1} and t_i, s the CDS spread paid by the protection buyer (before default, if it happens), and LGD the loss given default. The first term is the discounted accrued rate at default and represents the compensation the protection seller receives for the protection provided from the last t_i until default τ. The terms in the summation represent the CDS rate premium payments if there is no default: this is the premium received by the protection seller for the protection being provided. The final term is the payment of protection at default, if this happens before final t_j.

The t_j-maturity CDS price in $t_0 = 0$ according to risk-neutral valuation is

$$\text{CDS}_j\,(s,\text{LGD}) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_j(0)\right].$$

For computing the expectation is more convenient to separate the payments made by the protection buyer from the ones made by the protection seller. Also, it is assumed that default can occur at reset dates only, that is the first summand can be ignored (there are no accrued interests). Following Brigo and Mercurio (2006), the expected value of premium leg is equal to

$$\text{PremiumLeg}_j\,(s) = \mathbb{E}^{\mathbb{Q}}\left[s\sum_{i=1}^{j} DF(0,t_i)\alpha_i\mathbb{1}_{\{\tau\geq t_i\}}\right] = s\left[\sum_{i=1}^{j} P(0,t_i)\alpha_i\mathbb{Q}\left(\tau\geq t_i\right)\right],$$

where $P(t_i,t_j)$ is the t_i-value of a zero-coupon bond with maturity $t_j \geq t_i$, under the assumption of independence between the discount factor and the default time. From the perspective of the protection seller,

$$\text{ProtecLeg}_j\,(\text{LGD}) = \mathbb{E}^{\mathbb{Q}}\left[DF(0,\tau)\text{LGD}\mathbb{1}_{\{0<\tau\leq t_j\}}\right] = \text{LGD}\int_0^{t_j} P(0,t)\,\mathrm{d}\mathbb{Q}\left(\tau\geq t\right)$$

Hence the value of the CDS in $t_0 = 0$ is given by

$$\begin{aligned}\text{CDS}_j\,(s,\text{LGD}) &= \text{PremiumLeg}_j\,(s) - \text{ProtecLeg}_j\,(\text{LGD})\\ &= s\left[\sum_{i=1}^{j} P(0,t_i)\alpha_i\mathbb{Q}\left(\tau\geq t_i\right)\right] - \text{LGD}\int_0^{t_j} P(0,t)\,\mathrm{d}\mathbb{Q}\left(\tau\geq t\right)\end{aligned}$$

If we assume that in t_0 the term structure of the risk-free interest rates is known and a deterministic function of the maturity only, $r_0(t)$, then the previous expression simplifies as

$$\text{CDS}_j\,(s,\text{LGD}) = s\left[\sum_{i=1}^{j} e^{-r_0(t_i)t_i}\alpha_i\mathbb{Q}\left(\tau\geq t_i\right)\right] - \text{LGD}\int_0^{t_j} e^{-r_0(t)t}\,\mathrm{d}\mathbb{Q}\left(\tau\geq t\right).$$

At time t_0, provided that default has not occurred yet, the market sets the spread s to a value, s_j^{MID}, which makes the CDS fair at time t_0 that is

$$s_j^{\text{MID}} := \left\{s > r_0(t_j) : \text{CDS}_j\,(s,\text{LGD}) = 0\right\}$$

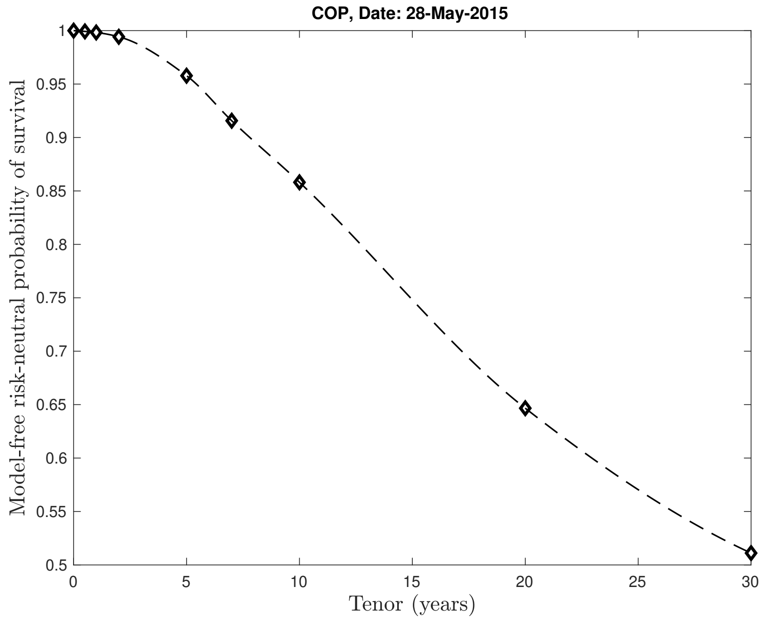


Figure D.1: Term-structure of risk-neutral survival probabilities for COP on 28/05/2015, LGD = 60%.

for different maturities t_j. Having assigned a level of loss given default (usually LGD $= \{0.5, 0.6, 0.8\}$), the set of equations

$$\text{PremiumLeg}_j \left(s_j^{\text{MID}}, \mathbb{Q}\left(\tau \geq t\right)\right) = \text{ProtecLeg}_j \left(\text{LGD}, \mathbb{Q}\left(\tau \geq t\right)\right)$$

can be solved in $\mathbb{Q}$, starting from the CDS quotation with shortest tenor, and recursively solving for the spread with longer maturities. Therefore, the market implied risk-neutral survival probabilities $\mathbb{Q}\left(\tau \geq t\right)$,with $t \in (t_j, t_{j+1}]$, can be found. Figure D.1 displays the term-structure of the risk neutral probability of survival for ConocoPhillips (COP) observed on May, 28th 2015.

The data on corporate CDS spreads are usually available for maturities $t_j \in \{6m, 1y, 2y, 5y, 7y, 10y, 20y, 30y\}$. The extraction of the risk-neutral probability of surviving is therefore conducted as follows:

- start from $t_j = 6m$ and estimate the market implied survival probability $\mathbb{Q}\left(\tau \geq t\right)$, with $t \in (0, 0.5]$ years;
- insert the estimated value into CDS legs formulas for $t_j = 1y$, and solve the same type of equation with $t_j = 1y$ to find the market implied survival probability $\mathbb{Q}\left(\tau \geq t\right)$, with $t \in (0.5, 1]$ years;
- repeat the same recursive procedure for all the tenors up to $t_j = 30$ years.

For further details see Brigo (2005).

Appendix E Estimating the Endogenous Default Barrier

The implied default barrier pays a crucial role in the compound option model dynamics and its value ultimately defines the default region in the asset value space. Given the risk-free and payout rates as well as the firm's capital structure, the values of the default thresholds are functions of the (unknown) volatility of the assets only. They are defined as

$$\bar{V}_i(\sigma_V) := \{v \in \mathbb{R}_+ : S_i^\star(v, \sigma_V) = F_i\}\,. \tag{E.1}$$

with $i \in I = \{1, \dots n\}$ where n is the number of bonds outstanding. That is, the implied barrier at t_i is defined as the value of the asset such that the continuation value of the equity, $S_i^\star$, is at least as large as the bond due, F_i. In order to determine the values of the sequence $\left(\bar{V}_i\right)_{i\in I}$, the problem must be solved starting from the latest maturity t_n and backwardly to the first payment date t_1.

As the most common instance for the given data is a company with three bond outstanding, I are going to present the estimation for the case $n = 3$. Trivially

$$\bar{V}_3 = F_3,$$

as in Merton (1974). This is the only default threshold that does not depend on the asset volatility (or any other model parameter). According to (E.1), as the continuation value of the equity is also a compound call option, the previous default point is defined as

$$\begin{aligned}\bar{V}_2(\sigma_V) &= \{v \in \mathbb{R}_+ : S_2^\star(v, \sigma_V) = F_2\} \\ &= \left\{v \in \mathbb{R}_+ : e^{-\varpi(t_3-t_2)} v \Phi\left(d_{3,2}^{\mathbb{M}}\right) - e^{-r(t_3-t_2)} F_3 \Phi\big(d_{3,2}^{\mathbb{Q}}\big) = F_2\right\}\end{aligned}$$

with

$$d_{3,2}^{\mathbb{M}} = \frac{\ln\left(v/F_3\right) + \left(r - \varpi + \sigma_V^2/2\right)(t_3 - t_2)}{\sigma_V\sqrt{t_3 - t_2}} \qquad d_{3,2}^{\mathbb{Q}} = d_{3,2}^{\mathbb{M}} - \sigma_V\sqrt{t_3 - t_2}.$$

This notation should make clear that different values of $\bar{V}_2$ are obtained for different values of σ_V[7]. Ultimately, if σ_V is known, estimating the barrier is equivalent to just solving a nonlinear integral equation[8].

Similarly, the value of the barrier at the first reimbursement date is

$$\begin{aligned}\bar{V}_1(\sigma_V) &= \{v \in \mathbb{R}_+ : S_1^\star(v, \sigma_V) = F_1\} \\ &= \left\{v \in \mathbb{R}_+ : e^{-\varpi(t_3-t_1)} v \Phi_2\left(d_{2,1}^{\mathbb{M}}, d_{3,1}^{\mathbb{M}}; \mathbf{\Gamma}\right) - e^{-r(t_3-t_1)} F_3 \Phi_2\big(d_{2,1}^{\mathbb{Q}}, d_{3,1}^{\mathbb{Q}}; \mathbf{\Gamma}\big) - e^{-r(t_2-t_1)} F_2 \Phi\big(d_{2,1}^{\mathbb{Q}}\big) = F_1\right\}\end{aligned}$$

with

$$\begin{aligned} d_{2,1}^{\mathbb{M}} &= \frac{\ln\left(v/\bar{V}_2\right) + \left(r - \varpi + \sigma_V^2/2\right)(t_2 - t_1)}{\sigma_V\sqrt{t_2 - t_1}} \qquad & d_{2,1}^{\mathbb{Q}} &= d_{2,1}^{\mathbb{M}} - \sigma_V\sqrt{t_2 - t_1}. \\ d_{3,1}^{\mathbb{M}} &= \frac{\ln\left(v/F_3\right) + \left(r - \varpi + \sigma_V^2/2\right)(t_3 - t_1)}{\sigma_V\sqrt{t_3 - t_1}} \qquad & d_{3,1}^{\mathbb{Q}} &= d_{3,1}^{\mathbb{M}} - \sigma_V\sqrt{t_3 - t_1}. \end{aligned}$$

and

$$\mathbf{\Gamma} = \begin{pmatrix} 1 & \sqrt{\frac{t_2-t_1}{t_3-t_1}} \\ \sqrt{\frac{t_2-t_1}{t_3-t_1}} & 1 \end{pmatrix}.$$

Again, $\bar{V}_1$ is found as the solution of a nonlinear equation which involves double integrals. The same procedure can be applied for $n > 3$; however, the computational cost becomes progressively more severe with increasing the number of bond outstanding[9].

[7]The same is true for different values of r, ϖ, ... However those parameters are assumed to be known and estimated with no error.

[8]I refer it as integral equation as the integration interval/hyperrectangular depends on v.

[9]The main reason being that the function Φ_n is calculated relying on Monte Carlo simulation for $n \geq 4$.

If the value of σ_V were known, the entire default barrier could be easily computed and the today-value of the assets could be easily found calibrating on the today-value of the equity,

$$S_0 = e^{-\varpi t_n} V_0 \Phi_n \left(\mathbf{d}^{\mathbb{M}}(V_0, \sigma_V); \mathbf{\Gamma}_n \right) - \sum_{k=1}^{n} e^{-r t_k} F_k \Phi_k \big(\mathbf{d}_k^{\mathbb{Q}}(V_0, \sigma_V); \mathbf{\Gamma}_k \big). \tag{E.2}$$

However, as both asset volatility and value are not observable, (E.2) can be seen as an equation in the two unknowns σ_V and V_0. In order to solve for those values, the dependence of the barrier on σ_V is reversely engineered in order to determine the implied asset volatility. Similarly to the computation of the Black-Scholes implied volatility, the observable option price

$$\begin{aligned} P_{0,\xi} = \xi \Bigg[& e^{-\varpi t_n} V_0 \Phi_{n+1} \left(\mathbf{d}_\xi^{\mathbb{M}}(V_0, \sigma_V); \mathbf{\Gamma}_{n+1,\xi} \right) - \sum_{k=i+1}^{n} e^{-r t_k} F_k \Phi_{k+1} \big(\mathbf{d}_{\xi,k+1}^{\mathbb{Q}}(V_0, \sigma_V); \mathbf{\Gamma}_{k+1,\xi} \big) \\ & - e^{-rT} K \Phi_{i+1} \big(\mathbf{d}_{\xi,T}^{\mathbb{Q}}(V_0, \sigma_V); \mathbf{\Gamma}_{T,\xi} \big) \Bigg] \end{aligned} \tag{E.3}$$

is used as second equation in order to determine the asset implied volatility and the corresponding asset value. Notably, only the value of the option and the equity are needed to determine the implied volatility of the assets $\sigma_V = \sigma_V(K, T)$ and construct the term-structure of the asset volatility's surface.

In addition, in order to make computations more efficient, (E.2) can be replaced by

$$S_0 = e^{-\varpi t_n} V_0 \Phi_n \left(\mathbf{d}^{\mathbb{M}}(V_0, \sigma_V); \mathbf{\Gamma}_n \right) - \sum_{k=1}^{n} e^{-r t_k} F_k \hat{\mathbb{Q}} \left(\tau > t_k \right), \tag{E.4}$$

where $\hat{\mathbb{Q}} \left(\tau > t_k \right)$ are the estimates of the risk-neutral probability of survival extracted from the CDS written on the same reference entity. These are estimated as in Appendix D, in which a simple model-free estimation procedure is provided.

Using (E.4) instead of (E.2) speeds up computation and serves as an indirect test on the integration of the CDS and option markets. If the two estimates of the implied volatility of the asset – obtained with and without the calibration on the CDS – are consistent, it can be inferred that options market participants incorporate information on default-related events into option prices consistently with the price of default quoted by CDS market participants.

At first glance, deriving the sensitivity of the default barrier with respect to volatility is not trivial for $n \geq 2$, as $V_i(\sigma_V)$ is implicitly defined via the integral equation (E.1). For illustrative purposes, consider the case of $n = 2$, where $\bar{V}_2 = F_2$ and

$$\bar{V}_1(\sigma_V) = \left\{ v \in \mathbb{R}_+ : e^{-\varpi(t_2 - t_1)} v \Phi \left(d_{2,1}^+ \right) - e^{-r(t_2 - t_1)} F_2 \Phi \big(d_{2,1}^- \big) = F_1 \right\}$$

with

$$d_{2,1}^+ = \frac{\ln \left(v / F_2 \right) + \left(r - \varpi + \sigma_V^2 / 2 \right) \left(t_2 - t_1 \right)}{\sigma_V \sqrt{t_2 - t_1}} \qquad d_{2,1}^- = d_{2,1}^+ - \sigma_V \sqrt{t_2 - t_1}.$$

Let $\sigma_V = x$ and $\bar{V}_1 = y$. In order to determine $y'(x)$, the Implicit Function Theorem can be used. Associating the curve

$$\Xi(x, y) = e^{-\varpi(t_2 - t_1)} y \Phi \left(d_{2,1}^+(x, y) \right) - e^{-r(t_2 - t_1)} F_2 \Phi \big(d_{2,1}^-(x, y) \big) - F_1 = 0$$

the derivative of the implicit function is

$$y'(x) = -\frac{\Xi_x}{\Xi_y}(x, y(x)).$$

Notice that the numerator is nothing but the Vega (see Appendix G) of the continuation value of the equity $S_1^\star$ (as a function of the asset volatility and the barrier) whilst the denominator is its Delta with respect the default threshold $\bar{V}_1$

(see Appendix F). That is

$$\bar{V}_1' = -\frac{\nu_{S^\star}^{(1)}}{\Delta_{S^\star}^{(1)}} = -\frac{\phi\left(d_{2,1}^+\right)}{\Phi\left(d_{2,1}^+\right)}\bar{V}_1\sqrt{t_2 - t_1}.$$

The same reasoning applies to any $n \geq 2$, and in general[10]

$$\bar{V}_i' = \begin{cases} -\nu_{S^\star}^{(i)}/\Delta_{S^\star}^{(i)} & \text{if } i < n \\ 0 & \text{if } i = n. \end{cases}$$

It can be shown numerically that, provided a reasonable set of parameters[11], the function $\bar{V}(\sigma_V)$ is positive and decreasing, displaying an inflation point. See Figure E.1 and E.2 for graphical inspection. The reason why the barrier lowers as σ_V increases can be intuitively explained as follows. As equity is a compound call option, by standard option pricing arguments, an increase in the volatility leads to an increase of the option premium, i.e. the equity value, which ultimately makes the firm 'safer'. As default events are measured based on the distance between the continuation value of the equity and the face value of the bond expiring, an increase in volatility lowers the default threshold as the equity has increased accordingly. This is a structural property of using a geometric Brownian motion for the dynamics of the assets and the equity as a compound call option on the value of the assets. The model in Merton (1974) displays similar features.

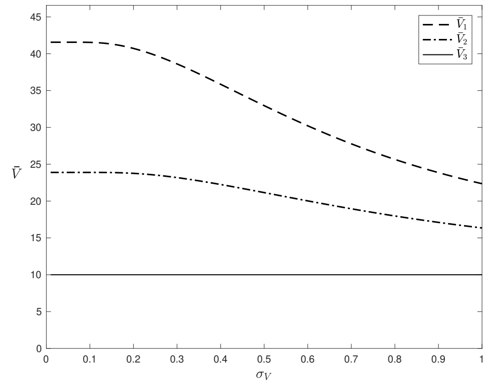
Additional references are Geske (1977) and Geske et al. (2016).

[10]It can be also shown that

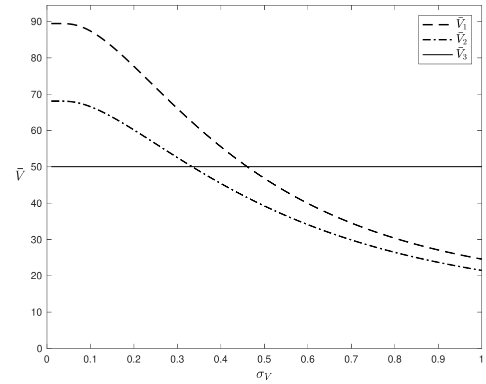
$$V'' = -\frac{\text{Delta}^2 \cdot \text{Volga} - 2 \cdot \text{Delta} \cdot \text{Vega} \cdot \text{Vanna} + \text{Vega}^2 \cdot \text{Gamma}}{\text{Delta}^3}.$$

Notice that $\nu_{S^\star}$ is obtained as the Vega of the equity in Appendix F having set the sensitivity of the barrier to zero.

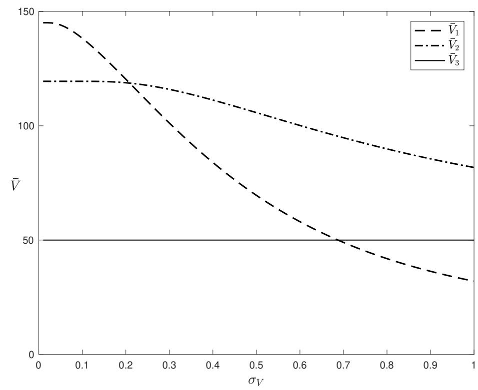
[11]That is for $\sigma_V \in (0, 1)$ and $F_2 < 10 \cdot F_1$.



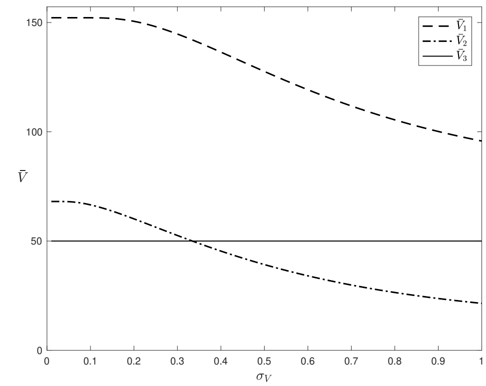
(a) $F_1 = F_2 = F_3 = 10$



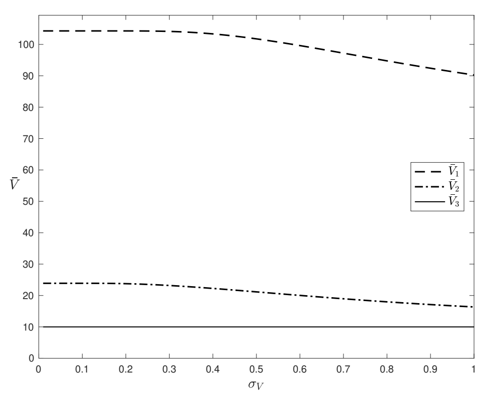
(b) $F_1 = F_2 = 10, F_3 = 50$



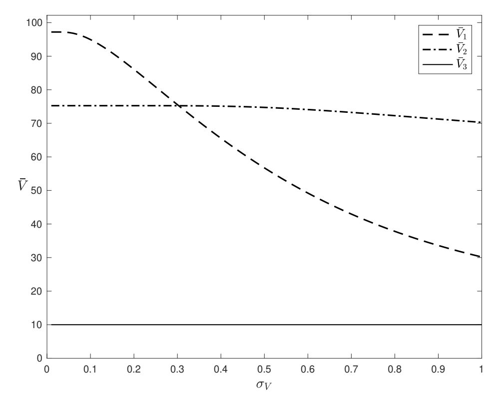
(c) $F_1 = 10, F_2 = F_3 = 50$



(d) $F_1 = F_3 = 50, F_2 = 10$

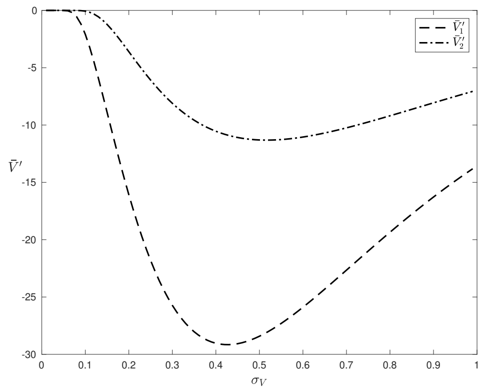


(e) $F_1 = 50, F_2 = F_3 = 10$

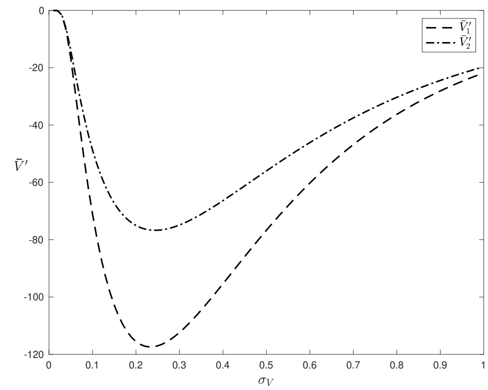


(f) $F_1 = F_3 = 10, F_2 = 50$

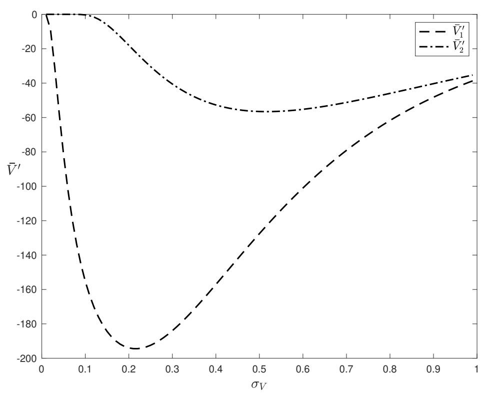
Figure E.1: $\bar{V}_i(\sigma_V)$, $i = \{1, 2, 3\}$, for $t_1 = 1, t_2 = 5, t_3 = 10$, $r = 0.03$ and $\varpi = 0.05$.



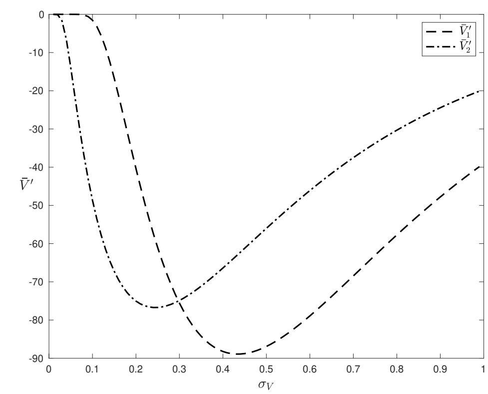
(a) $F_1 = F_2 = F_3 = 10$



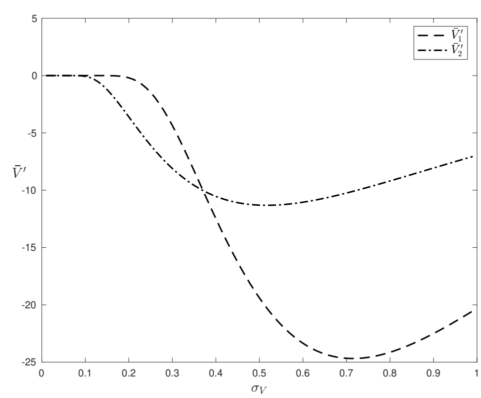
(b) $F_1 = F_2 = 10, F_3 = 50$



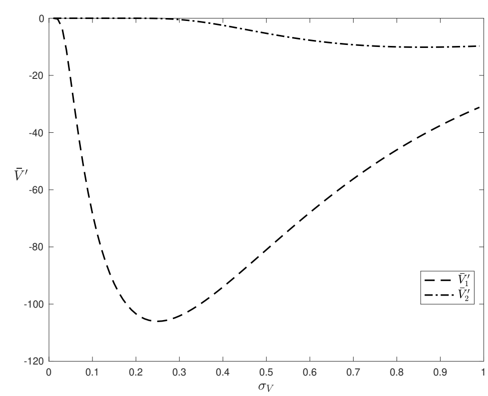
(c) $F_1 = 10, F_2 = F_3 = 50$



(d) $F_1 = F_3 = 50, F_2 = 10$



(e) $F_1 = 50, F_2 = F_3 = 10$



(f) $F_1 = F_3 = 10, F_2 = 50$

Figure E.2: $\bar{V}_i'(\sigma_V)$, $i = \{1, 2, 3\}$, for $t_1 = 1, t_2 = 5, t_3 = 10$, $r = 0.03$ and $\varpi = 0.05$.

Appendix F The Delta of the Equity

In order to compute the sensitivity of the equity with respect to changes in the asset value (herein, delta of the equity), the following result is needed.

Theorem F.1. *Let*

$$\Phi_k\left(\mathbf{d}(x);\mathbf{\Gamma}\right)=\int_{\Upsilon(x)}\phi_k(y_1,\ldots,y_i,\ldots,y_k;\mathbf{\Gamma})\mathrm{d}y_1\ldots\mathrm{d}y_i\ldots\mathrm{d}y_k$$

with $\mathbf{\Gamma}\in\mathcal{M}_+^k$ *and* $\Upsilon(x)=\bigcap_{i=1}^k\{y_i\in\mathbb{R}:y_i\leq d_i(x)\}$, *with* $\mathbf{d}(x):\mathbb{R}_+\to\mathbb{R}^k$, $d_i(x)=\frac{\ln x+a_i}{b_i}$ *with* $a_i\in\mathbb{R}$ *and* $b_i\in\mathbb{R}_+$. *Then*

$$\frac{\partial\Phi_k\left(\mathbf{d}(x);\mathbf{\Gamma}\right)}{\partial x}=\frac{1}{x}\sum_{i=1}^k\frac{1}{b_i}\int_{\bar{\Upsilon}_i(x)}\phi_k(y_1,\ldots,d_i(x),\ldots,y_k;\mathbf{\Gamma})\mathrm{d}y_1\ldots\mathrm{d}y_k,$$

where $\bar{\Upsilon}_i(x)=\Upsilon(x)\setminus\{y_i\leq d_i(x)\}$.

Proof. Let $z_i=d_i(x)$, with $i=\{1,\ldots,k\}$. Applying the chain rule, it follows

$$\frac{\partial\Phi_k(z_1,\ldots,z_k)}{\partial x}=\sum_{i=1}^k\frac{\partial\Phi_k}{\partial z_i}\frac{\partial z_i}{\partial x}$$

and by the virtue of the fundamental theorem of calculus

$$\frac{\partial\Phi_k}{\partial z_i}=\int_{-\infty}^{z_1}\cdots\int_{-\infty}^{z_{i-1}}\int_{-\infty}^{z_{i+1}}\cdots\int_{-\infty}^{z_k}\phi_k(y_1,\ldots,y_{i-1},z_i,y_{i+1},\ldots,y_k;\mathbf{\Gamma})\mathrm{d}y_1\ldots\mathrm{d}y_{i-1}\mathrm{d}y_{i+1}\ldots\mathrm{d}y_k.$$

As

$$\frac{\partial z_i}{\partial x}=\frac{1}{b_i x},$$

the result follows. □

Given n bond outstanding, the value of the equity $s=S(v)$ is given by (8), that is

$$s=e^{-\varpi t_n}v\Phi_n\left(\mathbf{d}^{\mathbb{M}}(v);\mathbf{\Gamma}_n\right)-\sum_{k=1}^n e^{-rt_k}F_k\Phi_k\left(\mathbf{d}_k^{\mathbb{Q}}(v);\mathbf{\Gamma}_k\right)$$

where $\mathbf{d}^{\mathbb{M}}(v):=\left(d_i^{\mathbb{M}}(v)\right)_{1\leq i\leq n}$ and $\mathbf{d}_k^{\mathbb{Q}}(v)=\left(d_i^{\mathbb{M}}(v)-\sigma_V\sqrt{t_i}\right)_{1\leq i\leq k}$ with

$$d_i^{\mathbb{M}}=\frac{\ln\left(v/\bar{V}_i\right)+\left(r-\varpi+\sigma_V^2/2\right)t_i}{\sigma_V\sqrt{t_i}}\qquad\text{and}\qquad\mathbf{\Gamma}_k=\begin{pmatrix}1&\sqrt{\frac{t_1}{t_2}}&\sqrt{\frac{t_1}{t_3}}&\cdots&\sqrt{\frac{t_1}{t_k}}\\&1&\sqrt{\frac{t_2}{t_3}}&\cdots&\sqrt{\frac{t_2}{t_k}}\\\cdots&\cdots&\cdots&\cdots&\cdots\\&&&1&\sqrt{\frac{t_{k-1}}{t_k}}\\&&&&1\end{pmatrix}.$$

Therefore, the delta of the equity is generally defined as

$$\Delta_S^{(n)}:=\frac{\partial s}{\partial v}=e^{-\varpi t_n}\left(\Phi_n\left(\mathbf{d}^{\mathbb{M}}(v);\mathbf{\Gamma}_n\right)+v\frac{\partial\Phi_n\left(\mathbf{d}^{\mathbb{M}}(v);\mathbf{\Gamma}_n\right)}{\partial v}\right)-\sum_{k=1}^n e^{-rt_k}F_k\frac{\partial\Phi_k\left(\mathbf{d}_k^{\mathbb{Q}}(v);\mathbf{\Gamma}_k\right)}{\partial v}.$$

The derivation of a semi-closed formula for the computation of the delta for a generic n is not straightforward. However, I explicitly develop analytical expressions for $n=\{1,2,3\}$ (which suffice for the actual calculations present in the paper). Also, despite $\Delta_S^{(n)}:\mathbb{R}_+\to(0,1)$, for all $n\in\mathbb{N}$, its numerical computation becomes progressively more intensive (as n grows).

For convenience of notation, the dependence on v in the integration intervals (the d's and related expressions) is omitted.

If $n = 1$ (also, let $t_1 = t$ and $F_1 = F$)

$$s = e^{-\varpi t} v \Phi(d^{\mathbb{M}}) - e^{-rt} F \Phi(d^{\mathbb{Q}})$$

with

$$d^{\mathbb{M}} = \frac{\ln(v/F) + \left(r - \varpi + \sigma_V^2/2\right) t}{\sigma_V \sqrt{t}} \qquad d^{\mathbb{Q}} = d^{\mathbb{M}} - \sigma_V \sqrt{t}.$$

Obviously, the delta is the same of Black and Scholes (1973). In fact

$$\begin{aligned}\frac{\partial s}{\partial v} &= e^{-\varpi t} \left(\Phi(d^{\mathbb{M}}) + v \frac{\partial \Phi(d^{\mathbb{M}})}{\partial v} \right) - e^{-rt} F \frac{\partial \Phi(d^{\mathbb{Q}})}{\partial v} \\ &= e^{-\varpi t} \left(\Phi(d^{\mathbb{M}}) + v \phi(d^{\mathbb{M}}) \frac{1}{v \sigma_V \sqrt{t}} \right) - e^{-rt} F \phi(d^{\mathbb{Q}}) \frac{1}{v \sigma_V \sqrt{t}}\end{aligned}$$

as

$$\phi(d^{\mathbb{Q}}) = \phi(d^{\mathbb{M}} - \sigma_V \sqrt{t}) = \frac{v e^{-\varpi t}}{F e^{-rt}} \phi(d^{\mathbb{M}}).$$

it follows

$$\boxed{\Delta_S^{(1)} = e^{-\varpi t} \Phi(d^{\mathbb{M}})}.$$

See Figures F.1a and F.1b for a graphical analysis of the delta in the case of one bond outstanding.

For $n = 2$

$$s = e^{-\varpi t_2} v \Phi_2\left(\mathbf{d}^{\mathbb{M}}; \mathbf{\Gamma}\right) - e^{-rt_1} F_1 \Phi\left(d_1^{\mathbb{Q}}\right) - e^{-rt_2} F_2 \Phi_2\left(\mathbf{d}_2^{\mathbb{Q}}; \mathbf{\Gamma}\right)$$

with

$$\mathbf{\Gamma} = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix} \qquad \text{and} \qquad \gamma = \sqrt{\frac{t_1}{t_2}}$$

$$\begin{aligned}\mathbf{d}^{\mathbb{M}} &= \begin{pmatrix} d_1^{\mathbb{M}} & d_2^{\mathbb{M}} \end{pmatrix} = \begin{pmatrix} \frac{\ln \frac{v}{V_1} + \left(r - \varpi + \frac{\sigma_V^2}{2}\right) t_1}{\sigma_V \sqrt{t_1}} & \frac{\ln \frac{v}{F_2} + \left(r - \varpi + \frac{\sigma_V^2}{2}\right) t_2}{\sigma_V \sqrt{t_2}} \end{pmatrix} \\ \mathbf{d}_2^{\mathbb{Q}} &= \begin{pmatrix} d_1^{\mathbb{Q}} & d_2^{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} \frac{\ln \frac{v}{V_1} + \left(r - \varpi - \frac{\sigma_V^2}{2}\right) t_1}{\sigma_V \sqrt{t_1}} & \frac{\ln \frac{v}{F_2} + \left(r - \varpi - \frac{\sigma_V^2}{2}\right) t_2}{\sigma_V \sqrt{t_2}} \end{pmatrix}.\end{aligned}$$

Here the delta is

$$\frac{\partial s}{\partial v} = e^{-\varpi t_2} \left(\Phi_2\left(\mathbf{d}^{\mathbb{M}}; \mathbf{\Gamma}\right) + v \frac{\partial \Phi_2\left(\mathbf{d}^{\mathbb{M}}; \mathbf{\Gamma}\right)}{\partial v} \right) - e^{-rt_1} F_1 \frac{\partial \Phi\left(d_1^{\mathbb{Q}}\right)}{\partial v} - e^{-rt_2} F_2 \frac{\partial \Phi_2\left(\mathbf{d}_2^{\mathbb{Q}}; \mathbf{\Gamma}\right)}{\partial v}.$$

In order to effectively compute the delta of the equity for $n = 2$, I need to find an expression for the partial derivative of the bivariate CDF. Based on Theorem F.1 (for convenience of notation the dependence on the measure is also omitted), it follows

$$\begin{aligned}\frac{\partial \Phi_2\left(\mathbf{d}; \mathbf{\Gamma}\right)}{\partial v} &= \frac{\partial \Phi_2\left(\mathbf{d}; \mathbf{\Gamma}\right)}{\partial d_1} \frac{\partial d_1}{\partial v} + \frac{\partial \Phi_2\left(\mathbf{d}; \mathbf{\Gamma}\right)}{\partial d_2} \frac{\partial d_2}{\partial v} \\ &= \frac{1}{v} \left(\frac{1}{\sigma_V \sqrt{t_1}} \int_{-\infty}^{d_2} \frac{1}{2\pi \sqrt{1-\gamma^2}} \exp\left(-\frac{1}{2} \frac{x^2 - 2\gamma d_1 x + d_1^2}{1-\gamma^2} \right) \mathrm{d}x \right. \\ &\quad \left. + \frac{1}{\sigma_V \sqrt{t_2}} \int_{-\infty}^{d_1} \frac{1}{2\pi \sqrt{1-\gamma^2}} \exp\left(-\frac{1}{2} \frac{d_2^2 - 2\gamma d_2 y + y^2}{1-\gamma^2} \right) \mathrm{d}y \right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{v}\left(\frac{1}{\sigma_V\sqrt{t_1}}\frac{\exp\left(-\frac{d_1^2}{2}\right)}{\sqrt{2\pi}}\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi(1-\gamma^2)}}\exp\left(-\frac{1}{2}\frac{(x-\gamma d_1)^2}{1-\gamma^2}\right)\mathrm{d}x\right.\\
&\quad\left.+\frac{1}{\sigma_V\sqrt{t_2}}\frac{\exp\left(-\frac{d_2^2}{2}\right)}{\sqrt{2\pi}}\int_{-\infty}^{d_1}\frac{1}{\sqrt{2\pi(1-\gamma^2)}}\exp\left(-\frac{1}{2}\frac{(y-\gamma d_2)^2}{1-\gamma^2}\right)\mathrm{d}y\right)\\
&= \frac{1}{v}\left(\frac{\phi(d_1)}{\sigma_V\sqrt{t_1}}\Phi\left(\frac{d_2-\gamma d_1}{\sqrt{1-\gamma^2}}\right)+\frac{\phi(d_2)}{\sigma_V\sqrt{t_2}}\Phi\left(\frac{d_1-\gamma d_2}{\sqrt{1-\gamma^2}}\right)\right).
\end{aligned}
$$

Setting

$$
\begin{aligned}
\mathfrak{d}_2^{\mathbb{M}} &:= \frac{d_2^{\mathbb{M}}-\gamma d_1^{\mathbb{M}}}{\sqrt{1-\gamma^2}} = \frac{\ln\frac{\bar{V}_1}{F_2}+\left(r-\varpi+\frac{\sigma_V^2}{2}\right)(t_2-t_1)}{\sigma_V\sqrt{t_2-t_1}}\\
\mathfrak{d}_2^{\mathbb{Q}} &:= \frac{d_2^{\mathbb{Q}}-\gamma d_1^{\mathbb{Q}}}{\sqrt{1-\gamma^2}} = \mathfrak{d}_2^{\mathbb{M}}-\sigma_V\sqrt{t_2-t_1}
\end{aligned}
\tag{F.1}
$$

and[12]

$$
\mathfrak{d}_1^{\mathbb{M}} := \frac{d_1^{\mathbb{M}}-\gamma d_2^{\mathbb{M}}}{\sqrt{1-\gamma^2}} = \frac{\ln\left(\frac{v}{\bar{V}_1}\right)t_2-\ln\left(\frac{v}{F_2}\right)t_1}{\sigma_V\sqrt{t_1t_2(t_2-t_1)}} = \frac{d_1^{\mathbb{Q}}-\gamma d_2^{\mathbb{Q}}}{\sqrt{1-\gamma^2}} := \mathfrak{d}_1^{\mathbb{Q}}
\tag{F.2}
$$

it follows

$$
\begin{aligned}
\frac{\partial s}{\partial v} &= e^{-\varpi t_2}\left(\Phi_2\left(\mathbf{d}^{\mathbb{M}};\mathbf{\Gamma}\right)+\frac{\phi(d_1^{\mathbb{M}})}{\sigma_V\sqrt{t_1}}\Phi\left(\mathfrak{d}_2^{\mathbb{M}}\right)+\frac{\phi(d_2^{\mathbb{M}})}{\sigma_V\sqrt{t_2}}\Phi\left(\mathfrak{d}_1^{\mathbb{M}}\right)\right)\\
&\quad - e^{-rt_1}\frac{F_1}{v\sigma_V\sqrt{t_1}}\phi(d_1^{\mathbb{Q}}) - e^{-rt_2}\frac{F_2}{v}\left(\frac{\phi(d_1^{\mathbb{Q}})}{\sigma_V\sqrt{t_1}}\Phi\big(\mathfrak{d}_2^{\mathbb{Q}}\big)+\frac{\phi(d_2^{\mathbb{Q}})}{\sigma_V\sqrt{t_2}}\Phi\big(\mathfrak{d}_1^{\mathbb{Q}}\big)\right).
\end{aligned}
$$

Finally, using

$$
\phi(d_1^{\mathbb{Q}}) = \phi(d_1^{\mathbb{M}}-\sigma_V\sqrt{t_1}) = \frac{ve^{-\varpi t_1}}{\bar{V}_1e^{-rt_1}}\phi(d_1^{\mathbb{M}}) \quad \text{and} \quad \phi(d_2^{\mathbb{Q}}) = \phi(d_2^{\mathbb{M}}-\sigma_V\sqrt{t_2}) = \frac{ve^{-\varpi t_2}}{F_2e^{-rt_2}}\phi(d_2^{\mathbb{M}})
$$

and (F.2), the previous expression can be written as

$$
\boxed{\Delta_S^{(2)} = e^{-\varpi t_2}\left[\Phi_2\left(\mathbf{d}^{\mathbb{M}};\mathbf{\Gamma}\right)+\frac{\phi(d_1^{\mathbb{M}})}{\sigma_V\sqrt{t_1}}\left(\Phi\left(\mathfrak{d}_2^{\mathbb{M}}\right)-\frac{F_2e^{-r(t_2-t_1)}}{\bar{V}_1e^{-\varpi(t_2-t_1)}}\Phi\big(\mathfrak{d}_2^{\mathbb{Q}}\big)\right)\right]-e^{-rt_1}\frac{F_1}{v\sigma_V\sqrt{t_1}}\phi(d_1^{\mathbb{Q}})}
$$

See Figures F.1c and F.1d for a graphical analysis of the delta in the case of two bonds outstanding.

Finally, if $n=3$

$$
s = e^{-\varpi t_3}v\Phi_3\left(\mathbf{d}^{\mathbb{M}};\mathbf{\Gamma_3}\right)-e^{-rt_1}F_1\Phi\big(d_1^{\mathbb{Q}}\big)-e^{-rt_2}F_2\Phi_2\big(\mathbf{d}_2^{\mathbb{Q}};\mathbf{\Gamma_2}\big)-e^{-rt_3}F_3\Phi_3\big(\mathbf{d}_3^{\mathbb{Q}};\mathbf{\Gamma_3}\big)
$$

[12] Just notice that

$$
\frac{d_1^{\mathbb{Q}}-\gamma d_2^{\mathbb{Q}}}{\sqrt{1-\gamma^2}} = \frac{d_1^{\mathbb{M}}-\gamma d_2^{\mathbb{M}}}{\sqrt{1-\gamma^2}}
$$

as

$$
\begin{aligned}
d_1^{\mathbb{M}}-d_1^{\mathbb{Q}} &= \gamma\left(d_2^{\mathbb{M}}-d_2^{\mathbb{Q}}\right)\\
\sigma_V\sqrt{t_1} &= \sqrt{\frac{t_1}{t_2}}\sigma_V\sqrt{t_2}.
\end{aligned}
$$

with

$$\mathbf{\Gamma_3} = \begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & 1 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & 1 \end{pmatrix}, \qquad \mathbf{\Gamma_2} = \begin{pmatrix} 1 & \gamma_{12} \\ \gamma_{12} & 1 \end{pmatrix} \qquad \text{and} \qquad \gamma_{ij} = \sqrt{\frac{t_i}{t_j}}, \text{ with } i \leq j$$

$$\mathbf{d}^{\mathbb{M}} = \begin{pmatrix} d_1^{\mathbb{M}} & d_2^{\mathbb{M}} & d_3^{\mathbb{M}} \end{pmatrix} = \begin{pmatrix} \frac{\ln \frac{v}{V_1} + \left(r - \varpi + \frac{\sigma_V^2}{2}\right) t_1}{\sigma_V \sqrt{t_1}} & \frac{\ln \frac{v}{V_2} + \left(r - \varpi + \frac{\sigma_V^2}{2}\right) t_2}{\sigma_V \sqrt{t_2}} & \frac{\ln \frac{v}{F_3} + \left(r - \varpi + \frac{\sigma_V^2}{2}\right) t_3}{\sigma_V \sqrt{t_3}} \end{pmatrix}$$

and

$$\mathbf{d}_3^{\mathbb{Q}} = \begin{pmatrix} \mathbf{d}_2^{\mathbb{Q}} & d_3^{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} d_1^{\mathbb{Q}} & d_2^{\mathbb{Q}} & d_3^{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} \frac{\ln \frac{v}{V_1} + \left(r - \varpi - \frac{\sigma_V^2}{2}\right) t_1}{\sigma_V \sqrt{t_1}} & \frac{\ln \frac{v}{V_2} + \left(r - \varpi - \frac{\sigma_V^2}{2}\right) t_2}{\sigma_V \sqrt{t_2}} & \frac{\ln \frac{v}{F_3} + \left(r - \varpi - \frac{\sigma_V^2}{2}\right) t_3}{\sigma_V \sqrt{t_3}} \end{pmatrix}.$$

Here the delta is equal to

$$\begin{aligned} \frac{\partial s}{\partial v} = {} & e^{-\varpi t_3} \left(\Phi_3\left(\mathbf{d}^{\mathbb{M}}; \mathbf{\Gamma_3}\right) + v \frac{\partial \Phi_3\left(\mathbf{d}^{\mathbb{M}}; \mathbf{\Gamma_3}\right)}{\partial v} \right) - e^{-r t_1} F_1 \frac{\partial \Phi\left(d_1^{\mathbb{Q}}\right)}{\partial v} - e^{-r t_2} F_2 \frac{\partial \Phi_2\left(\mathbf{d}_2^{\mathbb{Q}}; \mathbf{\Gamma_2}\right)}{\partial v} \\ & - e^{-r t_3} F_3 \frac{\partial \Phi_3\left(\mathbf{d}_3^{\mathbb{Q}}; \mathbf{\Gamma_3}\right)}{\partial v}. \end{aligned}$$

Again, to compute the delta of the equity for $n = 3$, I need to find an expression for the partial derivative of the trivariate CDF. Using Theorem F.1, it follows

$$\begin{aligned} \frac{\partial \Phi_3\left(\mathbf{d}; \mathbf{\Gamma}\right)}{\partial v} &= \frac{\partial \Phi_3\left(\mathbf{d}; \mathbf{\Gamma}\right)}{\partial d_1} \frac{\partial d_1}{\partial v} + \frac{\partial \Phi_3\left(\mathbf{d}; \mathbf{\Gamma}\right)}{\partial d_2} \frac{\partial d_2}{\partial v} + \frac{\partial \Phi_3\left(\mathbf{d}; \mathbf{\Gamma}\right)}{\partial d_3} \frac{\partial d_3}{\partial v} \\ &= \frac{1}{v} \left(\frac{1}{\sigma_V \sqrt{t_1}} \int_{-\infty}^{d_2} \int_{-\infty}^{d_3} \frac{1}{\sqrt{(2\pi)^3 \det \mathbf{\Gamma}}} \exp\left(-\frac{\tau_1 x^2 + \tau_4 y^2 + \tau_9 d_1^2 + 2\tau_2 x y + 2\tau_6 d_1 y}{2} \right) \mathrm{d}x \, \mathrm{d}y \right. \\ &\quad + \frac{1}{\sigma_V \sqrt{t_2}} \int_{-\infty}^{d_1} \int_{-\infty}^{d_3} \frac{1}{\sqrt{(2\pi)^3 \det \mathbf{\Gamma}}} \exp\left(-\frac{\tau_1 x^2 + \tau_4 d_2^2 + \tau_9 z^2 + 2\tau_2 d_2 x + 2\tau_6 d_2 z}{2} \right) \mathrm{d}x \, \mathrm{d}z \\ &\quad \left. + \frac{1}{\sigma_V \sqrt{t_3}} \int_{-\infty}^{d_1} \int_{-\infty}^{d_2} \frac{1}{\sqrt{(2\pi)^3 \det \mathbf{\Gamma}}} \exp\left(-\frac{\tau_1 d_3^2 + \tau_4 y^2 + \tau_9 z^2 + 2\tau_2 d_3 y + 2\tau_6 y z}{2} \right) \mathrm{d}y \, \mathrm{d}z \right) \\ &= \frac{1}{v} \left(\frac{I_1}{\sigma_V \sqrt{t_1}} + \frac{I_2}{\sigma_V \sqrt{t_2}} + \frac{I_3}{\sigma_V \sqrt{t_3}} \right) \end{aligned}$$

where $\det \mathbf{\Gamma} = \frac{(t_2 - t_1)(t_3 - t_2)}{t_2 t_3}$ and

$$\mathbf{\Gamma}^{-1} = \begin{pmatrix} \frac{t_2}{t_2 - t_1} & -\frac{\sqrt{t_1 t_2}}{t_2 - t_1} & 0 \\ -\frac{\sqrt{t_1 t_2}}{t_2 - t_1} & \frac{t_2 (t_3 - t_1)}{(t_2 - t_1)(t_3 - t_2)} & -\frac{\sqrt{t_2 t_3}}{t_3 - t_2} \\ 0 & -\frac{\sqrt{t_2 t_3}}{t_3 - t_2} & \frac{t_3}{t_3 - t_2} \end{pmatrix} = \begin{pmatrix} \tau_1 & \tau_2 & 0 \\ \tau_2 & \tau_4 & \tau_6 \\ 0 & \tau_6 & \tau_9 \end{pmatrix}.$$

All the double integrals can be computed recognising appropriate bivariate Gaussian random vector and re-expressing the integrals as an appropriate bivariate normal CDF, i.e.

$$\int_{-\infty}^{a} \int_{-\infty}^{b} \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \exp\left(-\frac{\left(\frac{w_1 - \mu_1}{\sigma_1}\right)^2 + \left(\frac{w_2 - \mu_2}{\sigma_2}\right)^2 - 2\rho \left(\frac{w_1 - \mu_1}{\sigma_1}\right)\left(\frac{w_2 - \mu_2}{\sigma_2}\right)}{2\left(1 - \rho^2\right)} \right) \mathrm{d}w_1 \, \mathrm{d}w_2.$$

Solution of I_1

In order to find the appropriate random vector $\mathbf{W_1} \sim \mathcal{N}(\boldsymbol{\mu_1}, \boldsymbol{\Sigma_1})$, I need to determine $\Theta_1 = \{\boldsymbol{\mu_1}, \boldsymbol{\Sigma_1}\} = \{\mu_1, \mu_2, \sigma_1, \sigma_2, \rho\}$ such that

$$\frac{\left(\frac{w_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{w_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{w_1-\mu_1}{\sigma_1}\right)\left(\frac{w_2-\mu_2}{\sigma_2}\right)}{1-\rho^2} = \tau_1 w_1^2 + \tau_4 w_2^2 + 2\tau_2 w_1 w_2 + 2\tau_6 d_1 w_2 + \tilde{a}_1 \tag{F.3}$$

and re-express the density as normalised based on its covariance matrix (notice that $\tilde{a}_1$ is a free parameter). Expanding the left-hand side of (F.3)

$$\frac{1}{1-\rho^2}\left[\frac{w_1^2}{\sigma_1^2} + \frac{w_2^2}{\sigma_2^2} - 2\frac{\rho}{\sigma_1\sigma_2}w_1w_2 + \frac{2}{\sigma_1}\left(\rho\frac{\mu_2}{\sigma_2} - \frac{\mu_1}{\sigma_1}\right)w_1 + \frac{2}{\sigma_2}\left(\rho\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\right)w_2 + \left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right]$$

the following conditions must be met

$$\begin{aligned}
\frac{1}{(1-\rho^2)\sigma_1^2} &= \tau_1 \\
\frac{1}{(1-\rho^2)\sigma_2^2} &= \tau_4 \\
-\frac{\rho}{(1-\rho^2)\sigma_1\sigma_2} &= \tau_2 \\
\frac{1}{(1-\rho^2)\sigma_1}\left(\rho\frac{\mu_2}{\sigma_2} - \frac{\mu_1}{\sigma_1}\right) &= 0 \\
\frac{1}{(1-\rho^2)\sigma_2}\left(\rho\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\right) &= \tau_6 d_1 \\
\frac{1}{1-\rho^2}\left[\left(\frac{\mu_1}{\sigma_1}\right)^2 + \left(\frac{\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right] &= \tilde{a}_1.
\end{aligned}$$

The first three conditions allow to find σ_1, σ_2 and ρ as

$$\rho = -\frac{\tau_2}{\sqrt{\tau_1\tau_4}} \qquad \sigma_1^2 = \frac{1}{\tau_1(1-\rho^2)} = \frac{\tau_4}{\tau_1\tau_4 - \tau_2^2} \qquad \sigma_2^2 = \frac{1}{\tau_4(1-\rho^2)} = \frac{\tau_1}{\tau_1\tau_4 - \tau_2^2}.$$

The forth condition, imposes

$$\frac{\mu_1}{\sigma_1} = \rho\frac{\mu_2}{\sigma_2}$$

which can be substituted into the fifth condition to find μ_2 as

$$\mu_2 = -\tau_6\sigma_2^2 d_1 = -\frac{\tau_6}{\tau_4(1-\rho^2)}d_1 = -\frac{\tau_1\tau_6}{\tau_1\tau_4 - \tau_2^2}d_1.$$

Finally, μ_1 is found as

$$\mu_1 = \rho\frac{\mu_2\sigma_1}{\sigma_2} = \frac{\tau_2\tau_6}{\tau_1\tau_4(1-\rho^2)}d_1 = \frac{\tau_2\tau_6}{\tau_1\tau_4 - \tau_2^2}d_1,$$

and

$$\tilde{a}_1 = \frac{\tau_1\tau_6^2}{\tau_1\tau_4 - \tau_2^2}d_1^2.$$

Therefore

$$\begin{aligned}
I_1 &= \sqrt{\frac{\det\boldsymbol{\Sigma_1}}{\det\boldsymbol{\Gamma}}}\frac{\exp\left(-\frac{\tau_9 d_1^2 - \tilde{a}_1}{2}\right)}{\sqrt{2\pi}}\int_{-\infty}^{d_2}\int_{-\infty}^{d_3}\frac{1}{2\pi\sqrt{\det\boldsymbol{\Sigma_1}}}\exp\left(-\frac{(\mathbf{w_1}-\boldsymbol{\mu_1})^\top\boldsymbol{\Sigma_1}^{-1}(\mathbf{w_1}-\boldsymbol{\mu_1})}{2}\right)\mathrm{d}\mathbf{w_1} \\
&= \sqrt{\frac{\det\boldsymbol{\Sigma_1}}{\det\boldsymbol{\Gamma}}}\phi\left(\sqrt{a_1}d_1\right)N_2\left(d_2, d_3; \boldsymbol{\mu_1}, \boldsymbol{\Sigma_1}\right)
\end{aligned}$$

with

$$a_1 = \tau_9 - \frac{\tau_1 \tau_6^2}{\tau_1 \tau_4 - \tau_2^2}$$

and

$$\det \mathbf{\Sigma_1} = \sigma_1^2 \sigma_2^2 \left(1 - \rho^2\right) = \frac{1}{\tau_1 \tau_4 - \tau_2^2}.$$

Solution of I_2

The second integral is simpler to solve as there is no xz term. In fact, it can be expressed as the CDFs of two univariate Gaussian (independent) random variables as

$$\begin{aligned}
I_2 &= \int_{-\infty}^{d_1} \int_{-\infty}^{d_3} \frac{1}{\sqrt{(2\pi)^3 \det \mathbf{\Gamma}}} \exp\left(-\frac{\tau_1 x^2 + \tau_4 d_2^2 + \tau_9 z^2 + 2\tau_2 d_2 x + 2\tau_6 d_2 z}{2}\right) \mathrm{d}x \, \mathrm{d}z \\
&= \frac{\sigma_x \sigma_z}{\sqrt{\det \mathbf{\Gamma}}} \frac{\exp\left(-\frac{a_2 d_2^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(-\frac{1}{2}\left(\frac{x - \mu_x}{\sigma_x}\right)^2\right) \mathrm{d}x \int_{-\infty}^{d_3} \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{1}{2}\left(\frac{z - \mu_z}{\sigma_z}\right)^2\right) \mathrm{d}z
\end{aligned}$$

with

$$\begin{aligned}
\mu_x &= -\frac{\tau_2 d_2}{\tau_1}, & \sigma_x^2 &= \frac{1}{\tau_1}, \\
\mu_z &= -\frac{\tau_6 d_2}{\tau_9}, & \sigma_z^2 &= \frac{1}{\tau_9}, \\
a_2 = \tau_4 - \frac{\tau_2^2}{\tau_1} &- \frac{\tau_6^2}{\tau_9}, & \mathbf{\Sigma_2} &= \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_z^2 \end{pmatrix}.
\end{aligned}$$

Therefore

$$I_2 = \sqrt{\frac{\det \mathbf{\Sigma_2}}{\det \mathbf{\Gamma}}} \phi\left(\sqrt{a_2} d_2\right) \Phi\left(\frac{\tau_1 d_1 + \tau_2 d_2}{\sqrt{\tau_1}}\right) \Phi\left(\frac{\tau_9 d_3 + \tau_6 d_2}{\sqrt{\tau_9}}\right)$$

with

$$\det \mathbf{\Sigma_2} = \sigma_x^2 \sigma_z^2 = \frac{1}{\tau_1 \tau_9}.$$

Alternatively, the integral can also be expressed as

$$I_2 = \sqrt{\frac{\det \mathbf{\Sigma_2}}{\det \mathbf{\Gamma}}} \phi\left(\sqrt{a_2} d_2\right) N_2\left(d_1, d_3; \boldsymbol{\mu_2}, \mathbf{\Sigma_2}\right),$$

where $\boldsymbol{\mu_2} = \begin{pmatrix} \mu_x & \mu_y \end{pmatrix}^\top$.

Solution of I_3

The procedure to solve the last integral is the same used for I_1. Consider the random vector $\mathbf{W_3} \sim \mathcal{N}\left(\boldsymbol{\mu_3}, \mathbf{\Sigma_3}\right)$. Again, I need to determine $\Theta_3 = \{\boldsymbol{\mu_3}, \mathbf{\Sigma_3}\} = \{\mu_1, \mu_2, \sigma_1, \sigma_2, \rho\}$ such that

$$\frac{\left(\frac{w_1 - \mu_1}{\sigma_1}\right)^2 + \left(\frac{w_2 - \mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{w_1 - \mu_1}{\sigma_1}\right)\left(\frac{w_2 - \mu_2}{\sigma_2}\right)}{1 - \rho^2} = \tau_4 w_1^2 + \tau_9 w_2^2 + 2\tau_2 d_3 w_1 + 2\tau_6 w_1 w_2 + \tilde{a}_3.$$

Thus, the following conditions must be met

$$\begin{aligned}
\frac{1}{(1-\rho^2)\sigma_1^2} &= \tau_4 \\
\frac{1}{(1-\rho^2)\sigma_2^2} &= \tau_9 \\
-\frac{\rho}{(1-\rho^2)\sigma_1\sigma_2} &= \tau_6 \\
\frac{1}{(1-\rho^2)\sigma_1}\left(\rho\frac{\mu_2}{\sigma_2}-\frac{\mu_1}{\sigma_1}\right) &= \tau_2 d_3 \\
\frac{1}{(1-\rho^2)\sigma_2}\left(\rho\frac{\mu_1}{\sigma_1}-\frac{\mu_2}{\sigma_2}\right) &= 0 \\
\frac{1}{1-\rho^2}\left[\left(\frac{\mu_1}{\sigma_1}\right)^2+\left(\frac{\mu_2}{\sigma_2}\right)^2-2\rho\frac{\mu_1\mu_2}{\sigma_1\sigma_2}\right] &= \tilde{a}_3.
\end{aligned}$$

The first three conditions allow to find σ_1, σ_2 and ρ as

$$\rho = -\frac{\tau_6}{\sqrt{\tau_4\tau_9}} \qquad \sigma_1^2 = \frac{1}{\tau_4\left(1-\rho^2\right)} = \frac{\tau_9}{\tau_4\tau_9-\tau_6^2} \qquad \sigma_2^2 = \frac{1}{\tau_9\left(1-\rho^2\right)} = \frac{\tau_4}{\tau_4\tau_9-\tau_6^2}.$$

The fifth condition, imposes

$$\frac{\mu_2}{\sigma_2} = \rho\frac{\mu_1}{\sigma_1}$$

which can be substituted into the forth condition to find μ_1 as

$$\mu_1 = -\tau_2\sigma_1^2 d_3 = -\frac{\tau_2}{\tau_4\left(1-\rho^2\right)}d_3 = -\frac{\tau_2\tau_9}{\tau_4\tau_9-\tau_6^2}d_3.$$

Finally, μ_2 is found as

$$\mu_2 = \rho\frac{\mu_1\sigma_2}{\sigma_1} = \frac{\tau_2\tau_6}{\tau_4\tau_9(1-\rho^2)}d_3 = \frac{\tau_2\tau_6}{\tau_4\tau_9-\tau_6^2}d_3,$$

and

$$\tilde{a}_3 = \frac{\tau_9\tau_2^2}{\tau_4\tau_9-\tau_6^2}d_3^2.$$

Therefore

$$\begin{aligned}
I_3 &= \sqrt{\frac{\det\mathbf{\Sigma_3}}{\det\mathbf{\Gamma}}}\frac{\exp\left(-\frac{\tau_1 d_3^2-\tilde{a}_3}{2}\right)}{\sqrt{2\pi}}\int_{-\infty}^{d_1}\int_{-\infty}^{d_2}\frac{1}{2\pi\sqrt{\det\mathbf{\Sigma_3}}}\exp\left(-\frac{(\mathbf{w_3}-\boldsymbol{\mu_3})^\top\mathbf{\Sigma_3}^{-1}(\mathbf{w_3}-\boldsymbol{\mu_3})}{2}\right)\mathrm{d}\mathbf{w_3} \\
&= \sqrt{\frac{\det\mathbf{\Sigma_3}}{\det\mathbf{\Gamma}}}\phi\left(\sqrt{a_3}d_3\right)N_2\left(d_1,d_2;\boldsymbol{\mu_3},\mathbf{\Sigma_3}\right)
\end{aligned}$$

with

$$a_3 = \tau_1 - \frac{\tau_9\tau_2^2}{\tau_4\tau_9-\tau_6^2}$$

and

$$\det\mathbf{\Sigma_3} = \sigma_1^2\sigma_2^2\left(1-\rho^2\right) = \frac{1}{\tau_4\tau_9-\tau_6^2}.$$

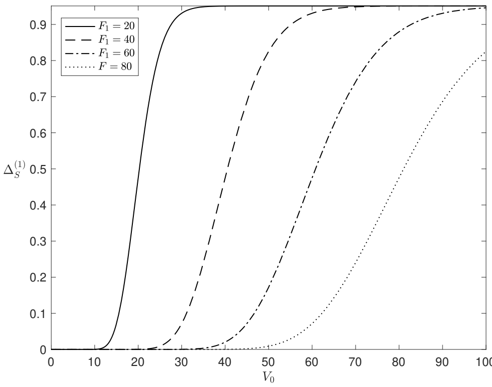
Hence, the delta of the equity in the case $n=3$ is

$$\begin{aligned}
\frac{\partial s}{\partial v} = {}& e^{-\varpi t_3}\left(\Phi_3\left(\mathbf{d}^{\mathbb{M}};\mathbf{\Gamma_3}\right)+\frac{I_1^{\mathbb{M}}}{\sigma_V\sqrt{t_1}}+\frac{I_2^{\mathbb{M}}}{\sigma_V\sqrt{t_2}}+\frac{I_3^{\mathbb{M}}}{\sigma_V\sqrt{t_3}}\right)-e^{-rt_3}\frac{F_3}{v}\left(\frac{I_1^{\mathbb{Q}}}{\sigma_V\sqrt{t_1}}+\frac{I_2^{\mathbb{Q}}}{\sigma_V\sqrt{t_2}}+\frac{I_3^{\mathbb{Q}}}{\sigma_V\sqrt{t_3}}\right) \\
&-e^{-rt_2}\frac{F_2}{v}\left(\frac{\phi(d_1^{\mathbb{Q}})}{\sigma_V\sqrt{t_1}}\Phi\big(\mathfrak{d}_2^{\mathbb{Q}}\big)+\frac{\phi(d_2^{\mathbb{Q}})}{\sigma_V\sqrt{t_2}}\Phi\big(\mathfrak{d}_1^{\mathbb{Q}}\big)\right)-e^{-rt_1}\frac{F_1}{v\sigma_V\sqrt{t_1}}\phi(d_1^{\mathbb{Q}})
\end{aligned}$$

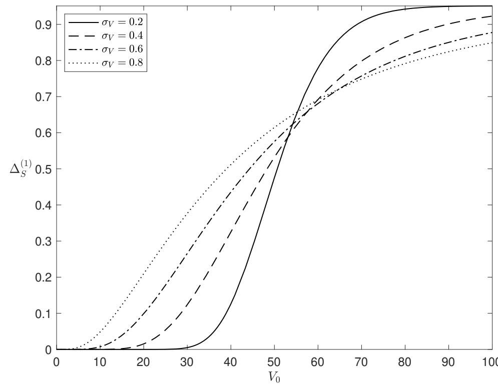
Writing the three integrals explicitly, it follows

$$\boxed{\begin{aligned}\Delta_S^{(3)} = e^{-\varpi t_3} & \left(\Phi_3\left(\mathbf{d}^{\mathbb{M}}; \mathbf{\Gamma_3}\right) + \frac{1}{\sigma_V \sqrt{\det \mathbf{\Gamma_3}}} \sum_{i=1}^{3} \sqrt{\frac{\det \mathbf{\Sigma}_{\boldsymbol{i}}}{t_i}} \phi\left(\sqrt{a_i} d_i^{\mathbb{M}}\right) N_2\left(\mathbf{d}^{\mathbb{M}} \setminus d_i^{\mathbb{M}}; \boldsymbol{\mu}_{\boldsymbol{i}}^{\mathbb{M}}, \mathbf{\Sigma}_{\boldsymbol{i}}\right)\right) \\ & - e^{-r t_3} \frac{F_3}{v} \frac{1}{\sigma_V \sqrt{\det \mathbf{\Gamma_3}}} \sum_{i=1}^{3} \sqrt{\frac{\det \mathbf{\Sigma}_{\boldsymbol{i}}}{t_i}} \phi\big(\sqrt{a_i} d_i^{\mathbb{Q}}\big) N_2\big(\mathbf{d}_3^{\mathbb{Q}} \setminus d_i^{\mathbb{Q}}; \boldsymbol{\mu}_{\boldsymbol{i}}^{\mathbb{Q}}, \mathbf{\Sigma}_{\boldsymbol{i}}\big) \\ & - e^{-r t_2} \frac{F_2}{v} \left(\frac{\phi(d_1^{\mathbb{Q}})}{\sigma_V \sqrt{t_1}} \Phi\big(\mathfrak{d}_2^{\mathbb{Q}}\big) + \frac{\phi(d_2^{\mathbb{Q}})}{\sigma_V \sqrt{t_2}} \Phi\big(\mathfrak{d}_1^{\mathbb{Q}}\big) \right) - e^{-r t_1} \frac{F_1}{v \sigma_V \sqrt{t_1}} \phi(d_1^{\mathbb{Q}}),\end{aligned}}$$

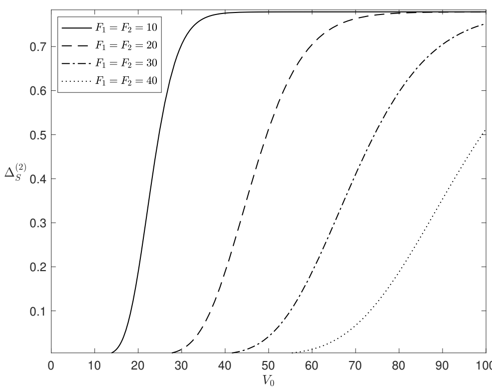
where $\mathbf{d} \setminus d_i$ must be intended as the vector obtained from $\mathbf{d}$ by removing the element d_i (and keeping the order of the other elements unchanged). See Figures F.1e and F.1f for a graphical analysis of the delta in the case of three bonds outstanding.



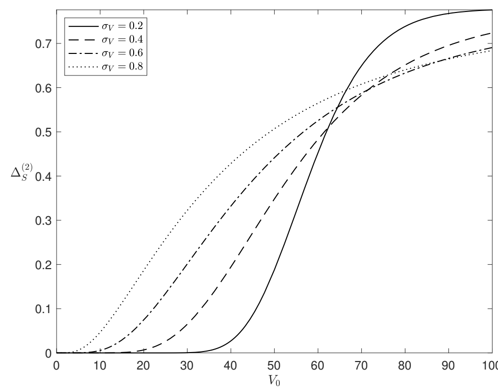
(a) $\sigma_V = 0.2, t_1 = 1$



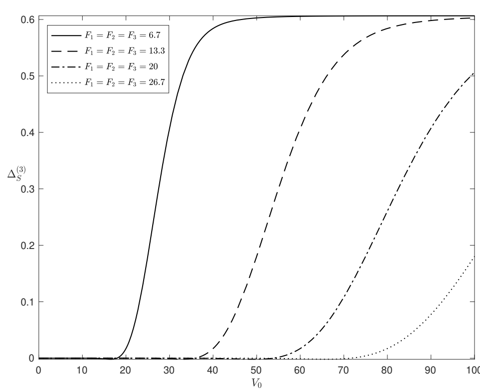
(b) $F_1 = 50, t_1 = 1$



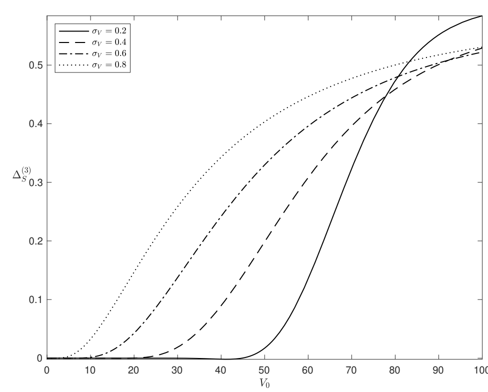
(c) $\sigma_V = 0.2, t_1 = 1, t_2 = 5$



(d) $F_1 = F_2 = 25, t_1 = 1, t_2 = 5$



(e) $\sigma_V = 0.2, t_1 = 1, t_2 = 5, t_3 = 10$



(f) $F_1 = F_2 = F_3 = 50/3, t_1 = 1, t_2 = 5, t_3 = 10$

Figure F.1: Sensitivity of equity with respect to financial leverage (left) and asset volatility (right). $r = 0.03$, $\varpi = 0.05$ throughout.

Appendix G The Vega of the Equity

In order to study the vega of the equity, the following result is needed.

Theorem G.1. *Let*

$$\Phi_k\left(\mathbf{d}(x);\mathbf{\Gamma}\right)=\int_{\Upsilon(x)}\phi_k(y_1,\ldots,y_i,\ldots,y_k;\mathbf{\Gamma})\mathrm{d}y_1\ldots\mathrm{d}y_i\ldots\mathrm{d}y_k$$

with $\mathbf{\Gamma}\in\mathcal{M}_+^k$ *and* $\Upsilon(x)=\bigcap_{i=1}^k\{y_i\in\mathbb{R}:y_i\leq d_i(x)\}$, *with* $\mathbf{d}(x):\mathbb{R}_+\to\mathbb{R}^k$, $d_i(x)=b_ix\pm\frac{a_i}{x}$ *with* $a_i:\mathbb{R}_+\to\mathbb{R}$ *and* $b_i\in\mathbb{R}_+$. *Then*

$$\frac{\partial\Phi_k\left(\mathbf{d}(x);\mathbf{\Gamma}\right)}{\partial x}=\sum_{i=1}^k\left(b_i\mp\frac{a_i(x)}{x^2}\right)\int_{\bar{\Upsilon}_i(x)}\phi_k(y_1,\ldots,d_i(x),\ldots,y_k;\mathbf{\Gamma})\mathrm{d}y_1\ldots\mathrm{d}y_k,$$

where $\bar{\Upsilon}_i(x)=\Upsilon(x)\setminus\{y_i\leq d_i(x)\}$.

Proof. It follows by the same arguments of Theorem F.1 with

$$\frac{\partial d_i}{\partial x}=b_i\mp\frac{a_i(x)}{x^2}.$$

□

Given n bond outstanding, the value of the equity $s=S(\sigma_V)$ is given by (8), that is

$$s=e^{-\varpi t_n}V_0\Phi_n\left(\mathbf{d}^{\mathbb{M}}(\sigma_V);\mathbf{\Gamma}_n\right)-\sum_{k=1}^n e^{-rt_k}F_k\Phi_k\left(\mathbf{d}_k^{\mathbb{Q}}(\sigma_V);\mathbf{\Gamma}_k\right)$$

where $\mathbf{d}^{\mathbb{M}}(\sigma_V):=\left(d_i^{\mathbb{M}}(\sigma_V)\right)_{1\leq i\leq n}$ and $\mathbf{d}_k^{\mathbb{Q}}(\sigma_V)=\left(d_i^{\mathbb{M}}(\sigma_V)-\sigma_V\sqrt{t_i}\right)_{1\leq i\leq k}$ with

$$d_i^{\mathbb{M}}=\frac{\ln\left(V_0/\bar{V}_i\right)+\left(r-\varpi+\sigma_V^2/2\right)t_i}{\sigma_V\sqrt{t_i}}\qquad\text{and}\qquad\mathbf{\Gamma}_k=\begin{pmatrix}1&\sqrt{\frac{t_1}{t_2}}&\sqrt{\frac{t_1}{t_3}}&\cdots&\sqrt{\frac{t_1}{t_k}}\\&1&\sqrt{\frac{t_2}{t_3}}&\cdots&\sqrt{\frac{t_2}{t_k}}\\\cdots&\cdots&\cdots&\cdots&\cdots\\&&&1&\sqrt{\frac{t_{k-1}}{t_k}}\\&&&&1\end{pmatrix},$$

and

$$\bar{V}_i:=\left\{v\in\mathbb{R}_+:S_i^\star(v)=F_i\right\}.$$

The vega of the equity is defined in genaral as

$$\nu_S^{(n)}:=\frac{\partial s}{\partial\sigma_V}=e^{-\varpi t_n}V_0\frac{\partial\Phi_n\left(\mathbf{d}^{\mathbb{M}}(\sigma_V);\mathbf{\Gamma}_n\right)}{\partial\sigma_V}-\sum_{k=1}^n e^{-rt_k}F_k\frac{\partial\Phi_k\left(\mathbf{d}_k^{\mathbb{Q}}(\sigma_V);\mathbf{\Gamma}_k\right)}{\partial\sigma_V}.$$

In the same fashion of Appendix F, I calculate the vega of the equity for $n=\{1,2,3\}$. For convenience of notation, the dependence on σ_V in the integration intervals (the d's and related expressions) is omitted.

If $n=1$ (also, let $t_1=t$ and $F_1=F$)

$$s=e^{-\varpi t}V_0\Phi(d^{\mathbb{M}})-e^{-rt}F\Phi(d^{\mathbb{Q}})$$

with

$$d^{\mathbb{M}}=\frac{\ln\left(V_0/F\right)+\left(r-\varpi+\sigma_V^2/2\right)t}{\sigma_V\sqrt{t}}\qquad d^{\mathbb{Q}}=d^{\mathbb{M}}-\sigma_V\sqrt{t}.$$

Obviously, the vega is the same of Black and Scholes (1973). In fact

$$\begin{aligned}\frac{\partial s}{\partial \sigma_V} &= e^{-\varpi t} V_0 \frac{\partial \Phi(d^{\mathbb{M}})}{\partial \sigma_V} - e^{-rt} F \frac{\partial \Phi(d^{\mathbb{Q}})}{\partial \sigma_V} \\ &= e^{-rt} V_0 \phi\left(d^{\mathbb{M}}\right)\left(-\frac{d^{\mathbb{Q}}}{\sigma_V}\right) - e^{-\varpi t} F \phi\left(d^{\mathbb{Q}}\right)\left(-\frac{d^{\mathbb{M}}}{\sigma_V}\right) \\ &= e^{-\varpi t} V_0 \phi\left(d^{\mathbb{M}}\right) \frac{\sigma_V \sqrt{t} - d^{\mathbb{M}}}{\sigma_V} + e^{-rt} F \phi\left(d^{\mathbb{Q}}\right) \frac{d^{\mathbb{M}}}{\sigma_V}\end{aligned}$$

as

$$\phi(d^{\mathbb{Q}}) = \phi(d^{\mathbb{M}} - \sigma_V \sqrt{t}) = \frac{V_0 e^{-\varpi t}}{F e^{-rt}} \phi(d^{\mathbb{M}}).$$

it follows

$$\boxed{\nu_S^{(1)} = e^{-\varpi t} \phi(d^{\mathbb{M}}) V_0 \sqrt{t}}.$$

See Figures G.1a and G.1b for a graphical analysis of the vega in the case of one bond outstanding.

For $n = 2$

$$s = e^{-\varpi t_2} V_0 \Phi_2\left(\mathbf{d}^{\mathbb{M}}; \mathbf{\Gamma}\right) - e^{-rt_1} F_1 \Phi\big(d_1^{\mathbb{Q}}\big) - e^{-rt_2} F_2 \Phi_2\big(\mathbf{d}_2^{\mathbb{Q}}; \mathbf{\Gamma}\big)$$

with

$$\mathbf{\Gamma} = \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix} \qquad \text{and} \qquad \gamma = \sqrt{\frac{t_1}{t_2}}$$

$$\begin{aligned}\mathbf{d}^{\mathbb{M}} &= \begin{pmatrix} d_1^{\mathbb{M}} & d_2^{\mathbb{M}} \end{pmatrix} = \begin{pmatrix} \frac{\ln \frac{V_0}{\bar{V}_1} + \left(r - \varpi + \frac{\sigma_V^2}{2}\right) t_1}{\sigma_V \sqrt{t_1}} & \frac{\ln \frac{V_0}{F_2} + \left(r - \varpi + \frac{\sigma_V^2}{2}\right) t_2}{\sigma_V \sqrt{t_2}} \end{pmatrix} \\ \mathbf{d}_2^{\mathbb{Q}} &= \begin{pmatrix} d_1^{\mathbb{Q}} & d_2^{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} \frac{\ln \frac{V_0}{\bar{V}_1} + \left(r - \varpi - \frac{\sigma_V^2}{2}\right) t_1}{\sigma_V \sqrt{t_1}} & \frac{\ln \frac{V_0}{F_2} + \left(r - \varpi - \frac{\sigma_V^2}{2}\right) t_2}{\sigma_V \sqrt{t_2}} \end{pmatrix}.\end{aligned}$$

Here the delta is

$$\frac{\partial s}{\partial \sigma_V} = e^{-\varpi t_2} V_0 \frac{\partial \Phi_2\left(\mathbf{d}^{\mathbb{M}}; \mathbf{\Gamma}\right)}{\partial \sigma_V} - e^{-rt_1} F_1 \frac{\partial \Phi\big(d_1^{\mathbb{Q}}\big)}{\partial \sigma_V} - e^{-rt_2} F_2 \frac{\partial \Phi_2\big(\mathbf{d}_2^{\mathbb{Q}}; \mathbf{\Gamma}\big)}{\partial \sigma_V}.$$

In order to effectively compute the vega of the equity for $n = 2$, I need to find an expression for the partial derivative of the bivariate CDF. Furthermore, notice that $\bar{V}_1$ is an implicit function of σ_V. Analytical expression for $\partial \bar{V}_1 / \partial \sigma_V = \bar{V}_1'$ are available in Appendix E. Based on Theorem G.1 (for convenience of notation the dependence on the measure is written as $\{\mathbb{M}, \mathbb{Q}\} = \{+, -\}$), it follows

$$\begin{aligned}\frac{\partial \Phi_2\left(\mathbf{d}^{\pm}; \mathbf{\Gamma}\right)}{\partial \sigma_V} &= \frac{\partial \Phi_2\left(\mathbf{d}^{\pm}; \mathbf{\Gamma}\right)}{\partial d_1^{\pm}} \frac{\partial d_1^{\pm}}{\partial \sigma_V} + \frac{\partial \Phi_2\left(\mathbf{d}^{\pm}; \mathbf{\Gamma}\right)}{\partial d_2^{\pm}} \frac{\partial d_2^{\pm}}{\partial \sigma_V} \\ &= -\frac{1}{\sigma_V}\left[\left(d_1^{\mp} + \frac{\bar{V}_1'}{\bar{V}_1 \sqrt{t_1}}\right) \int_{-\infty}^{d_2^{\pm}} \frac{1}{2\pi\sqrt{1-\gamma^2}} \exp\left(-\frac{1}{2} \frac{x^2 - 2\gamma d_1^{\pm} x + {d_1^{\pm}}^2}{1-\gamma^2}\right) \mathrm{d}x \right. \\ &\quad \left. + d_2^{\mp} \int_{-\infty}^{d_1^{\pm}} \frac{1}{2\pi\sqrt{1-\gamma^2}} \exp\left(-\frac{1}{2} \frac{{d_2^{\pm}}^2 - 2\gamma d_2^{\pm} y + y^2}{1-\gamma^2}\right) \mathrm{d}y \right] \\ &= -\frac{1}{\sigma_V}\left[\left(d_1^{\mp} + \frac{\bar{V}_1'}{\bar{V}_1 \sqrt{t_1}}\right) \frac{\exp\left(-\frac{{d_1^{\pm}}^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{d_2^{\pm}} \frac{1}{\sqrt{2\pi(1-\gamma^2)}} \exp\left(-\frac{1}{2} \frac{\left(x - \gamma d_1^{\pm}\right)^2}{1-\gamma^2}\right) \mathrm{d}x \right. \\ &\quad \left. + d_2^{\mp} \frac{\exp\left(-\frac{{d_2^{\pm}}^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{d_1^{\pm}} \frac{1}{\sqrt{2\pi(1-\gamma^2)}} \exp\left(-\frac{1}{2} \frac{\left(y - \gamma d_2^{\pm}\right)^2}{1-\gamma^2}\right) \mathrm{d}y \right]\end{aligned}$$

$$= -\frac{1}{\sigma_V}\left[\left(d_1^{\mp} + \frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)\phi\left(d_1^{\pm}\right)\Phi\left(\frac{d_2^{\pm} - \gamma d_1^{\pm}}{\sqrt{1-\gamma^2}}\right) + d_2^{\mp}\phi\left(d_2^{\pm}\right)\Phi\left(\frac{d_1^{\pm} - \gamma d_2^{\pm}}{\sqrt{1-\gamma^2}}\right)\right].$$

Using (F.1) and (F.2), and rearranging, it follows

$$\boxed{\begin{aligned}\nu_S^{(2)} = \frac{1}{\sigma_V}\Bigg[& e^{-rt_2}F_2\left(\left(d_1^{\mathbb{M}} + \frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)\phi\big(d_1^{\mathbb{Q}}\big)\Phi\big(\mathfrak{d}_2^{\mathbb{Q}}\big) + d_2^{\mathbb{M}}\phi\big(d_2^{\mathbb{Q}}\big)\Phi\big(\mathfrak{d}_1^{\mathbb{Q}}\big)\right) + e^{-rt_1}F_1\left(d_1^{\mathbb{M}} + \frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)\phi\big(d_1^{\mathbb{Q}}\big)\\ & -e^{-\varpi t_2}V_0\left(\left(d_1^{\mathbb{Q}} + \frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)\phi\left(d_1^{\mathbb{M}}\right)\Phi\left(\mathfrak{d}_2^{\mathbb{M}}\right) + d_2^{\mathbb{Q}}\phi\left(d_2^{\mathbb{M}}\right)\Phi\left(\mathfrak{d}_1^{\mathbb{M}}\right)\right)\Bigg]\end{aligned}}$$

See Figures G.1c and G.1d for a graphical analysis of the delta in the case of two bonds outstanding.

Finally, if $n = 3$

$$s = e^{-\varpi t_3}V_0\Phi_3\left(\mathbf{d}^{\mathbb{M}};\mathbf{\Gamma_3}\right) - e^{-rt_1}F_1\Phi\big(d_1^{\mathbb{Q}}\big) - e^{-rt_2}F_2\Phi_2\big(\mathbf{d}_2^{\mathbb{Q}};\mathbf{\Gamma_2}\big) - e^{-rt_3}F_3\Phi_3\big(\mathbf{d}_3^{\mathbb{Q}};\mathbf{\Gamma_3}\big)$$

with

$$\mathbf{\Gamma_3} = \begin{pmatrix} 1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & 1 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & 1 \end{pmatrix}, \qquad \mathbf{\Gamma_2} = \begin{pmatrix} 1 & \gamma_{12} \\ \gamma_{12} & 1 \end{pmatrix} \qquad \text{and} \qquad \gamma_{ij} = \sqrt{\frac{t_i}{t_j}},\ \text{with } i \le j$$

$$\mathbf{d}^{\mathbb{M}} = \begin{pmatrix} d_1^{\mathbb{M}} & d_2^{\mathbb{M}} & d_3^{\mathbb{M}} \end{pmatrix} = \begin{pmatrix} \frac{\ln\frac{V_0}{V_1} + \left(r-\varpi+\frac{\sigma_V^2}{2}\right)t_1}{\sigma_V\sqrt{t_1}} & \frac{\ln\frac{V_0}{V_2} + \left(r-\varpi+\frac{\sigma_V^2}{2}\right)t_2}{\sigma_V\sqrt{t_2}} & \frac{\ln\frac{V_0}{F_3} + \left(r-\varpi+\frac{\sigma_V^2}{2}\right)t_3}{\sigma_V\sqrt{t_3}} \end{pmatrix}$$

and

$$\mathbf{d}_3^{\mathbb{Q}} = \begin{pmatrix} \mathbf{d}_2^{\mathbb{Q}} & d_3^{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} d_1^{\mathbb{Q}} & d_2^{\mathbb{Q}} & d_3^{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} \frac{\ln\frac{V_0}{V_1} + \left(r-\varpi-\frac{\sigma_V^2}{2}\right)t_1}{\sigma_V\sqrt{t_1}} & \frac{\ln\frac{V_0}{V_2} + \left(r-\varpi-\frac{\sigma_V^2}{2}\right)t_2}{\sigma_V\sqrt{t_2}} & \frac{\ln\frac{V_0}{F_3} + \left(r-\varpi-\frac{\sigma_V^2}{2}\right)t_3}{\sigma_V\sqrt{t_3}} \end{pmatrix}.$$

Here the vega is equal to

$$\frac{\partial s}{\partial \sigma_V} = e^{-\varpi t_3}V_0\frac{\partial\Phi_3\left(\mathbf{d}^{\mathbb{M}};\mathbf{\Gamma_3}\right)}{\partial\sigma_V} - e^{-rt_1}F_1\frac{\partial\Phi\big(d_1^{\mathbb{Q}}\big)}{\partial\sigma_V} - e^{-rt_2}F_2\frac{\partial\Phi_2\big(\mathbf{d}_2^{\mathbb{Q}};\mathbf{\Gamma_2}\big)}{\partial\sigma_V} - e^{-rt_3}F_3\frac{\partial\Phi_3\big(\mathbf{d}_3^{\mathbb{Q}};\mathbf{\Gamma_3}\big)}{\partial\sigma_V}.$$

Again, to compute the delta of the equity for $n = 3$, I need to find an expression for the partial derivative of the trivariate CDF. Using Theorem G.1 (for convenience of notation the dependence on the measure is written as $\{\mathbb{M},\mathbb{Q}\} = \{+,-\}$),it follows

$$\begin{aligned}\frac{\partial\Phi_3\left(\mathbf{d}^{\pm};\mathbf{\Gamma}\right)}{\partial\sigma_V} &= \frac{\partial\Phi_3\left(\mathbf{d}^{\pm};\mathbf{\Gamma}\right)}{\partial d_1^{\pm}}\frac{\partial d_1^{\pm}}{\partial\sigma_V} + \frac{\partial\Phi_3\left(\mathbf{d}^{\pm};\mathbf{\Gamma}\right)}{\partial d_2^{\pm}}\frac{\partial d_2^{\pm}}{\partial\sigma_V} + \frac{\partial\Phi_3\left(\mathbf{d}^{\pm};\mathbf{\Gamma}\right)}{\partial d_3^{\pm}}\frac{\partial d_3^{\pm}}{\partial\sigma_V}\\ &= -\frac{1}{\sigma_V}\Bigg[\left(d_1^{\mp} + \frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)\int_{-\infty}^{d_2^{\pm}}\int_{-\infty}^{d_3^{\pm}}\frac{1}{\sqrt{(2\pi)^3\det\mathbf{\Gamma}}}\exp\left(-\frac{\tau_1x^2 + \tau_4y^2 + \tau_9{d_1^{\pm}}^2 + 2\tau_2xy + 2\tau_6d_1^{\pm}y}{2}\right)\mathrm{d}x\,\mathrm{d}y\\ &\quad + \left(d_2^{\mp} + \frac{\bar{V}_2'}{\bar{V}_2\sqrt{t_2}}\right)\int_{-\infty}^{d_1^{\pm}}\int_{-\infty}^{d_3^{\pm}}\frac{1}{\sqrt{(2\pi)^3\det\mathbf{\Gamma}}}\exp\left(-\frac{\tau_1x^2 + \tau_4{d_2^{\pm}}^2 + \tau_9z^2 + 2\tau_2d_2^{\pm}x + 2\tau_6d_2^{\pm}z}{2}\right)\mathrm{d}x\,\mathrm{d}z\\ &\quad + d_3^{\mp}\int_{-\infty}^{d_1^{\pm}}\int_{-\infty}^{d_2^{\pm}}\frac{1}{\sqrt{(2\pi)^3\det\mathbf{\Gamma}}}\exp\left(-\frac{\tau_1{d_3^{\pm}}^2 + \tau_4y^2 + \tau_9z^2 + 2\tau_2d_3^{\pm}y + 2\tau_6yz}{2}\right)\mathrm{d}y\,\mathrm{d}z\Bigg]\\ &= -\frac{1}{\sigma_V}\left[\left(d_1^{\mp} + \frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)I_1 + \left(d_2^{\mp} + \frac{\bar{V}_2'}{\bar{V}_2\sqrt{t_2}}\right)I_2 + d_3^{\mp}I_3\right]\end{aligned}$$

where $\det \mathbf{\Gamma} = \frac{(t_2-t_1)(t_3-t_2)}{t_2 t_3}$ and

$$\mathbf{\Gamma}^{-1} = \begin{pmatrix} \frac{t_2}{t_2-t_1} & -\frac{\sqrt{t_1 t_2}}{t_2-t_1} & 0 \\ -\frac{\sqrt{t_1 t_2}}{t_2-t_1} & \frac{t_2(t_3-t_1)}{(t_2-t_1)(t_3-t_2)} & -\frac{\sqrt{t_2 t_3}}{t_3-t_2} \\ 0 & -\frac{\sqrt{t_2 t_3}}{t_3-t_2} & \frac{t_3}{t_3-t_2} \end{pmatrix} = \begin{pmatrix} \tau_1 & \tau_2 & 0 \\ \tau_2 & \tau_4 & \tau_6 \\ 0 & \tau_6 & \tau_9 \end{pmatrix}.$$

All the double integrals can computed in the same fashion described in Appendix F.

Solution of I_1

$$I_1 = \sqrt{\frac{\det \mathbf{\Sigma_1}}{\det \mathbf{\Gamma}}}\phi\left(\sqrt{a_1}d_1^{\pm}\right) N_2\left(d_2^{\pm}, d_3^{\pm}; \boldsymbol{\mu}_{\mathbf{1}}^{\pm}, \mathbf{\Sigma_1}\right)$$

with

$$\mathbf{\Sigma_1} = \frac{1}{\tau_1\tau_4 - \tau_2^2}\begin{pmatrix} \tau_4 & -\tau_2 \\ -\tau_2 & \tau_1 \end{pmatrix}, \quad \boldsymbol{\mu}_{\mathbf{1}}^{\pm} = -\det \mathbf{\Sigma_1}\begin{pmatrix} -\tau_2\tau_6 \\ \tau_1\tau_6 \end{pmatrix} d_1^{\pm} \quad \text{and} \quad a_1 = \tau_9 - \det \mathbf{\Sigma_1}\tau_1\tau_6^2$$

Solution of I_2

$$I_2 = \sqrt{\frac{\det \mathbf{\Sigma_2}}{\det \mathbf{\Gamma}}}\phi\left(\sqrt{a_2}d_2^{\pm}\right) N_2\left(d_1^{\pm}, d_3^{\pm}; \boldsymbol{\mu}_{\mathbf{2}}^{\pm}, \mathbf{\Sigma_2}\right) = \sqrt{\frac{\det \mathbf{\Sigma_2}}{\det \mathbf{\Gamma}}}\phi\left(\sqrt{a_2}d_2^{\pm}\right)\Phi\left(\frac{\tau_1 d_1^{\pm} + \tau_2 d_2^{\pm}}{\sqrt{\tau_1}}\right)\Phi\left(\frac{\tau_9 d_3^{\pm} + \tau_6 d_2^{\pm}}{\sqrt{\tau_9}}\right).$$

with

$$\mathbf{\Sigma_2} = \frac{1}{\tau_1\tau_9}\begin{pmatrix} \tau_9 & 0 \\ 0 & \tau_1 \end{pmatrix}, \quad \boldsymbol{\mu}_{\mathbf{2}}^{\pm} = -\det \mathbf{\Sigma_2}\begin{pmatrix} \tau_2\tau_9 \\ \tau_1\tau_6 \end{pmatrix} d_2^{\pm} \quad \text{and} \quad a_2 = \tau_4 - \det \mathbf{\Sigma_2}\left(\tau_2^2\tau_9 + \tau_1\tau_6^2\right).$$

Solution of I_3

$$I_1 = \sqrt{\frac{\det \mathbf{\Sigma_3}}{\det \mathbf{\Gamma}}}\phi\left(\sqrt{a_3}d_3^{\pm}\right) N_2\left(d_1^{\pm}, d_2^{\pm}; \boldsymbol{\mu}_{\mathbf{3}}^{\pm}, \mathbf{\Sigma_3}\right)$$

with

$$\mathbf{\Sigma_3} = \frac{1}{\tau_4\tau_9 - \tau_6^2}\begin{pmatrix} \tau_9 & -\tau_6 \\ -\tau_6 & \tau_4 \end{pmatrix}, \quad \boldsymbol{\mu}_{\mathbf{3}}^{\pm} = -\det \mathbf{\Sigma_3}\begin{pmatrix} \tau_2\tau_9 \\ -\tau_2\tau_6 \end{pmatrix} d_1^{\pm} \quad \text{and} \quad a_3 = \tau_1 - \det \mathbf{\Sigma_3}\tau_2^2\tau_9.$$

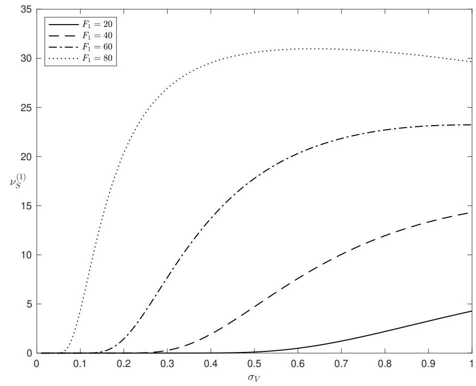
Hence, the vega of the equity in the case $n = 3$ is

$$\begin{aligned}\frac{\partial s}{\partial \sigma_V} = \frac{1}{\sigma_V}\Bigg[&e^{-rt_3}F_3\left(\left(d_1^{\mathbb{M}} + \frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)I_1^{\mathbb{Q}} + \left(d_2^{\mathbb{M}} + \frac{\bar{V}_2'}{\bar{V}_2\sqrt{t_2}}\right)I_2^{\mathbb{Q}} + d_3^{\mathbb{M}}I_3^{\mathbb{Q}}\right) \\ &- e^{-\varpi t_3}V_0\left(\left(d_1^{\mathbb{Q}} + \frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)I_1^{\mathbb{M}} + \left(d_2^{\mathbb{Q}} + \frac{\bar{V}_2'}{\bar{V}_2\sqrt{t_2}}\right)I_2^{\mathbb{M}} + d_3^{\mathbb{Q}}I_3^{\mathbb{M}}\right) \\ &+ e^{-rt_2}F_2\left(\left(d_1^{\mathbb{M}} + \frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)\phi(d_1^{\mathbb{Q}})\Phi(\mathfrak{d}_2^{\mathbb{Q}}) + \left(d_2^{\mathbb{M}} + \frac{\bar{V}_2'}{\bar{V}_2\sqrt{t_2}}\right)\phi(d_2^{\mathbb{Q}})\Phi(\mathfrak{d}_1^{\mathbb{Q}})\right) \\ &+e^{-rt_1}F_1\left(d_1^{\mathbb{M}} + \frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)\phi(d_1^{\mathbb{Q}})\Bigg]\end{aligned}$$

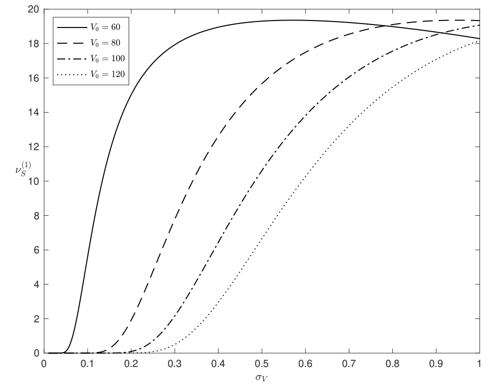
Writing the three integrals explicitly, it follows

$$\boxed{\begin{aligned}\nu_S^{(3)} = \frac{1}{\sigma_V}\Bigg[& e^{-rt_3}F_3\frac{1}{\sqrt{\det\boldsymbol{\Gamma_3}}}\sum_{i=1}^{3}\left(d_i^{\mathbb{M}}+\frac{\bar{V}_i'}{\bar{V}_i\sqrt{t_i}}\right)\sqrt{\det\boldsymbol{\Sigma_i}}\phi\big(\sqrt{a_i}d_i^{\mathbb{Q}}\big)N_2\left(\mathbf{d}_3^{\mathbb{Q}}\setminus d_i^{\mathbb{Q}};\boldsymbol{\mu_i}^{\mathbb{Q}},\boldsymbol{\Sigma_i}\right)\\ &- e^{-\varpi t_3}V_0\frac{1}{\sqrt{\det\boldsymbol{\Gamma_3}}}\sum_{i=1}^{3}\left(d_i^{\mathbb{Q}}+\frac{\bar{V}_i'}{\bar{V}_i\sqrt{t_i}}\right)\sqrt{\det\boldsymbol{\Sigma_i}}\phi\left(\sqrt{a_i}d_i^{\mathbb{M}}\right)N_2\left(\mathbf{d}^{\mathbb{M}}\setminus d_i^{\mathbb{M}};\boldsymbol{\mu_i}^{\mathbb{M}},\boldsymbol{\Sigma_i}\right)\\ &+e^{-rt_2}F_2\left(\left(d_1^{\mathbb{M}}+\frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)\phi\big(d_1^{\mathbb{Q}}\big)\Phi\big(\mathfrak{d}_2^{\mathbb{Q}}\big)+\left(d_2^{\mathbb{M}}+\frac{\bar{V}_2'}{\bar{V}_2\sqrt{t_2}}\right)\phi\big(d_2^{\mathbb{Q}}\big)\Phi\big(\mathfrak{d}_1^{\mathbb{Q}}\big)\right)\\ &+e^{-rt_1}F_1\left(d_1^{\mathbb{M}}+\frac{\bar{V}_1'}{\bar{V}_1\sqrt{t_1}}\right)\phi\big(d_1^{\mathbb{Q}}\big)\Bigg],\end{aligned}}$$

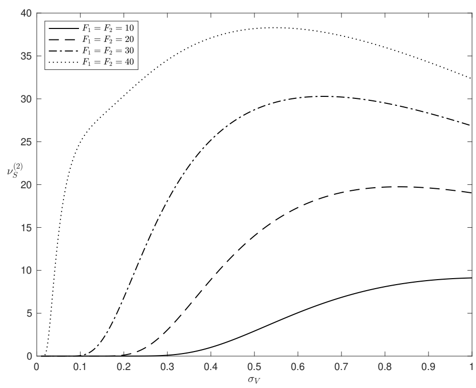
where $\mathbf{d}\setminus d_i$ must be intended as the vector obtained from $\mathbf{d}$ by removing the element d_i (and keeping the order of the other elements unchanged) and $\bar{V}_3' = 0$ (by construction). See Figures G.1e and G.1f for a graphical analysis of the delta in the case of three bonds outstanding.



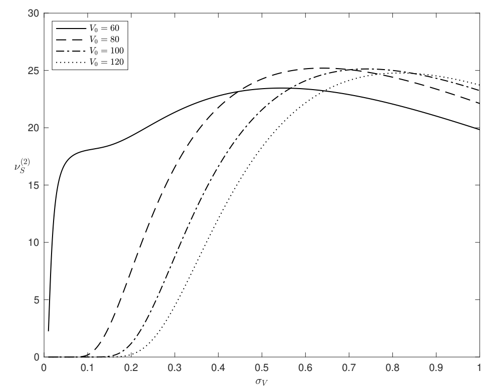
(a) $V_0 = 100, t = 1$



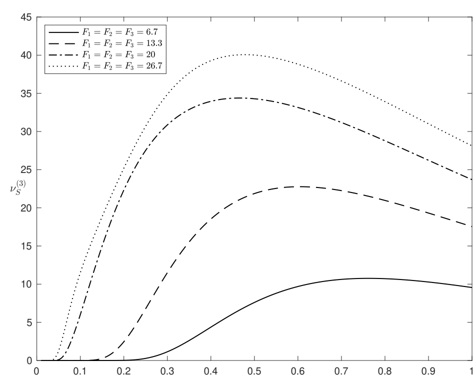
(b) $F = 50, t = 1$



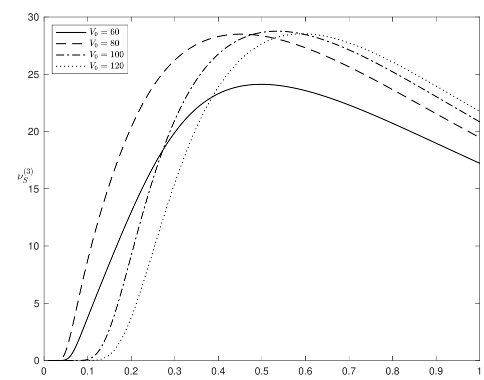
(c) $V_0 = 100, t_1 = 1, t_2 = 5$



(d) $F_1 = F_2 = 25, t_1 = 1, t_2 = 5$



(e) $V_0 = 100, t_1 = 1, t_2 = 5, t_3 = 10$



(f) $F_1 = F_2 = F_3 = 50/3, t_1 = 1, t_2 = 5, t_3 = 10$

Figure G.1: Sensitivity of equity with respect to asset volatility under different aggregation schemes for debt (left) and leverage (right). $r = 0.03$, $\varpi = 0.05$ throughout.

Appendix H Robustness checks for different values of LGD

Regressand						Adj–R^2:	0.9433
$\overline{AICR}'_1$							
Regressors	Coefficient	Robust Standard Error	t–stat	p–value			
$\overline{AICR}_1$	0.7748793	0.0052173	148.52	0.000	***		
α_1	0.0007165	0.0000297	24.13	0.000	***		

(a): LGD = 60%

Regressand						Adj–R^2:	0.8788
$\overline{AICR}'_{-1}$							
Regressors	Coefficient	Robust Standard Error	t–stat	p–value			
$\overline{AICR}_{-1}$	0.9196883	0.0242480	37.93	0.000	***		
α_{-1}	0.0001626	0.0000705	2.31	0.021	**		

(b): LGD = 60%

Regressand						Adj–R^2:	0.9484
$\overline{AICR}'_1$							
Regressors	Coefficient	Robust Standard Error	t–stat	p–value			
$\overline{AICR}_1$	0.7647910	0.0056719	134.84	0.000	***		
α_1	0.0008249	0.0000275	29.99	0.000	***		

(c): LGD = 80%

Regressand						Adj–R^2:	0.5552
$\overline{AICR}'_{-1}$							
Regressors	Coefficient	Robust Standard Error	t–stat	p–value			
$\overline{AICR}_{-1}$	0.6130806	0.0379853	16.14	0.000	***		
α_{-1}	0.0005543	0.0001077	5.15	0.000	***		

(d): LGD = 80%

Table H.1: Estimation of regression (16) for different values of LGD.
(a): Estimates of the pooled panel regression of average $AICR$ obtained from call options ($\xi = 1$) and CDS regressed onto average $AICR$ obtained from call options only. Number of observations: 15,470. F–stat: 22,058.80 (p–value: 0.0000).
(b): Estimates of the pooled panel regression of average $AICR$ obtained from put options ($\xi = -1$) and CDS regressed onto average $AICR$ obtained from put options only. Number of observations: 15,027. F–stat: 1,438.57 (p–value: 0.0000).
(c): Estimates of the pooled panel regression of average $AICR$ obtained from call options ($\xi = 1$) and CDS regressed onto average $AICR$ obtained from call options only. Number of observations: 15,470. F–stat: 18,181.34 (p–value: 0.0000).
(d): Estimates of the pooled panel regression of average $AICR$ obtained from put options ($\xi = -1$) and CDS regressed onto average $AICR$ obtained from put options only. Number of observations: 15,027. F–stat: 260.50 (p–value: 0.0000).
A sandwich estimator for panel data is used to obtain robust standard errors. Significance levels: 10% (*), 5% (**), 1% (***).

Regressand: $\overline{AICR}'_1$ Adj–R^2: 0.0590

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0075571	0.0032449	2.33	0.102	
α_1	-0.001002	0.0019052	-0.53	0.635	
Industry-FE	✓				
Year-FE	✓				

(a): LGD = 60%

Regressand: $\overline{AICR}'_{-1}$ Adj–R^2: 0.5676

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0365895	0.0057882	6.32	0.008	***
α_{-1}	-0.0112664	0.0051351	-2.19	0.116	
Industry-FE	✓		-	0.379	
Year-FE	✓	-	-	-	-

(b): LGD = 60%

Regressand: $\overline{AICR}'_1$ Adj–R^2: 0.0622

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0076758	0.002566	2.99	0.058	*
α_1	-0.0011136	0.0013743	-0.81	0.477	
Industry-FE	✓				
Year-FE	✓				

(c): LGD = 80%

Regressand: $\overline{AICR}'_{-1}$ Adj–R^2: 0.5980

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0303678	0.0093729	3.24	0.048	**
α_{-1}	-0.0086424	0.0082499	-1.05	0.372	
Industry-FE	✓				
Year-FE	✓				

(d): LGD = 80%

Table H.2: Estimation of regression (17) for different values of LGD.
(a): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto average $AICR$ calculated over call options and CDSs, with LGD = 60%. Number of observations: 15,470.
(b): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs, with LGD = 60%. Number of observations: 15,027.
(c): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto average $AICR$ calculated over call options and CDSs, with LGD = 80%. Number of observations: 15,470.
(d): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs, with LGD = 80%. Number of observations: 15,027.
Standard errors are adjusted for four clusters based on industry. Significance levels: 10% (*), 5% (**), 1% (***).

Regressand					Adj–R^2: 0.3653
η_1					
Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0051749	0.0025327	2.04	0.134	
α_1	-0.0012296	0.0020341	-0.60	0.588	
Industry-FE	✓				
Year-FE	✓	-			

(a): LGD = 60%

Regressand					Adj–R^2: 0.0923
η_{-1}					
Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0054215	0.0026694	2.03	0.135	
α_{-1}	-0.0028967	0.0023997	-1.21	0.314	
Industry-FE	✓				
Year-FE	✓				

(b): LGD = 60%

Regressand					Adj–R^2: 0.4673
η_1					
Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0054288	0.001427	3.80	0.032	**
α_1	-0.0014958	0.0010836	-1.38	0.261	
Industry-FE	✓				
Year-FE	✓	-			

(c): LGD = 80%

Regressand					Adj–R^2: 0.1645
η_{-1}					
Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0105748	0.0075426	1.40	0.255	
α_{-1}	-0.0040323	0.0067281	-0.60	0.591	
Industry-FE	✓				
Year-FE	✓				

(d): LGD = 80%

Table H.3: Estimation of regression (18) for different values of LGD.
(a): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto the residuals obtain from regression (16) (calls), for LGD = 60%. Number of observations: 15,470.
(b): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto the residuals obtain from regression (16) (puts), for LGD = 60%. Number of observations: 15,027.
(c): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto the residuals obtain from regression (16) (calls), for LGD = 80%. Number of observations: 15,470.
(d): Estimates of the fixed-effects panel regression of market model-implied leverage LEV onto the residuals obtain from regression (16) (puts), for LGD = 80%. Number of observations: 15,027.
Standard errors are adjusted for four clusters based on industry. Significance levels: 10% (*), 5% (**), 1% (***).

Regressand				Adj–R^2:	0.6608
$\overline{AICR}'_{-1}$					
Regressors	**Coefficient**	**Robust Standard Error**	**t–stat**	**p–value**	
LEV	0.0391051	0.0012387	31.57	0.000	***
α_{-1}	-0.0145449	0.0011321	-12.85	0.000	***
Year-FE	✓				

(a): Financials, LGD = 60%

Regressand				Adj–R^2:	0.2128
$\overline{AICR}'_{-1}$					
Regressors	**Coefficient**	**Robust Standard Error**	**t–stat**	**p–value**	
LEV	0.0069186	0.0004261	16.24	0.000	***
α_{-1}	-0.0007640	0.0001489	-5.13	0.000	***
Year-FE	✓				

(b): Mining, Energy and Utilities, LGD = 60%

Regressand				Adj–R^2:	0.1173
$\overline{AICR}'_{-1}$					
Regressors	**Coefficient**	**Robust Standard Error**	**t–stat**	**p–value**	
LEV	0.0014455	0.0000861	16.78	0.000	***
α_{-1}	-0.0000211	0.0000163	-1.29	0.197	
Year-FE	✓				

(c): Manufacturing, LGD = 60%

Regressand				Adj–R^2:	0.2450
$\overline{AICR}'_{-1}$					
Regressors	**Coefficient**	**Robust Standard Error**	**t–stat**	**p–value**	
LEV	0.0075047	0.0009004	8.34	0.000	***
α_{-1}	-0.0007342	0.0001427	-5.14	0.000	***
Year-FE	✓				

(d): Retail, Wholesale and Services, LGD = 60%

Table H.4: Estimation of regression (17) over the four sub-samples for LGD = 60%.
(a): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Financials. Number of observations: 1,938. F–stat = 199.90 (p–value = 0.000).
(b): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Mining, Energy and Utilities. Number of observations: 1,916. F–stat = 73.60 (p–value = 0.000).
(c): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Manufacturing. Number of observations: 6,515. F–stat = 80.21 (p–value = 0.000).
(d): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Retail, Wholesale and Services. Number of observations: 4,658. F–stat = 20.63 (p–value = 0.000).
A sandwich estimator for panel data is used in order to obtain robust standard errors. Significance levels: 10% (*), 5% (**), 1% (***).

Regressand				Adj–R^2:	0.6608
$\overline{AICR}'_{-1}$					
Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0391051	0.0012387	31.57	0.000	***
α_{-1}	-0.0145449	0.0011321	-12.85	0.000	***
Year-FE	✓				

(a): Financials, LGD = 80%

Regressand				Adj–R^2:	0.2128
$\overline{AICR}'_{-1}$					
Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0069186	0.0004261	16.24	0.000	***
α_{-1}	-0.0007640	0.0001489	-5.13	0.000	***
Year-FE	✓				

(b): Mining, Energy and Utilities, LGD = 80%

Regressand				Adj–R^2:	0.1173
$\overline{AICR}'_{-1}$					
Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0014455	0.0000861	16.78	0.000	***
α_{-1}	-0.0000211	0.0000163	-1.29	0.197	
Year-FE	✓				

(c): Manufacturing, LGD = 80%

Regressand				Adj–R^2:	0.2450
$\overline{AICR}'_{-1}$					
Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
LEV	0.0075047	0.0009004	8.34	0.000	***
α_{-1}	-0.0007342	0.0001427	-5.14	0.000	***
Year-FE	✓				

(d): Retail, Wholesale and Services, LGD = 80%

Table H.5: Estimation of regression (17) over the four sub-samples for LGD = 80%.
(a): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Financials. Number of observations: 1,938. F–stat = 199.90 (p–value = 0.000).
(b): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Mining, Energy and Utilities. Number of observations: 1,916. F–stat = 73.60 (p–value = 0.000).
(c): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Manufacturing. Number of observations: 6,515. F–stat = 80.21 (p–value = 0.000).
(d): Estimates of the year-fixed effect panel regression of market model-implied leverage LEV onto average $AICR$ calculated over put options and CDSs of Retail, Wholesale and Services. Number of observations: 4,658. F–stat = 20.63 (p–value = 0.000).
A sandwich estimator for panel data is used in order to obtain robust standard errors. Significance levels: 10% (*), 5% (**), 1% (***).

Regressand					Adj–R^2:	0.0005
$\Delta\text{Skew}_{T<1}$						
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
$\overline{AICR}'_{-1,T<1}$	1.479315	0.9954766	1.49	0.142		
$\alpha_{T<1}$	-0.0038017	0.0009416	-4.04	0.000	***	
firm-FE	✓					
year-FE	✓					

(a): Predictive regression for short-term skew, LGD = 60%

Regressand					Adj–R^2:	0.0004
$\Delta\text{Skew}_{T>1}$						
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
$\overline{AICR}'_{-1,T>1}$	0.3945692	0.1372269	2.88	0.005	***	
$\alpha_{T>1}$	-0.0106972	0.0017378	-6.16	0.000	***	
firm-FE	✓					
year-FE	✓					

(b): Predictive regression for long-term skew, LGD = 60%

Regressand					Adj–R^2:	0.0007
$\Delta\text{Skew}_{T<1}$						
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
$\overline{AICR}'_{-1,T<1}$	1.567852	1.051431	1.49	0.141		
$\alpha_{T<1}$	-0.0038733	0.000953	-4.06	0.000	***	
firm-FE	✓					
year-FE	✓					

(c): Predictive regression for short-term skew, LGD = 80%

Regressand					Adj–R^2:	0.0005
$\Delta\text{Skew}_{T>1}$						
Regressors	Coefficient	Robust Standard Error	t–stat	p–value		
$\overline{AICR}'_{-1,T>1}$	0.4403912	0.1533718	2.87	0.006	***	
$\alpha_{T>1}$	-0.011145	0.0018258	-6.10	0.000	***	
firm-FE	✓					
year-FE	✓					

(d): Predictive regression for long-term skew, LGD = 80%

Table H.6: Estimation of regression (20) for different values of LGD.
(a): Predictive regregression for short-term skew based on the average $AICR$ calculated over short-term put options and CDSs (LGD = 60%). Number of observations: 7,656. F–stat = 8.72 (p–value = 0.000).
(b): Predictive regregression for long-term skew based on the average $AICR$ calculated over long-term put options and CDSs (LGD = 60%). Number of observations: 6,818. F–stat = 11.47 (p–value = 0.000).
(c): Predictive regregression for long-term skew based on the average $AICR$ calculated over short-term put options and CDSs (LGD = 80%). Number of observations: 7,656. F–stat = 8.73 (p–value = 0.000).
(d): Predictive regregression for long-term skew based on the average $AICR$ calculated over long-term put options and CDSs (LGD = 80%). Number of observations: 6,818 F–stat = 11.54 (p–value = 0.000).
A sandwich estimator for panel data is used to obtain robust standard errors. Significance levels: 10% (*), 5% (**), 1% (***).

Regressand					
$\Delta\text{Skew}_{T>1}$			Adj–R^2:	0.0018	

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
$\overline{AICR}'_{-1,T>1}$	0.3746640	0.1414883	2.65	0.029	**
$\alpha_{T>1}$	-0.0116986	0.0109631	-1.07	0.317	
firm-FE	✓				
year-FE	✓				

(a): Long-term skew of Financials, LGD = 60%

Regressand					
$\Delta\text{Skew}_{T>1}$			Adj–R^2:	0.0014	

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
$\overline{AICR}'_{-1,T>1}$	0.5889985	0.6279188	0.94	0.379	
$\alpha_{T>1}$	-0.0164705	0.0029668	-5.55	0.001	***
firm-FE	✓				
year-FE	✓				

(b): Long-term skew of Mining, Energy and Utilities, LGD = 60%

Regressand					
$\Delta\text{Skew}_{T>1}$			Adj–R^2:	0.0007	

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
$\overline{AICR}'_{-1,T>1}$	-4.338104	3.015046	-1.44	0.166	
$\alpha_{T>1}$	-0.0073019	0.0022992	-3.18	0.005	***
firm-FE	✓				
year-FE	✓				

(c): Long-term skew of Manufacturing, LGD = 60%

Regressand					
$\Delta\text{Skew}_{T>1}$			Adj–R^2:	0.0006	

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
$\overline{ICR}'_{-1,T>1}$	1.2258928	0.8286194	1.48	0.151	
$\alpha_{T>1}$	-0.0107916	0.0027597	-3.91	0.001	***
firm-FE	✓				
year-FE	✓				

(d):Long-term skew of Retail, Wholesale and Services, LGD = 60%

Table H.7: Estimation of regression (20) over the four sub-samples for LGD = 60%.
(a): Predictive regregression for short-term skew based on the average $AICR$ calculated over long-term put options and CDSs of Financials. Number of observations: 810. F–stat = 10.99 (p–value = 0.002).
(b): Predictive regregression for short-term skew based on the average $AICR$ calculated over long-term put options and CDSs of Mining, Energy and Utilities. Number of observations: 791. F–stat = 24.87 (p–value = 0.000).
(c): Table 5b: Predictive regregression for short-term skew based on the average $AICR$ calculated over long-term put options and CDSs of Manufacturing. Number of observations: 2,305 F–stat = 4.17 (p–value = 0.010).
(d): Predictive regregression for short-term skew based on the average $AICR$ calculated over long-term put options and CDSs of Retail, Wholesale and Services. Number of observations: 2,912 F–stat = 5.78 (p–value = 0.001).
A sandwich estimator for panel data is used to obtain robust standard errors. Significance levels: 10% (*), 5% (**), 1% (***).

Regressand					Adj–R^2:	0.0022
$\Delta\text{Skew}_{T>1}$						

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
$\overline{AICR}'_{-1,T>1}$	0.4297893	0.1668369	2.58	0.033	**
$\alpha_{T>1}$	-0.0145834	0.0121469	-1.20	0.264	
firm-FE	✓				
year-FE	✓				

(a): Predictive regression for long-term skew of Financials, LGD = 80%

Regressand					Adj–R^2:	0.0017
$\Delta\text{Skew}_{T>1}$						

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
$\overline{AICR}'_{-1,T>1}$	0.2832847	0.0658556	4.30	0.004	***
$\alpha_{T>1}$	-0.0194797	0.002694	-7.23	0.000	***
firm-FE	✓				
year-FE	✓				

(b): Long-term skew of Mining, Energy and Utilities, LGD = 80%

Regressand					Adj–R^2:	0.0006
$\Delta\text{Skew}_{T>1}$						

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
$\overline{ICR}'_{-1,T>1}$	-2.560192	2.574604	-0.99	0.333	
$\alpha_{T>1}$	-0.0078215	0.0022739	-3.44	0.003	***
firm-FE	✓				
year-FE	✓				

(c): Long-term skew of Manufacturing, LGD = 80%

Regressand					Adj–R^2:	0.0006
$\Delta\text{Skew}_{T>1}$						

Regressors	Coefficient	Robust Standard Error	t–stat	p–value	
$\overline{ICR}'_{-1,T>1}$	1.310575	0.6199556	2.11	0.044	**
$\alpha_{T>1}$	-0.0110623	0.0027377	-4.04	0.000	***
firm-FE	✓				
year-FE	✓				

(d): Long-term skew of Retail, Wholesale and Services, LGD = 80%

Table H.8: Estimation of regression (20) over the four sub-samples for LGD =80%.
(a): Predictive regregression for short-term skew based on the average $AICR$ calculated over long-term put options and CDSs of Financials. Number of observations: 810. F–stat = 7.63 (p–value = 0.007).
(b): Predictive regregression for short-term skew based on the average $AICR$ calculated over long-term put options and CDSs of Mining, Energy and Utilities. Number of observations: 791. F–stat = 32.75 (p–value = 0.000).
(c): Table 5b: Predictive regregression for short-term skew based on the average $AICR$ calculated over long-term put options and CDSs of Manufacturing. Number of observations: 2,305 F–stat = 4.04 (p–value = 0.011).
(d): Predictive regregression for short-term skew based on the average $AICR$ calculated over long-term put options and CDSs of Retail, Wholesale and Services. Number of observations: 2,912 F–stat = 6.77 (p–value = 0.000).
A sandwich estimator for panel data is used to obtain robust standard errors. Significance levels: 10% (*), 5% (**), 1% (***).