The Global Factor Structure of Exchange Rates *

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Abstract

We provide a model-free framework to study the global factor structure of exchange rates. To this end, we propose a new methodology to estimate model-free global stochastic discount factors (SDFs) pricing large cross-sections of international assets, such as stocks, bonds, and currencies, independently of the currency denomination and in the presence of trading frictions. We derive a unique mapping between the optimal portfolios of global investors trading in international markets with frictions and international SDFs, which allows us to recover such SDFs from asset return data alone. Trading frictions shrink portfolio weights of some assets to zero, leading to endogenously segmented markets and robust properties of international SDFs. From the cross-section of numéraire invariant SDFs, we extract one local risk factor (*currency basket*) and one global risk factor (*global SDF*). We show that the global SDF factor alone provides an excellent in- and out-of-sample fit for the cross-section of international asset returns across all denominations, significantly improving upon the performance of benchmark factor models. Finally, we estimate the cost to obtain the portfolio home bias observed in the data and find it to be small.

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Canonical models in international finance, which match salient features of international asset returns, posit that stochastic discount factors (SDFs) are driven by both global and country-specific shocks.[1] The consensus view is that two global factors, dollar and carry, account for a significant fraction of the systematic variation of exchange rates, see, e.g., Verdelhan (2018). One common feature of this literature is that the nature of global risks is imposed ex ante into the construction of international SDFs, with global risks that enter all SDFs of all investors in all countries. In this paper, we take an agnostic view and develop a new model-free framework to study the nature and source of global exchange rate factors which emerge endogenously in a setting where investors face trading frictions. Moreover unlike earlier literature that identifies global risk factors from the cross-section of currencies themselves, we extract global factors from international asset returns such as stocks and bonds.

One common assumption underlying the literature studying global factor structures of exchange rates is that markets are complete and agents can trade without frictions in global international markets. Under this assumption, the rate of appreciation of the exchange rate (X) is uniquely recovered from the ratio of the foreign (M_f) and the domestic SDF (M_d): $X = M_f/M_d$. This identity is called the asset market view of exchange rates (AMV) and implies that exchange rate shocks always reflect corresponding foreign or domestic SDF shocks. Validity of the AMV is convenient for studying global factor structures of exchange rates because of at least two reasons. First, it is well known that in order to address the volatility puzzle in exchange rates, international SDFs need to be almost perfectly correlated, which gives rise to a strong and parsimonious factor structure.[2] Second, if one assumes that the AMV holds, a factor that explains a significant part of the SDF variation in one currency is *by construction*, a global exchange rate factor that explains a significant part of the SDF variation in all other currencies. By triangular arbitrage, such SDF factors will explain bilateral exchange rate movements with respect to *any* currency numéraire.

There are various important aspects to consider when relying on the AMV to understand global factor structures of exchange rates. First, as we deviate from the complete market assumption and allow markets to be incomplete, most domestic and foreign SDFs do not satisfy the AMV, see, e.g., Backus, Foresi, and Telmer (2001). Second, SDF properties, and hence, the factor structure of exchange rates depend crucially on the underlying assumptions about the menu of assets available to investors for trade. For example, as we increase the degree of segmentation across countries, SDFs become less correlated, and hence the factor structure weaker. In this case, the importance of global factors decreases whereas local factors may matter more. These issues, however, make the ensuing global factor structures dependent on ad hoc assumptions about the degree of financial

[1] For example, while exposure to country-specific risk factors can explain the negative coefficient in uncovered interest rate parity regressions, it cannot explain the cross-section of carry trade returns. Lustig, Roussanov, and Verdelhan (2011) find that the carry factor, a high minus low interest rate sorted currency portfolio, is a direct measure of a global common risk factor. This factor can explain two thirds of the cross-sectional variation of exchange rates. Lustig, Roussanov, and Verdelhan (2014) extend this work and find that US specific exposure to global risk, a dollar factor, is the main driver of currency return predictability. While the carry is linked to changes in global equity volatility, the dollar factor returns are correlated with the average growth rate of consumption across countries and the rate of US-specific component of industrial production growth.

[2] Recall from the AMV that (log) exchange rate changes are equal to the difference in (log) foreign and domestic SDFs. Taking variances on both sides and assuming exchange rate volatilities around 15% per year implies an almost perfect correlation for international SDFs; see, e.g., Brandt, Cochrane, and Santa-Clara (2006).

market integration, which in most cases may be economically hard to justify.

In this paper, we study global factor structures of exchange rates by means of a new model-free methodology, which allows SDFs to price large cross-sections of international assets, such as stocks, bonds, and currencies, in presence of real-world frictions and independent of the currency denomination. We contribute to the literature in three ways. First, incorporating frictions allows us to endogenize the optimal set of assets selected by global investors for trading, because barriers to trade can induce optimal portfolio weights on some assets to shrink to zero. This produces endogenous international market segmentation without imposing any assumptions.

Second, we theoretically identify market structures under which model-free SDFs satisfy the AMV and are numéraire invariant even in the presence of market frictions. The numéraire invariance makes these SDFs uniquely suited to estimate *global* exchange rate factors. We then show that SDFs can always be decomposed into two factors: one that pertains to the common global factor given by the average negative log return of the global optimal portfolio across all currency denominations and second, by a currency basket which measures the average appreciation of any currency. In case of symmetric markets, this representation is exact.

Third, from the cross-section of these model-free international SDFs, we extract global SDF factors and currency baskets and show that they are directly related to dollar and carry, thus providing model-free support to the model-based evidence in Lustig, Roussanov, and Verdelhan (2011, 2014) and Verdelhan (2018). This is important as our SDFs are estimated from stocks and bonds and not currencies. As we argue, the two-factor structure proposed in these papers has wider appeal beyond just explaining exchange rate changes. Indeed, using our two factors we then price the cross-section of international asset returns and find that they substantially improve upon the performance of benchmark factor models of exchange rates, especially out-of-sample.

As it is well-known that standard methods to estimate SDFs using cross-sections of assets can lead to spurious estimates when the size of the cross-section increases, we rely on a new methodology for estimating model-free SDFs, which can endogenously produce sparse SDFs and sparse optimal portfolios. For instance, Kozak, Nagel, and Santosh (2019) shrink the coefficients of low variance principal components of characteristics-based factors using various penalization techniques from the machine learning literature. Different from this literature, we endogenously obtain sparse SDFs by imposing various forms of economically relevant financial market frictions.[3]

Our framework is general as it can incorporate various forms of frictions, such as proportional transaction costs, margin or collateral constraints, and short-sell constraints, while at the same time ensuring that model-free SDFs are consistent with the absence of arbitrage in asset markets with frictions. It is well-known that in frictionless and arbitrage-free markets, asset prices can be fully characterized by an SDF that only depends on asset returns, which at the same time prices all assets exactly, see, e.g., Ross (1978) and Hansen and Richard (1987), among many others. However, in the presence of frictions, linear SDF pricing implies in general non zero pricing errors that reflect the underlying structure of market frictions.[4] Earlier literature has often treated pricing errors as

[3]This approach is particularly appealing in the context of international financial markets, given the evidence of currency-biased portfolios of international investors; see, e.g., Maggiori, Neiman, and Schreger (2019).

[4]Non zero pricing errors may also emerge in frictionless arbitrage-free market settings under the notion of "asymptotic

evidence of SDF misspecification, by expressing them in terms of the least squares distance between an SDF and the family of SDFs that price correctly all assets. We instead explicitly work under the assumption of arbitrage-free markets with frictions and the resulting pricing error structures. In this setting, we characterize model-free SDFs that minimize established notions of dispersion in terms of the optimal portfolios of global investors constrained by market frictions, which allows us to recover global model-free SDFs from asset return data alone.

When relaxing the assumptions of complete and frictionless markets, violations of the AMV arise for the vast majority of international SDFs, which implies factor structures of exchange rates that are not exact. While this is expected, we theoretically show that there exist model-free SDF families which satisfy the AMV in international asset markets even in the presence of frictions as long as these frictions are symmetric. Market symmetry in frictions means that frictions are identical across currency denominations.[5] The family of model-free SDFs satisfying the AMV in symmetric international asset markets with frictions is the family of minimum entropy SDFs. We establish two powerful properties for these SDFs. First, we show that these SDFs are by construction numéraire-invariant, meaning that an optimal SDF pricing assets well in one currency also prices assets well in any other currency (after a simple multiplication with the exchange rate). Second, as long as markets are symmetric, optimal portfolios of global investors are identical leading to an exact factor structure of exchange rates. Therefore, in symmetric markets, the family of minimum entropy SDFs establish a benchmark to understand both the nature of global exchange rate risks and the implications of market friction asymmetries for the properties of these risks.

We study the global factor structure of exchange rates using a large cross-section of bonds and equities on developed countries, and we extract numéraire invariant minimum entropy SDFs under different market setting assumptions with varying transaction costs. When we assume that investors can trade internationally the full menu of assets in frictionless markets, we find that the ensuing SDFs are volatile and that they satisfy a virtually perfect single-factor structure with perfectly correlated SDFs. Moreover, the numéraire invariant optimal portfolio of global investors also implies positions in single assets that may be hard to maintain in practice without taking massively levered long and short positions. When we impose symmetric international market frictions instead, it already implies a number of zero portfolio weights on some assets and segmentation arises endogenously. Similar to the frictionless case, international cross-country SDF correlations are still almost perfect, indicating once more a very strong factor structure.

When we move from symmetric to asymmetric market settings, we find that SDF volatilities drop by 20% relative to the symmetric frictions case and that the ensuing optimal portfolios of global investors become more sparse. While lower than under the symmetric market setting, minimum entropy SDF correlations are still very high and common factor structures very strong. The optimal portfolios of global investors in markets with frictions typically further shrink average international portfolio weights and reveal an interesting factor composition: Global investors always trade a

arbitrage" as in Ross (1976)' Arbitrage Pricing Theory (APT). Such settings are also compatible with our methodology.

[5] For instance, when we assume a symmetric market, we either impose no frictions at all or we impose the exact same friction across all countries. Asymmetric markets, on the other hand, imply differences in the friction structure of some countries, such as, e.g., lower trade barriers for investing in local assets than in foreign assets.

carry, i.e., go long in high interest rate currencies and go short in low interest rate currencies, and they go long in US equity, while virtually completely ignoring long-term bonds. These findings connect to a larger literature in international finance documenting priced systematic dollar and carry risks. However, different from this literature our approach is completely model-free, it extracts SDFs directly from a broad cross-section of international asset returns (not only from a cross-section of exchange rates) and it allows us to interpret global factor compositions economically from the optimal portfolios of global investors.

From the cross-section of model-free international SDFs, we extract a global factor and a country level currency basket factor by estimating (i) the global factor as the average SDF calculated from the cross-section of model-free SDFs and (ii) the currency basket factor as the average appreciation of a currency relative to a basket of the remaining currencies. Using these two factors, we price the cross-section of international asset returns, including stock, bond, and currency returns, both in- and out-of-sample. The out-of-sample analysis is useful to control for overfitting effects, as we know, e.g., that in a market setting with no frictions in-sample model-free SDF pricing errors are zero by construction. Using rolling training periods of ten years for SDF estimation and one year rolling windows for testing, we find that the out-of-sample performance of model-free SDFs is excellent. Indeed, our currency independent global SDF risk factor alone explains up to 90% of the in-sample cross-sectional variation and more than 75% of the out-of-sample variation in the cross-section of currencies, stock and bond returns across all denominations. Hence, given this global risk-factor the local currency basket does not provide substantial additional improvement in explaining the cross section of international assets. We also find that the global SDF factor captures both the with-in and cross-asset class variation.

Our theoretical framework posits a two factor structure for global SDFs which in case of symmetric markets is exact. This naturally relates our framework to the findings of Verdelhan (2018) who argues that two factors, carry and dollar, explain a significant fraction of the systematic share in exchange rate changes. Indeed, while by construction the USD currency basket equivalent to the dollar factor, we find that our global SDF factor is correlated to carry risk factor. Prima facie this may seem surprising given that our SDFs are estimated from bonds and stocks and not currencies as in earlier literature. This, however, implies that carry and dollar may have appeal beyond explaining just currency risk premia but international asset returns more generally.

As a last exercise, we use our framework to study the effects of trading frictions on currency home bias. Maggiori, Neiman, and Schreger (2019) argue, that currency home bias may reflect a combination of financial frictions but also behavioral biases such as inattention that effectively segment financial markets. Since the latter cost are usually hard to quantify, we take our framework and ask how much asymmetry in frictions do we have to impose in order to match the almost perfect home bias that we observe in the data?[6] To this end, we take our asymmetric market friction setting and increase the cost to trade foreign assets. We find that in order to achieve almost perfect home bias, trading foreign assets needs to be around six times more expensive than trading local assets. Given that the average bid ask spread is around 2bps, this implies that trading foreign assets entails a

[6] See Coeurdacier and Rey (2013) for a discussion of the implied costs.

transaction cost of around 10bps in unobservable costs. Overall, different from earlier literature that has argued that transaction costs need to be unrealistically high to explain the home bias in the data, we conclude that these costs can be relatively small.

Literature Review: Our paper contributes to a growing literature in international finance studying global factors. The seminal work of Lustig, Roussanov, and Verdelhan (2011, 2014) and Verdelhan (2018) documents that two factors, carry and dollar, explain a significant share of the systematic variation in exchange rates. Panayotov (2020) studies global risk in an extended version of Lustig, Roussanov, and Verdelhan (2014), in which the US SDF has a larger exposure to global risk than all other international SDFs. Maurer, Tô, and Tran (2019) extract two principal exchange rate components from the cross-section of all cross-currency returns related to dollar and carry, in order to construct country specific SDFs. These SDFs are shown to price international equity returns well in sample. Aloosh and Bekaert (2019) reduce the cross-section of currencies by means of currency baskets that measure the average appreciation of each currency against all other currencies. They then apply clustering techniques to the cross-section of currency baskets and identify two clusters, one related to the dollar and another related to the Euro. Lustig and Richmond (2019) model gravity in the cross-section of exchange rates and find factor structures related to physical, cultural, and institutional distances between countries. Jiang and Richmond (2019) link trade networks between countries to exchange rates comovement in order to explain the existence of the global dollar and carry factors.

Our paper is different from this literature along several dimensions. First, it relies on common SDF factors that are extracted from a family of model-free, numéraire invariant SDFs, which satisfy the AMV in arbitrage-free international asset markets with frictions. Second, our methodology allows us to extract global factors jointly from a cross-section of returns including international equities and bonds, in addition to exchange rate returns. Third, our factors are directly related to the optimal portfolios of global investors in international asset markets with frictions, which provide additional unique insights into the factor composition, in terms of the portfolio exposure of global investors to various international assets. Fourth, the generality of our methodology allows us to incorporate into our analysis several economically relevant forms of market frictions and endogenous market segmentation features, and to study their implications for global factor structures. In this context, we find that our two factors which are extracted from the cross-section of SDFs are related to carry and and dollar, in line with the findings of Lustig, Roussanov, and Verdelhan (2011, 2014). Finally, we show that our common factors jointly price cross-sections of returns including international equities and bonds, in addition to exchange rate returns, not just in- but also out-of-sample.

Our paper is related to another important recent literature which studies how to extract regularized SDFs from a large cross-section of asset returns. A common denominator of the various approaches in this literature is the introduction of different penalization devices that shrink the in-sample exposures of extracted SDFs to assets returns. These penalizations are designed to mitigate in-sample overfitting and to improve the out-of-sample pricing performance. Moreover, depending on the chosen penalization structure, they can induce different degrees of sparsity in extracted SDFs,

in order to reduce the number of relevant factors an SDF depends on. For example, Kozak, Nagel, and Santosh (2019) use an elastic net penalization to shrink the contributions of low-variance principal components of characteristics-based factors. This technique induces sparse linear SDFs, which depend on a small number of characteristics-based principal components, and is shown to produce a more robust out-of-sample pricing performance. Freyberger, Neuhierl, and Weber (2019) study the cross-section of expected stock returns, by means of an adaptive group lasso technique enclosing an additive nonparametric specification of nonlinear dependencies between expected returns and characteristics. Using this technique, they show that a small set of characteristics can explain the cross-section of out-of-sample expected returns. None of these papers makes use of penalizations directly motivated from corresponding transaction cost specifications as we do in our paper.[7]

A small but important literature studies model-free SDFs in markets with frictions. He and Modest (1995) and Luttmer (1996) extend the Hansen and Jagannathan (1991) minimum variance SDF setting by incorporating various specifications of sublinear transaction costs that give rise to generalized diagnostics for asset pricing models. Hansen, Heaton, and Luttmer (1995) provide the econometric tools for the evaluation of asset pricing models in such settings. A key theoretical finding in this literature is that the pricing functional sublinearity gives rise to SDFs with non zero pricing errors, which are tightly constrained by the given transaction cost structure. Korsaye, Quaini, and Trojani (2018) extend this theory to address minimum dispersion SDFs resulting from general convex pricing errors structures, which are uniquely characterized in terms of optimal portfolios of investors in markets with convex transaction costs.[8] We make use of this theory in order to identify financial market structures that deliver numéraire invariant model-free SDF families which satisfy the AMV. This result is key to identify parsimonious global exchange rate factors structures. Second, we exploit the theoretical relation between numéraire invariant model-free SDFs and the optimal portfolios of investors in markets with frictions in order to study economic factor compositions in terms of the SDF exposures to international asset returns.

Our work is also related to the literature documenting that even small transaction costs can induce substantial market segmentation. For instance, Bhamra, Coeurdacier, and Guibaud (2014) show in a two country economy with transaction costs that any cost bearing on foreign equity holdings has two opposite effects on portfolios. A direct effect, which reduces cross-border holdings via lower expected returns on foreign assets, and an indirect effect, which reduces the substitutability between national assets because of a lower return correlation, thus increasing the willingness to diversify internationally. Since the latter effect is negligible, the authors show that a large home bias can arise also from small frictions.

Outline of the paper: The rest of the paper is organized as follows. Section 1 provides the theoretical framework for studying model-free SDFs in international financial markets with frictions. Section 4

[7] DeMiguel, Martin-Utrera, Nogales, and Uppal (2019) extend a mean-variance portfolio problem with lasso penalization by an explicit additional penalization for characteristics-dependent proportional transaction costs. Their findings show that the resulting optimal portfolio depends on a larger number of characteristics, because combining characteristics reduces transaction costs involved with portfolio adjustments that often cancel out.

[8] These authors also develop the econometric theory for the empirical analysis of minimum dispersion SDFs with convex pricing errors and study the empirical properties of these SDFs in an APT context.

presents our main empirical findings. Section 5 concludes. All proofs are gathered in a separate appendix.

1 Stochastic Discount Factors and Market Frictions

In this section, we develop a model-free framework that allows us to study international SDFs in the presence of various forms of frictions, such as bid-ask spreads, short-selling, margin, or collateral constraints. In this framework, we obtain a unique one-to-one relation between minimum dispersion SDFs and the optimal portfolio returns of a corresponding portfolio choice problem with penalized portfolio weights. This mapping between SDFs and optimal portfolio returns allows us to systematically pin down the optimal portfolios of global investors when there are barriers to trade. We then characterize international frictions structures that give rise to families of numéraire invariant model-free SDFs satisfying the AMV. We show that these SDF families are naturally suited to study global factor structures and detail their tight links to the cross-section of exchange rates.

1.1 Model-Free SDFs in the Presence of Frictions

To keep notation simple, for the time being, we drop all references to particular countries and we will relax this later on. We consider a two period incomplete market economy in which investors can trade a set of N basis assets with payoff vector $\boldsymbol{Z} := (Z_n)_{n=1,\dots,N}$ at time 1, and a corresponding price vector $\boldsymbol{P} := (P_n)_{n=1,\dots,N}$ at time 0. We assume that each payoff Z_n belongs to the space $L^q(\Omega, \mathcal{F}, \mathbb{P})$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with finite q-th moment, for some $1 < q \leq \infty$. We partition the set of assets into two subsets indexed by $D \subset \{1, \dots, N\}$, and $S := \{1, \dots, N\} \setminus D$, of size N_D and N_S, respectively, and we denote by $(\boldsymbol{Z}_D, \boldsymbol{P}_D)$ and $(\boldsymbol{Z}_S, \boldsymbol{P}_S)$ the payoff and price vectors of assets subject to market frictions and assets not subject to any market friction. All prices and payoffs are denominated in a common currency (or numéraire).

We model transaction costs in our economy using a closed and sublinear transaction cost function h that quantifies the costs associated with any portfolio in our economy, based on the same common numéraire for payoffs and prices.[9] Intuitively, sublinear transaction costs reflect the fact that (i) the cost of a portfolio implemented in a single execution is no greater than the cost when the same portfolio is implemented in more than one execution; and (ii) investing in a portfolio A, which is a multiple of a portfolio B with a certain factor, entails a cost equivalent to the cost of portfolio B multiplied by the same factor. This specification of transaction costs encompasses all transaction cost structures relevant for our analysis, such as, e.g., short selling constraints, bid-ask spreads, and proportional transaction costs, such as leverage constraints.

Following Harrison and Kreps (1979), we introduce a price system $(\mathcal{Y}, \pi)$, consisting of traded payoff space $\mathcal{Y}$ and a price functional π defined on $\mathcal{Y}$. We define $\mathcal{Y}$ as the set of portfolio payoffs that can be traded with finite transaction costs, out of the set of all portfolio payoffs generated by

[9] A real valued function f on a real vector space V is called sublinear, if $f(x+y) \leq f(x) + f(y)$ for all $x, y \in V$ and $f(\lambda x) = \lambda f(x)$ for all $\lambda \geq 0$ and $x \in V$. f is called closed if its sublevel set $\{x \in dom(f) : f(x) \leq \alpha\}$ is closed for any $\alpha \in \mathbb{R}$.

portfolios of payoffs in vector $\boldsymbol{Z}$:

$$\mathcal{Y} := \{Y = \theta'\boldsymbol{Z} : \; h(\boldsymbol{\theta}_D) < \infty, \; \boldsymbol{\theta} \in \mathbb{R}^N\}, \tag{1}$$

where $\boldsymbol{\theta}$ is the vector of portfolio weights of all assets and $\boldsymbol{\theta}_D$ the sub-vector of portfolio weights in assets subject to market frictions. The pricing functional in this economy with frictions is then simply given by the payoff minimum replicating cost when accounting for transaction costs:

$$\pi(Y) := \inf_{\boldsymbol{\theta}\in\mathbb{R}^N} \left\{\boldsymbol{\theta}'\boldsymbol{P} + h(\boldsymbol{\theta}_D) \; : \; Y = \boldsymbol{\theta}'\boldsymbol{Z}\right\} \; . \tag{2}$$

We work under the assumption of an arbitrage-free price system $(\mathcal{Y}, \pi)$, where as usual, we define an arbitrage portfolio as a traded payoff $Y \geq 0$ such that $\pi(Y) \leq 0$ and $\mathbb{P}(Y > 0) > 0$.

The next Proposition provides the foundation to our approach based on SDFs with non-zero pricing errors, by showing that the assumption of absence of arbitrage in such market of is equivalent to the existence of such SDFs in our setting.

Proposition 1. *Price system $(\mathcal{Y}, \pi)$ defined in equations (1) and (2) is arbitrage-free if and only if there exists a strictly positive SDF $M \in L^p$ that satisfies the (unconditional) pricing conditions:*

$$\mathbb{E}[M\mathbf{R}_S] - \mathbf{1}_{N_S} = 0 \qquad \textit{and} \qquad \mathbb{E}[M\mathbf{R}_D] - \mathbf{1}_{N_D} \in C_h \; , \tag{3}$$

with

$$C_h := \{\boldsymbol{y} \in \mathbb{R}^{N_D} \; : \; \boldsymbol{y}'\boldsymbol{w} \leq h(\boldsymbol{w}) \textit{ for all } \boldsymbol{w} \in \mathbb{R}^{N_D}\} \; , \tag{4}$$

and the gross return vectors $\boldsymbol{R}_S$, $\boldsymbol{R}_D$ associated with payoff vectors $\boldsymbol{Z}_S$, $\boldsymbol{Z}_D$.

Proposition 1 is akin to the well-known Fundamental Theorem of Asset Pricing and it characterizes the existence of strictly positive SDFs in markets with frictions by a standard no-arbitrage condition. According to pricing relation (3), such SDFs price exactly all assets not subject to trading frictions, while they imply non zero pricing errors constrained by the closed convex set C_h for assets subject to trading frictions. Intuitively, constraining the pricing error in the set C_h implies that the profit with any portfolio $\boldsymbol{w}$ due to mispricing is lower or equal to the transaction cost of implementing such portfolio.

Notice that Proposition 1 extends earlier results in the literature for markets with pricing errors constrained by a convex cone; see, e.g., Cvitanić and Karatzas (1992), Hansen, Heaton, and Luttmer (1995), and Luttmer (1996), among others. Moreover, it also naturally contains the standard case of arbitrage-free markets with no frictions on any asset, which arises whenever $C_h = \{\mathbf{0}_{N_D}\}$, i.e., transaction costs on all assets are zero.

Proposition 1 constrains the set of relevant SDFs in our economy with frictions and it implies that the following set of non-negative SDFs contains at least one strictly positive SDF:

$$\mathcal{M}_h := \{M \in L^p \;\; : \; M \geq 0 \, , \, \mathbb{E}[M\boldsymbol{R}_S] - \mathbf{1}_{N_S} = 0 \text{ and } \mathbb{E}[M\boldsymbol{R}_D] - \mathbf{1}_{N_D} \in C_h\} \; . \tag{5}$$

Recall that since markets are incomplete, there exist many possible SDFs. In particular, set $\mathcal{M}_h$ can usually contain several non negative SDFs and we are interested in SDFs out of this set that

minimize established notions of dispersion in the literature. To this end, we measure SDF dispersion by standard integral functionals of the form $\mathbb{E}[\phi(\cdot)] : L^p \to (-\infty, \infty]$, where ϕ is a closed and convex function. This setting contains several well-known minimum dispersion SDFs in the literature, e.g., Hansen and Jagannathan (1991) minimum variance SDFs, minimum Kullback-Leibler divergence, and minimum entropy SDFs. We next define minimum dispersion SDFs in our economy with frictions.

Definition 1. Minimum dispersion SDFs. Given transaction cost function h, convex and closed dispersion function ϕ, and partition $S \cup F$ into assets subject and not subject to trading costs, a minimum dispersion SDF is the solution to following optimization problem:

$$\Pi_h := \inf_{M \in \mathcal{M}_h} \mathbb{E}[\phi(M)] \,. \tag{6}$$

By definition, a minimum dispersion SDF is the solution to an optimization problem of the form (6). Working directly with such optimization problems is difficult, however, as they represent infinite-dimensional optimization problems that depend on unobservable random variables. Therefore, we follow Korsaye, Quaini, and Trojani (2018) and first characterize minimum dispersion SDFs via the solutions of a penalized portfolio problem in $\mathbb{R}^N$, in which the portfolio payoff dispersion is minimized in presence of transaction costs. To this end, let:

$$\Delta_h := \min_{\theta \in \mathbb{R}^N} \left\{ \mathbb{E}[\phi_+^*(-\boldsymbol{\theta}'\boldsymbol{R})] + \boldsymbol{\theta}'\mathbf{1}_N + h(\boldsymbol{\theta}_D) \right\} \,, \tag{7}$$

where ϕ_+^* is the convex conjugate of the restriction of ϕ to the nonnegative real line.[10] From portfolio problem (7), we obtain the following explicit characterization of minimum dispersion SDFs in our economy with frictions.

Proposition 2. *Given the convex and closed transaction cost function h, strictly convex and closed dispersion function ϕ, and arbitrage-free price system $(\mathcal{Y}, \pi)$, it follows:*

1. $\Pi_h = -\Delta_h$.
2. *If $\boldsymbol{\theta}_0$ is the solution to problem (7), then the unique solution M_0 to problem (6) is given by*

$$M_0 = (\phi_+^*)'(-\boldsymbol{\theta}_0'\boldsymbol{R}) \,, \tag{8}$$

where $(\phi_+^*)'(y)$ is the derivative of ϕ_+^* at y.

In Proposition 2, the unique SDF solving problem Π_h, as a function of the optimal portfolio payoff solving the alternative problem Δ_h, is completely determined by the shape of ϕ_+^*, via its derivative at the solution of problem (7). On the other hand, the transaction cost function h impacts the structure of the optimal portfolio in problem Δ_h. Importantly, with Proposition 2 we are now able to directly estimate model-free minimum dispersion SDFs in markets with frictions from asset return data alone, since from equation (8) the minimum dispersion SDF can be directly estimated from the return vector

[10] Function ϕ_+ is defined by $\phi_+(y) = \phi(y)$ for $y \geq 0$ and by $\phi_+(y) = 0$ otherwise. Given a function f defined on $\mathbb{R}^m$, its convex conjugate is defined for $y \in \mathbb{R}^m$ by $f^*(y) := \sup_{w \in \mathbb{R}^m} \{w'y - f(w)\}$. The assumption that f is closed and convex insures a one to one relation between f and f^*.

$\boldsymbol{R}$ using the optimal portfolio weight $\boldsymbol{\theta}_0$ from problem (7).[11] For the class of dispersion functions ϕ studied in this paper and for a large class of penalizations $h(\boldsymbol{\theta}_D)$, including penalizations generated by market frictions as in this paper, estimator $\hat{\boldsymbol{\theta}}$ consistently estimates parameter $\boldsymbol{\theta}_0$. In the context of our applications, Proposition 2 provides us with a consistent estimator for minimum dispersion SDF M_0 which is given by

$$\hat{M}_{t+1} := (\phi_+^*)'(-\hat{\boldsymbol{\theta}}'\boldsymbol{R}_{t+1}) \, . \tag{10}$$

Equation (10) will serve as the basis to estimate international model-free SDFs in our empirical analysis of global exchange rate factors in Section 4.

Finally, notice that Proposition 2 allows for a fairly general dispersion function ϕ in the specification of the SDF dispersion $\mathbb{E}[\phi(M)]$. However, in the following, we focus on negative entropy. More specifically, negative entropy SDF-dispersion is obtained for $\phi(x) = -\ln(x)$ when $x > 0$ and $\phi(x) = +\infty$ when $x \leq 0$. This corresponds to a convex conjugate portfolio dispersion function given by $\phi_+^*(y) = -\ln(-y)$ for $y < 0$ and $\phi_+^*(y) = +\infty$ otherwise. The minimum entropy SDF from Proposition 2 reads explicitly:

$$M_0 = \frac{1}{\boldsymbol{\theta}_0'\boldsymbol{R}} \, . \tag{11}$$

The relevance of the negative entropy SDF-dispersion for our analysis comes from the fact that it defines a numéraire invariant optimization criterion in Definition 1 and hence a numéraire invariant minimum entropy SDF. This property makes minimum entropy SDFs particularly useful to study factor structures since the particular currency denomination does not matter.

1.2 Examples of Market Frictions

While our setup can potentially incorporate a large variety of market frictions, we focus on three such frictions: short-selling constraints, bid-ask spreads, and proportional transaction costs, where the latter can equivalently be interpreted as leverage restrictions.

1.2.1 Short-Sell Constraints

Earlier literature explores the effect of short-sell constraints on SDFs; see Luttmer (1996) and Hansen, Heaton, and Luttmer (1995), among others. Incorporating such constraints into our setup is very straightforward, as it is sufficient to specify as a transaction cost function the characteristic function of $\mathbb{R}_+^{N_D} = \{\boldsymbol{w} \in \mathbb{R}^{N_D} : \boldsymbol{w} \geq 0, \ k = 1, \ldots, N_D\}$:

$$h(\boldsymbol{\theta}_D) = \delta_{\mathbb{R}_+^{N_D}}(\boldsymbol{\theta}_D) := \begin{cases} 0 & \boldsymbol{\theta}_D \in \mathbb{R}_+^{N_D} \\ \infty & \text{else} \end{cases} \, . \tag{12}$$

[11] Korsaye, Quaini, and Trojani (2018) formally address the consistent estimation of parameter $\boldsymbol{\theta}_0$, which can be achieved with an extremum estimator that incorporates the penalization term $h(\boldsymbol{\theta}_D)$ from equation (7). More specifically, given a stationary return process $\{\boldsymbol{R}_{t+1}\}_{t\in\mathbb{N}}$, the empirical version of (7) defines the following estimator:

$$\hat{\boldsymbol{\theta}} := \arg\min_{\hat{\boldsymbol{\theta}} \in \mathbb{R}^N} \left\{ \frac{1}{T}\sum_{t=1}^{T} \phi_+^*(-\boldsymbol{\theta}'\boldsymbol{R}_{t+1}) + \boldsymbol{\theta}'\mathbf{1}_N + h(\boldsymbol{\theta}_D) \right\} \, . \tag{9}$$

Under such a specification of transaction costs, set C_h in equation (4) of Proposition 1 is equal to $\mathbb{R}_-^{N_D}$ and all pricing errors on returns of assets subject to trading frictions are negative:

$$\mathbb{E}[M\boldsymbol{R}_D] - \mathbf{1}_{N_D} \leq 0\,. \tag{13}$$

1.2.2 Proportional Transaction Costs

Proportional transaction costs are trading costs proportional to portfolio positions such as leverage constraints. We model such market frictions in a straightforward manner by applying portfolio weight norms. In particular, let $h(\cdot) := \tau||\cdot||$, for some $\tau \geq 0$ and a norm $||\cdot||$. Under such a specification of transaction costs, set C_h in equation (4) of Proposition 1 is given by:

$$C_h = \left\{\boldsymbol{y} \in \mathbb{R}^{N_D}\ :\ ||\boldsymbol{y}||_* \leq \tau\right\}\ , \tag{14}$$

where $||\cdot||_*$ is the dual norm of $||\cdot||$.[12] Hence, in the arbitrage-free economy with proportional transaction costs, the following pricing error inequality holds:

$$||\mathbb{E}[M\mathbf{R}_D] - \mathbf{1}_{N_D}||_* \leq \tau\,. \tag{15}$$

One possible specification of proportional transaction costs relies on a l_1-norm, defined by $||\boldsymbol{y}||_1 := \sum_{i=1}^{N_D}|y_i|$. This specification gives rise to a lasso-type penalty in optimal portfolio problem (7) and to a bounded maximum pricing error in equation (15).[13] It is well-known from the machine learning literature that a lasso penalization produces sparse solutions in a corresponding optimization problem. In our setting, this feature leads to optimal portfolios and minimum dispersion SDFs that depend only on a strict subset of asset returns.

1.2.3 Bid-Ask Spreads

A natural way to incorporate bid-ask spreads in our framework is to consider long positions $\boldsymbol{\theta}^L$, when one buys an asset at ask price at time 0 and sells it at bid price at time 1, and short positions $\boldsymbol{\theta}^S$, when one buys an asset at bid price at time 0 and sells at ask price at time 1. This corresponds to a setting with (i) no short-selling constraints on long position $\boldsymbol{\theta}^L$ and (ii) no buying constraints on short positions $\boldsymbol{\theta}^S$. Denoting by $\boldsymbol{\theta}_D = [\boldsymbol{\theta}_D^{L\prime}, \boldsymbol{\theta}_D^{S\prime}]'$ the extended portfolio vector of long and short position on each asset, we can easily incorporate these market frictions with following transaction cost function:

$$h(\boldsymbol{\theta}_D) := \delta_{\mathbb{R}_+^{N_D}}(\boldsymbol{\theta}_D^L) + \delta_{\mathbb{R}_-^{N_D}}(\boldsymbol{\theta}_D^S)\,. \tag{16}$$

Under this specification of transaction costs, set C_h in equation (4) of Proposition 1 is given by the Cartesian product of $\mathbb{R}_+^{N_D}$ and $\mathbb{R}_-^{N_D}$ ($C_h := \mathbb{R}_-^{N_D} \times \mathbb{R}_+^{N_D}$). Hence, pricing errors on long (short) positions are negative (positive):

$$\mathbb{E}[M\mathbf{R}_D^L] - \mathbf{1}_{N_D} \leq 0\,;\ \mathbb{E}[M\mathbf{R}_D^S] - \mathbf{1}_{N_D} \geq 0\,, \tag{17}$$

where $\mathbf{R}_D^L$ and $\mathbf{R}_D^S$ are the gross return vectors for long and short positions, respectively.

[12] The dual norm of norm $||\cdot||$ is defined by $||\boldsymbol{y}||_* = \sup_{\boldsymbol{w}}\{\boldsymbol{w}'\boldsymbol{y} :\ ||\boldsymbol{w}|| \leq 1\}$.

[13] If $||\cdot||$ is an l_p norm then $||\cdot||_*$ is an l_q norm, and viceversa, with p, q in $(1, \infty)$ and $1/p + 1/q = 1$. Moreover, l_1 and l_∞ norms are dual to each other. Finally, the l_2 norm is the only self-dual norm.

1.2.4 Combination of Frictions

It is straightforward to introduce different types of trading frictions for different subsets of assets. For example, one could introduce short-sell constraints for equities and at the same time impose bid-ask spreads on the carry trade. In such settings, set C_h in equation (4) of Proposition 1 for the global transaction cost constraints just becomes the Cartesian product of the individual sets reflecting the trading frictions applied to each subset of assets.

2 International SDFs

Our setting above can be naturally extended to account for the fact that global investors trade assets internationally in markets with possibly different currency denominations. To this end, consider an economy consisting of M countries, in which investors in each country can trade N assets, consisting of both local and foreign assets, denominated in the country's own currency. We denote with $\boldsymbol{R}^{(i)}$ the vector of gross returns denominated in country i's currency, i.e.,

$$\boldsymbol{R}^{(i)} := \{R_k^{(i)}\}_{k=1,\dots,N} \, , \tag{18}$$

where $R_k^{(i)}$ is the gross return of asset k in country (currency) i. Denoting with $X_j^{(i)}$ the gross exchange rate return, with the exchange rate defined as the price in country i currency of one unit of country j's currency, it then follows:

$$\boldsymbol{R}^{(i)} = X_j^{(i)} \boldsymbol{R}^{(j)}. \tag{19}$$

We partition the assets available to an investor in country i into two groups: assets not subject to any market frictions, with gross return vector $\boldsymbol{R}_S^{(i)}$, and assets subject to possibly different types of market frictions, with return vector $\boldsymbol{R}_D^{(i)}$. Market frictions in market i are specified as $h^{(i)}(\boldsymbol{\theta}_D^{(i)})$, based on a sublinear transaction cost function $h^{(i)}$ applied to the sub-vector $\boldsymbol{\theta}_D^{(i)}$ of portfolio weights on assets subject to market frictions, within global portfolio vector $\boldsymbol{\theta}_D^{(i)}$ of country i.

We work under the assumption of arbitrage-free markets with respect to each currency denomination. This implies from Proposition 1 that the following sets of admissible SDFs in economies denominated in country i's currency ($i = 1, \dots, M$) contain at least a strictly positive SDF:

$$\mathcal{M}_h^{(i)} := \left\{ M \in L^p \ \ : \ \ M \geq 0 \, , \, \mathbb{E}[M \boldsymbol{R}_S^{(i)}] - \boldsymbol{1}_{N_S} = 0 \text{ and } \mathbb{E}[M \boldsymbol{R}_D^{(i)}] - \boldsymbol{1}_{N_D} \in C_h^{(i)} \right\} \, , \tag{20}$$

where $C_h^{(i)}$ is the pricing error constraint set for investors in country i, defined in equation (4). Model-free minimum dispersion SDFs are then defined by:

$$\Pi_h^{(i)} := \inf_{M \in \mathcal{M}_h^{(i)}} \mathbb{E}[\phi^{(i)}(M)] \, , \tag{21}$$

for corresponding closed and strictly convex dispersion functions $\phi^{(i)}$. Furthermore, Proposition 2 applies and minimum dispersion SDFs are uniquely pinned down as

$$M_0^{(i)} = (\phi_+^{(i)*})'(-\boldsymbol{\theta}_0^{(i)\prime} \boldsymbol{R}^{(i)}) \, ,$$

where $\boldsymbol{\theta}_0^{(i)}$ presents the solution to the following optimal global portfolio problem for country i investors:

$$\Delta_h^{(i)} := \min_{\theta \in \mathbb{R}^{N_D}} \left\{ \mathbb{E}[\phi_+^{(i)*}(-\theta' \boldsymbol{R}^{(i)})] + \boldsymbol{\theta}' \mathbf{1}_N + h^{(i)}(\boldsymbol{\theta}_D) \right\} . \tag{22}$$

By construction, our setting allows us to consider potentially country-specific transaction cost specifications $h^{(i)}$ that can vary along different dimensions, such as the type of assets subject to transaction costs in each country or the type of transaction costs applied to each asset in a particular country. The resulting SDF can then be estimated from a stationary time series $\{\boldsymbol{R}_{t+1}^{(i)}\}_{t \in \mathbb{N}}$ of returns in currency denomination i, using the estimator provided in equation (10).

Country-specific transaction costs induce asymmetries in investors' behavior across countries. In contrast, identical specifications of transaction cost functions, identical dispersion metric and identical definitions of assets subject to transaction costs in each country lead to identical optimal portfolios across countries or market symmetry. To clarify these concepts, we define market symmetry as follows:

Definition 2. An international economy consisting of M countries with sets of assets subject to transaction costs indexed by $\{D^{(1)}, \ldots, D^{(M)}\}$ and transaction cost specification $\{h^{(1)}, \ldots, h^{(M)}\}$ is called symmetric if:

$$D^{(1)} = \ldots = D^{(M)} \text{ and } h^{(1)} = \ldots = h^{(M)} . \tag{23}$$

As we show later on, assumptions about the symmetry of international markets have significant implications for the factor structure of exchange rates.

2.1 Asset Market View of Exchange Rates in the Presence of Frictions

A key assumption of many asset pricing models in international finance is the Asset Market View (AMV) of exchange rates, which stipulates that the exchange rate between domestic and foreign currencies is uniquely pinned down by the ratio of foreign and domestic SDFs. It is well-known that under the assumption of complete frictionless markets, the AMV holds with respect to a unique pair of international SDFs. This results holds also more generally in incomplete frictionless markets for the pair of minimum entropy domestic and foreign SDFs as long as markets are integrated, meaning that investors can trade all assets internationally via the exchange rate market, see Sandulescu, Trojani, and Vedolin (2019).

In a quest to describe the factor structure of exchange rates, the AMV comes in very handy. Intuitively, validity of the AMV directly implies that a cross section of (log) exchange rates is exactly reproduced by a simple linear transformation of (log) international SDFs, meaning that factor structures in the cross-section of exchange rates are naturally explained by the factor structure in the cross section of SDFs. In addition, the fact that SDF pairs which satisfy the AMV need to be strongly correlated to explain the low exchange rate volatilities makes such SDF factor structures potentially parsimonious and hence economically insightful.

Notice, however, that even under the assumption of the AMV, many exact exchange rate factor representations are possible. Intuitively, since optimal SDFs are uniquely linked to the

optimal portfolios of international investors, various market frictions may impact these portfolios differentially. Unfortunately, this feature makes the estimated factor structures dependent on ad hoc assumptions on the set of common assets traded internationally.[14]

The theory developed in Section 1.1 allows us construct model-free minimum dispersion SDFs that (i) naturally endogenize the choice of the common assets traded internationally in markets with frictions and (ii) are sufficiently regularized by the imposed frictions to produce an accurate pricing performance not just in- but also out-of-sample. Therefore, this theory is a natural starting point to construct exchange rate factor structures with commonly traded international assets returns that are selected endogenously, based on economically plausible specifications for frictions. The missing link, in order to obtain exact exchange rate factor representations using minimum dispersion SDFs, is a result on the validity of the AMV in international asset markets with frictions. This result is provided next.

Proposition 3. *Consider a symmetric international economy consisting of M countries. Moreover, let $\phi^{(1)}(\cdot) = \ldots = \phi^{(M)}(\cdot) = -\ln(\cdot)$ and denote by $M_0^{(1)}, \ldots, M_0^{(M)}$ the resulting family of minimum entropy SDFs solving problem (21). It then follows, for any $i, j \in \{1, \ldots, M\}$:*

$$X_j^{(i)} = M_0^{(j)} / M_0^{(i)} \,, \tag{24}$$

i.e., the AMV holds with respect to all ensuing minimum entropy SDFs.

In Proposition 3, the key condition for the validity of the AMV with respect to all minimum entropy SDF pairs in international asset markets with frictions, is that international frictions are symmetric. Together with Proposition 1, this condition also immediately implies that pricing error constraints are as well *symmetric* across markets: $C_h^{(1)} = \ldots = C_h^{(M)}$. Another hidden implication of Proposition 3 is that the optimal portfolio returns are symmetric as well across markets, i.e.: $\boldsymbol{\theta}_0^{(1)} = \ldots = \boldsymbol{\theta}_0^{(M)}$, where $\boldsymbol{\theta}_0^{(i)}$ is the solution of problem (22) for market i under the conditions of Proposition 3. This feature is a consequence of the numéraire invariance of minimum entropy SDFs and it follows from the particular choice of the convex conjugate of dispersion function $\phi(\cdot) = -\ln(\cdot)$ in Proposition 2 and implies for any $i, j \in \{1, \ldots, M\}$:

$$X_j^{(i)} = \frac{(\boldsymbol{\theta}_0^{(j)} \boldsymbol{R}^{(j)})^{-1}}{(\boldsymbol{\theta}_0^{(i)} \boldsymbol{R}^{(i)})^{-1}} = \frac{(\boldsymbol{\theta}_0^{(j)} \boldsymbol{R}^{(j)})^{-1}}{(\boldsymbol{\theta}_0^{(i)} \boldsymbol{R}^{(j)})^{-1}} X_j^{(i)} \,, \tag{25}$$

i.e., $\boldsymbol{\theta}_0^{(j)} \boldsymbol{R}^{(j)} = \boldsymbol{\theta}_0^{(i)} \boldsymbol{R}^{(j)}$ state by state, and hence $\boldsymbol{\theta}_0^{(i)} = \boldsymbol{\theta}_0^{(j)}$.

One key implication of Proposition 3 for our analysis of the global factor structures of exchange rates is that the cross-section of $M \times (M-1)$ distinct exchange rates is always *exactly* described by a simple log linear transformation of the cross-section of M minimum entropy SDFs computed under symmetric international transaction cost structures.

[14]Moreover, under the assumption of frictionless markets, estimation of model-free SDFs tends to produce over-fitted estimates and spurious results when the size of the cross-section of asset return increases.

3 The Global Factor Structure of International SDFs

We now turn to the main focus of our paper and study the relation between exchange rate factors and the AMV for our families of global minimum dispersion SDFs $\{M_0^{(i)}\}_{i=1,...,M}$.

3.0.1 Asset Market View and Global Factors

In general, for an arbitrary choice of a family of minimum dispersion SDFs, the AMV is violated and deviations from it can be captured by a family of Backus, Foresi, and Telmer (2001) stochastic exchange rate wedges $\{\eta_j^{(i)}\}_{1\leq i,j\leq M}$, defined by:

$$X_j^{(i)} = \frac{M_0^{(j)}}{M_0^{(i)}} e^{\eta_j^{(i)}} . \tag{26}$$

After taking logs on both sides of this identity yields:

$$x_j^{(i)} = m_0^{(j)} - m_0^{(i)} + \eta_j^{(i)} , \tag{27}$$

where we denote with lower-case letters logs: $x_j^{(i)} := \ln X_j^{(i)}$. It immediately follows that whenever a family of minimum dispersion SDFs satisfies the AMV, all wedges are zero and log exchange rate returns satisfy an exact linear model, in which exchange rate shocks are completely spanned by SDF shocks:

$$x_j^{(i)} = m_0^{(j)} - m_0^{(i)} . \tag{28}$$

From Proposition 3, we can recover equation (28) for all families of minimum entropy SDFs estimated under the assumption of market symmetry. Accordingly, the cross section of $M \times (M-1)$ log exchange rate returns is completely explained by a cross-section of M such log minimum entropy SDFs. Furthermore, these SDFs need to be highly correlated, in order to be consistent also with the low exchange rate volatilities in the data. Such a feature naturally induces a parsimonious factors structure in minimum entropy SDFs constructed under symmetric market frictions.

In the more general situation of asymmetric market frictions and/or SDF measures of dispersion different from entropy, deviations from the AMV arise and log exchange rates in linear model (27) are only approximately related to log minimum dispersion SDFs, because of the existence of non zero stochastic wedges. In this case, an exact representation of the cross-section of $M \times (M-1)$ exchange rates requires in principle information from the cross section of M minimum dispersion SDFs and all $M \times (M-1)$ stochastic wedges. In this representation, the factor structures in exchange rates and SDFs depend on both the degree of variability of the wedges and possible factor structures in the cross-section of wedges that are not spanned by factors in the cross-sections of SDFs.[15] The last two wedge properties are endogenously determined by the degree of asymmetry in the market frictions assumed to construct minimum dispersion SDFs.

[15] Since wedges need themselves to be consistent with the pricing constraints satisfied by minimum dispersion SDFs in each currency denomination, wedges and SDFs are typically correlated.

3.0.2 Two-Factor SDF Representation

Proposition 2 posits that global factors in the cross-section of minimum dispersion SDFs are directly linked to the corresponding portfolios of global investors in markets with frictions. In this setting, the numéraire invariance of the portfolios of minimum entropy investors in symmetric markets plays a particular role, implying that:

$$m_0^{(i)} = -\ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(i)} = -\ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)} - x_j^{(i)} , \tag{29}$$

for each currency denomination $i \in \{1, \ldots, M\}$. In other words, the cross-section of these SDFs is exactly reproduced by the sum of the optimal return of a single global portfolio with respect to a reference currency j, and the cross-section of log exchange rate returns with respect to the reference currency. Therefore, the cross-section of these SDFs naturally contains a common factor $-\ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)}$ and additional potential factors reflecting common exchange rate factors not spanned by $-\ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)}$.

A second representation of minimum entropy SDFs in symmetric markets directly follows from identity (29), using weights $w_1, \ldots, w_M$ such that $\sum_{j=1}^{M} w_j = 1$:

$$m_0^{(i)} \quad = \quad \underbrace{\sum_{j=1}^{M} -w_j \ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)}}_{\text{Global SDF}} + \underbrace{\sum_{j=1}^{M} -w_j x_j^{(i)}}_{\text{Currency Basket}} \quad = \quad G + CB^{(i)} , \tag{30}$$

where G and $CB^{(i)}$ denote the global SDF and the local currency basket factors, respectively.[16] Here, every minimum entropy SDF is written as the sum of (i) a common factor given by the average negative log return of the global optimal portfolio across currency denominations and (ii) a currency basket factor $CB^{(i)}$ measuring the average appreciation of currency i relative to currencies $j \neq i$.[17] In this representation, the cross-section of SDFs naturally contains a common factor, given by the average negative log return of the global optimal portfolio across currencies, and additional potential factors reflecting common currency basket factors not spanned by this average return.

3.0.3 The Global Factor Structure of Exchange Rates Redux

The two factor SDF structure in equation (30) is closely related to the models of Lustig, Roussanov, and Verdelhan (2014) and Verdelhan (2018) who posit that country-level SDFs are driven by two factors: carry and dollar where the latter measures the average appreciation with respect to the dollar, similar to our currency baskets. In case of minimum entropy SDFs, this implies that, given the global SDF and currency basket factors, there cannot be any other factors since the exchange rate representation is exact, i.e., the AMV holds. Notice that, however, the currency basket is still "local" in the sense that it is the average appreciation of a local currency with respect to the rest of the world. Hereafter, we

[16] For SDFs minimizing dispersion measures beyond entropy or in presence of market asymmetry the factor model in (30) becomes an approximation due to stochastic wedge and reads $m_0^{(i)} = \sum_j w_j m_0^{(j)} + \sum_j -w_j \ln X_j^{(i)}$.

[17] Our notion of currency baskets is also related to Aloosh and Bekaert (2019) who use equal weighted baskets of all possible bilateral exchange rate with respect to a specific currency. Notice, however, that different from these authors, who focus exclusively on currency pricing in the cross-section, we price also stocks and bonds in the time-series, moreover, our SDFs are spanned by two distinct factors: the average negative log return and the common currency basket.

consider two factor representations of the local optimal SDFs, $M_0^{(i)}$. First we propose a two-factor approximation of the local SDF using both the global SDF and the currency basket factors, i.e.

$$\widetilde{M}_0^{(i)} = \exp\left(G + CB^{(i)}\right). \tag{31}$$

Notice that under international symmetric market setting the factor representation (31) is exact. Second, we propose a one-factor representation of the local SDFs using only the global SDF factor.

$$\overline{M}_0 = \exp\left(G\right). \tag{32}$$

Equations (32) and (31) serve as the main basis of our empirical analysis. In particular, we want to test how the two factors, global SDF and local currency baskets, that span international SDFs explain international assets in the cross-section.

4 Empirical Analysis

Proposition 2 and in particular equation (10) allows us to estimate minimum dispersion international SDFs directly from returns data in the presence of trading frictions. We study two different market settings with varying transaction costs in both symmetric and asymmetric markets and explore the impact of frictions on the properties of SDFs. Using SDFs denominated in different currencies, we then extract two global factors from equation (31) to price cross-sections of currencies as well as short- and long-term bonds and stocks.

4.1 Data

In our empirical analysis, we use monthly data between January 1988 and December 2015. We focus on the following developed markets: Australia, Canada, Euro Area, Japan, New Zealand, Switzerland, United Kingdom and United States.[18] We collect data on exchange rates, short- and long-term interest rates, and MSCI country equity indices' prices from Datastream. When we analyze a specific currency denomination, we treat the corresponding market as the domestic and all other markets as foreign. Hence, we do not consider bilateral trades but a global economy where global investors can trade all possible assets.

4.2 Market Settings

We study two different market settings, symmetric and asymmetric markets, and in each market impose different types of transaction costs.

SYMMETRIC MARKETS: For the first symmetric market setting, we assume that global investors can trade the full menu of assets and there are no frictions to trade. The second symmetric market setting arises when investors face proportional transaction costs. We assume that investors incur no transaction costs when trading short-term bonds globally (i.e., investors can borrow and lend at

[18] Before the introduction of the Euro, we take the Deutsche Mark in its place.

the short-term interest rate without any frictions) but face transaction costs when trading long-term bonds and equity. More specifically, we assume that transaction costs, modeled with $l_1 norm$, are proportional to their positions and in line with the size of bid-ask spreads.[19] To this end, the proportional transaction cost parameter τ is chosen such that we have comparable pricing errors implied on the returns based on mid-prices by (i) SDFs in an economy with proportional transaction costs and by (ii) SDFs in an economy where market frictions are quantified by bid-ask spreads.

ASYMMETRIC MARKETS: In the first asymmetric market setting, we assume that investors face bid-ask spreads when buying and selling international assets. To this end, we use average bid-ask spreads for exchange rates directly available from Datastream which are in the order of 2bps. For the long-term bonds, we also assume average bid-ask spreads of 2bps in line with Adrian, Fleming, and Vogt (2017) for the US and Bank of International Settlement (2016) for Japan and Germany ten-year bonds. For equity indices, we impose a 6bps spread.[20] The second asymmetric market setting assumes that local short-term bonds can be traded without any frictions whereas all foreign short-, as well as long-term bonds and equities face proportional transaction costs which are again consistent with the size of the bid-ask spread. In these settings the asymmetry is introduced through frictions that differentiate between home and foreign assets.

4.3 Properties of Model-Free International SDFs

As a first exercise, we study the properties of international SDFs, their comovement, and the corresponding optimal portfolio weights. To this end, we estimate equation (10) using the different transaction cost functions discussed before. Table 1 provides summary statistics on SDFs for different currency denomination for the four market settings.

[Insert Table 1 here.]

Panels A to D in Table 1 report summary statistics for SDFs in each currency denomination for the four market structures. While average SDFs are the same across the different market settings, volatilities decrease significantly as we impose market frictions. For example, while the average volatility in markets with no frictions is around 0.35, the volatility in asymmetric markets with transaction costs is only around 0.19, a 45% drop.

The cross-country correlations in the four market settings are presented in the lower panels of Table 1. Not very surprisingly, the correlations are almost perfect in symmetric markets. The intuition for this is that under the assumption of market symmetry, the AMV holds and "enforces" a high correlation among international SDFs, see Proposition 3. As we move to an asymmetric market setting where transaction costs vary among the different countries, we notice that the correlations are slightly lower but still all above 90%. This may be more surprising given that in this case, we have

[19] $h(\boldsymbol{\theta}_D) := \tau||\boldsymbol{\theta_D}||_1 = \sum_{i=1}^{N_D} |\theta_{Di}|$.

[20] It is in general impossible to know the exact bid and ask spread of assets. Luttmer (1996) uses bid and ask spreads of around 0.012% which corresponds to the tick size on the NYSE. Andersen, Bondarenko, Kyle, and Obizhaeva (2018) reports that the bid and ask spread on E-Mini Futures on the S&P500, one of top two most liquid exchange traded futures in the world, is around 0.25 index points.

violations of the AMV. This finding clearly indicates that, (i) stochastic wedges are not significant and (ii) a strong factor structure of exchange rates emerges even in presence of market frictions.

4.4 Optimal Portfolio Weights

In order to get a better understanding of the optimal SDFs for different currencies, we now study optimal portfolios. Our theoretical framework allows us to exactly identify optimal portfolio weights from agents' Euler equations. Figure 1 plots the optimal portfolio weights for symmetric markets with no frictions (upper panel) and symmetric proportional transaction costs (lower panel).

[Insert Figure 1 here.]

Recall that a direct consequence of Proposition 3 is that portfolio holdings have to be identical in symmetric markets. Therefore, we only plot the USD denominated portfolio weights. We notice that whenever investors can trade without frictions, portfolio weights can be very large both long and short. The larger positions reveal that investors borrow in the lower interest rate currencies such as JPY, USD, CAD and EUR and hold long positions in high interest rate countries such as NZD and AUD. Most positions in long-term bonds are short, with the exception of USD, EUR, and JPY. Global investors hold large long positions in USD, AUD, and CHF equity indices. The large positions indicate that without taking large levered positions it may be hard to maintain this portfolio.

When we impose symmetric proportional transaction costs on investors, given by an l_1 norm, some weights on assets are zero, portfolio selection is sparse, and the size of the positions shrink, see the lower panel in Figure 1. More specifically, most portfolio weights on the long-term bonds are zero. This is not very surprising given that currency risk premia at the long-end of the term structure are small, see, e.g., Lustig, Stathopolous, and Verdelhan (2019). Investors also drop most of the equity indices except for a long position in the USD and CHF and a short position in the JPY. Interestingly, most of the wealth is held in short-term bonds. In particular, as in the no frictions case, global investors trade a "carry." Short positions are in typical funding currencies, whereas long positions are in investment currencies. Overall, we conclude that even small transaction costs which restrain the leverage that investors can take, can have big impacts on the optimal portfolios held for global investors.

[Insert Figures 2 and 3 here.]

We can now study the asymmetric market settings. Figures 2 and 3 plot the cases where we assume asymmetric bid-ask spreads and proportional transaction costs, respectively. Because in asymmetric markets portfolio weights vary across the different currency denominations, we plot all currencies separately. As in the symmetric proportional transaction cost case, portfolios are more sparse and investors hold positions that resemble the carry and long equity. The most sparse portfolio is when investors face asymmetric proportional transaction costs as also reflected in the low SDF volatility in Table 1. Even though in asymmetric markets, portfolio weights are not enforced to be the same, positions look highly similar. As in the previous symmetric market cases, investors engage in carry trades: shorting the US dollar, CAD, EUR or CHF and go long in the NZD and AUD. In addition, investors trade long USD and CHF equity.

4.5 The Global Factor Structure of Exchange Rates

We now turn to the main focus of our paper, the factor structure of exchange rates. Different from earlier literature, which primarily estimates factors from the cross-section of currencies (or currency portfolios) directly, we use SDFs which are estimated from international stock and bond data. Moreover, in addition to pricing currencies, we also price stocks and bonds.

As a first observation, notice that the almost perfect correlation among international SDFs implies a very parsimonious SDF factor structure. Second, recall that our SDF representation decomposes each SDF into two distinct factors: the portfolio of optimal returns and a local currency basket, defined in equation (30), or a global currency basket, defined in equation (31). We can then use these two factors to price the cross-section of international assets both in- and out-of-sample.

Figure 4 plots the time-series of the global SDF risk factor in equation(32) for the four market settings. While we notice a much higher volatility for the global factor in market settings without any frictions, especially during crises, there is overall a high comovement among the different settings. They all increase during bad economic times such as recessions or during times of disruptions in financial markets such as the dot com bubble burst or the Lehman default. Interestingly, we notice that global SDFs spike during US specific crisis events. For example, in all four market settings, we find that global SDFs exhibit a massive spike in August 2011 during the US downgrade from AAA to AA+ by S&P.

[Insert Figure 4 here.]

Figure 5 plots the time-series of the local currency baskets. While the global SDF is by construction currency independent, notice that local currency baskets are country specific. The global SDF factor correlates very differently to the local currency baskets. For example in the asymmetric proportional transaction cost setting it correlates negatively to the high interest rate currency baskets, NZD (-43%) and AUD (-35%), and it positively correlates to CHF (14%), EUR (25%) and JPY (31%). It is instead almost uncorrelated to USD (5%), GBP (3%) and CAD (1%). Notice that in the case of USD the local currency basket is equivalent to the Dollar factor of Lustig, Roussanov, and Verdelhan (2014) and Verdelhan (2018).

[Insert Figure 5 here.]

We analyze two factor models for the international asset returns, (1) a one-factor model where the risk factor depends only the global SDF, $\overline{M}_0 = \exp(G)$ and (2) a one-factor model where the risk factor depends on both the global SDF and local currency basket, i.e. $\widetilde{M}_0^{(i)} = \exp(G + CB^{(i)})$. We then run two-step Fama and MacBeth (1973) regressions both in- and out-of-sample for the cross-section of currencies and international stocks and bonds both separately and jointly. In addition to the two factors, we also present regression results using the local country SDF only.

4.5.1 Factor Premia: In-Sample

Tables 2 to 3 present in-sample estimated factor premia for USD denominated cross-section of currency returns (Table 2) and stocks and bonds (Table 3) using two-step Fama and MacBeth (1973)

regressions. Column 1 of each table reports the price of risk relative to the local minimum entropy SDF, $M_0^{(i)}$. Column 2 reports the risk premium relative to the risk factor, $\overline{M}_0$, given by the global SDF, see equation (32). Column 3 instead reports the results for the factor model where the risk factor, $\widetilde{M}_0^{(i)}$, uses both the global SDF and local currency baskets, see equation (31). Recall that in symmetric market settings the later is an exact representation of the local SDF.

As by construction pricing errors are zero (i.e., expected asset returns are perfectly matched) in absence of market frictions, R^2s are 100% for the local SDFs. As we introduce market frictions and market asymmetry the pricing ability of the factor model declines. All estimated factor risk premia are highly statistically significant for the two factors. While not very surprising, we find that the factor, $\widetilde{M}_0^{(i)}$, given by the approximation of the local SDF with both global SDF and local currency basket explains the same percentage of the cross-sectional variation of currency returns and all assets as the local SDF. Surprisingly instead, we find that the factor model where the risk factor, $\overline{M}_0$, is given by the currency independent global factor akone is enough to explain more than 90% of the cross-sectional variation of all assets in all market settings except in the asymmetric proportional transaction case where the model explains 85% of the variation of the currency returns. In fact, adding the local currency basket does not provide substantial additional explanatory power.

[Insert Tables 2 to 3 here.]

Notice that the R^2s in Table 3 capture not only with-in asset class variation but also across asset class variation. To show this Figure 6 plots the expected excess return as function of exposure to the global SDF risk factor, $\overline{M}_0$ for the USD denominated economy. In all four settings we can notice that the global factor model captures the cross asset class variation. Expectedly, as market frictions and asymmetry are introduced the assets are not aligned across the dashed line representing the price of risk as in the case with no market frictions. As we will see in the next section the later settings will entail a better out-of-sample pricing ability. Moreover in presence of asymmetry and market frictions, the variation in the risk exposure increases as can be seen in the X-axis of the figures. This will become even more evident in the out-of-sample case.

[Insert Figure 6 here.]

To show that the pricing ability of the global SDF is consistent across all denominations we plot the cross-sectional R^2 for different economies in the asymmetric proportional transaction cost setting. Across all denominations the global SDF alone explains above 65% of the cross-sectional variation for long-term bonds and 75% for equities and currency returns. While the long-term bonds represent the asset class with lower R^2s, the currency returns represent instead the asset class with the highest R^2s except in the CAD denominated economy. The top-left figure confirms the ability of the global SDF in capturing the cross asset class variation. In general, the local currency basket does not provide additional explanatory power except for the so-called *commodity currencies*, AUD, CAD and NZD.

[Insert Figure 7 here.]

We can compare these results to earlier literature which constructs global risk factors for currency returns from the cross-section of individual currencies or currency portfolios. For example, in the two-factor specification of Lustig, Roussanov, and Verdelhan (2011) the authors achieve an R^2 of around 60%, Menkhoff, Sarno, Schmeling, and Schrimpf (2012) find 90% R^2 for a two-factor specification with dollar and exchange rate volatility. Notice that our SDFs are extracted also from bonds and stocks and not only currency returns.

4.5.2 Factor Premia: Out-Of-Sample

An economically more interesting exercise than in-sample pricing is whether our factor models are able to price the cross-section of international assets out-of-sample. To this end we use a training 10 year window up to say year y to estimate the optimal weight $\hat{\boldsymbol{\theta}}_y$ solving problem (7) and the next year $y+1$ to compute the out-of-sample SDF as $\hat{M}_{y+1} = \phi'_*(\hat{\boldsymbol{\theta}}'_y \boldsymbol{R}_{y+1})$. We then construct the out-of-sample SDF time-series $\{\hat{M}_t\}_t$ in a rolling window manner. The out-of-sample global SDF is computed using local out-of-sample SDFs and equation (32). We then evaluate the out-of-sample pricing performance of our factor models by calculating the cross-sectional R^2.

Tables 4 to 5 present the out-of-sample results for the four market settings. The results are mixed for the cross-section of FX returns: While estimated factor premia are significant in some of the specifications, they are insignificant in the no frictions case. This, however, is not very surprising given the highly volatile nature of the SDFs documented in Table 1. As a comparison, we can gauge the estimated factor premia in the case where we impose proportional transaction costs in a symmetric market setting. In this case, we significantly improve on the cross-sectional pricing abilities of the local SDF as well as the global SDF with significantly increased R^2s.

We can further improve upon the pricing performance by imposing asymmetric market settings as the ensuing SDFs feature more robust properties, e.g., lower volatilities. For example, when pricing only the currency returns, proportional transaction cost entails better R^2s and statistical significance in estimating the risk premium. In fact, when we assume that markets are asymmetric and global investors face proportional transaction costs, the global SDF alone explains 81% of the variation in the currency returns which is twice the R^2 in absence of market frictions. Pricing the full cross-section of assets further improves R^2: Global and currency basket factors explain between 70% to 80% of the cross-sectional variation of international stocks, bonds and currency returns.

[Insert Tables 4 to 5 here.]

Figure 8 plots the expected excess return as function of the exposure to out-of-sample global SDF risk factor for the USD denominated economy. The ability of global SDF alone to capture the variation within and across asset classes increases as frictions and asymmetry are introduced. Moreover, as in the in-sample case but more evident here, the variation of the exposure to the global SDF risk factor increases with market friction and asymmetry.

[Insert Figure 8 here.]

We can compare these numbers to those in the literature. For example, Aloosh and Bekaert (2019) report R^2s of 60% with an associated RMSE of 0.11 when pricing the cross-section of currency returns with a currency basket. However, notice that their estimates are all in- and not out-of-sample. Our out-of-sample R^2s are 20% larger with a RMSE which is similar in size. We overall conclude that global SDF is alone significant priced risk factor with excellent out of sample performance not just for currencies but also stocks and bonds.

4.6 Home Bias in Global Portfolios

Proposition 3 posits that portfolios are identical across different currency denominations as long as markets are symmetric. This, however, is in strong contradiction to empirical evidence that documents a strong currency home bias in international portfolios. For example, French and Poterba (1991) and Lewis (1999), and more recently Camanho, Hau, and Rey (2019), show that the proportion of domestic stocks invested in portfolios exceeds their country's relative market capitalization in the world. This home bias phenomenon extends to bonds and is found to be even more pronounced, see, e.g., Maggiori, Neiman, and Schreger (2019). To rationalize home bias, Coeurdacier and Rey (2013) review three explanations: (i) hedging motives in frictionless markets, (ii) asset trade costs in international financial markets (such as transaction costs, differences in tax treatments between national and foreign assets or differences in legal frameworks), and (iii) information frictions and behavioral biases.[21]

While we can measure transaction costs such as bid-ask spreads, it is harder to quantify other types of asset trade costs such as the cost to trade via an intermediary or the role of international taxation. Intangible costs such as information frictions and behavioral biases are even harder to quantify, see, e.g., Coeurdacier and Rey (2013) for a discussion.[22] It is therefore natural to assume that the observed costs such as bid-ask spreads represent a lower bound to the true costs of trading foreign assets.

In the following, we can use our framework to estimate the unobservable cost of trading foreign assets such that portfolio holdings line up with the home bias observed in the data. To this end, we incrementally increase foreign transaction costs relative to local transaction costs and study its effect on optimal portfolio weights.

[Insert Figure 9 here.]

Figure 9 plots the optimal portfolio weights for each currency assuming that transaction costs on the foreign assets are six times as big as on the local assets.[23] As we note, there is an almost perfect home bias in equities and long-term bonds for each currency. For nearly all currencies, local

[21] Most studies that empirically explore the effect on asset trade costs conclude that the costs would need to be unrealistically high to explain the level of home bias observed in the data. For example, French and Poterba (1991) argue in a mean-variance framework that these costs must be several hundred basis points. However, a different strand of the literature argues that if diversification benefits are small across countries, then these costs can be small and still explain home bias, see, e.g., Martin and Rey (2004) and Bhamra, Coeurdacier, and Guibaud (2014).

[22] Van Nieuwerburgh and Veldkamp (2009) study a model of home bias with informational frictions in international markets and link information asymmetry to earning forecasts, investors behavior, or pricing errors.

[23] To save space, we do not present the intermediate figures in the paper.

investors short the local long-term bond and have a long position in the local equity index. At the same time, we notice that across all currency denominations, investors trade the carry. For example, US investors short the US short-term bond while holding a long position in the Australian short-term bond. Australian global investors, on the other hand, short the Japanese Yen and buy New Zealand short-term bonds. This implies that independent of the foreign transaction cost, investors trade carry.

To get a sense of the implied cost to achieve home bias, recall that the average bid-ask spread is around 2bps in our data sample which implies that the "hidden" costs are around 14bps. As mentioned earlier, these costs can include differences in taxation, behavioral or informational costs, as well as intermediation costs. Overall, we conclude that even small transaction costs can lead to highly currency biased portfolios.

5 Conclusions

This paper develops a theoretical model-free framework that allows us to identify global risk factors from a large cross-section of international assets such as stocks, bonds, and currencies when investors face barriers to trade. Intuitively, limiting the allocation of wealth that is invested into risky assets leads to sparse portfolios, endogenously segmented markets, and hence robust properties of international SDFs.

Our main theoretical contribution is twofold. First, under the assumption of market symmetry, we show that we can always uniquely recover the exchange rate appreciation from the ratio of foreign and domestic SDFs even in the presence of frictions. This result is very useful when studying the global factor structure of exchange rates. Intuitively, the fact that the cross-section of (log) exchange rate changes are exactly reproduced by the cross-section of (log) SDFs, directly implies that the factor structure of exchange rates can be described by the factor structure of international SDFs. In addition because our SDFs are numéraire-invariant, it does not matter whether exchange rates are vis-à-vis one currency or another. Second, we characterize international SDFs using two factor representations, (1) a one-factor model where the factor, *global SDF*, is currency-independent and pertains to the average optimal portfolio of global investors and (2) a one-factor model where the factor pertains to both to the global SDF and a *currency basket* given by the average appreciation of local currency against the foreign currencies.

We then use this framework to estimate international SDFs from the cross-section of stocks, and short- and long-term bonds for developed countries. When international agents face no barriers to trade, SDFs need to exactly price all assets which leads to volatile SDFs and perfect correlation, because the AMV holds. Imposing market frictions significantly reduces the volatility and we also find correlations to be nearly perfect, the reason being that investors hold almost identical portfolios. Closer inspection of these portfolios reveals that global investors take their biggest exposures in the classical carry trade and long positions in USD and CHF equities and short position in JPY equity. Long-term bonds, on the other hand, enter only in relatively small short positions in international investors' portfolios.

We analyze the ability of our factor representations in pricing separately and jointly the cross-section of currency returns, bonds and equities. Both the in-sample and out-of sample results suggest

that the currency-independent global SDF risk factor alone captures most of the cross-sectional variation both within and across asset classes. Moreover, introducing market frictions and market asymmetries allows to have a better out-of sample fit of the cross-section of international assets as a result of robust minimum entropy SDFs. Indeed, when international investors face asymmetric proportional transaction costs, the global SDF factor alone, captures up to 90% of the in-sample and 75% of the out-of-sample cross-sectional variation across all denominations.

Finally, we estimate the cost to obtain the home bias in portfolio allocation observed in the data, measured as the discrepancy between transaction costs one faces in investing in home and foreign assets, and find it to be very small.

Figures

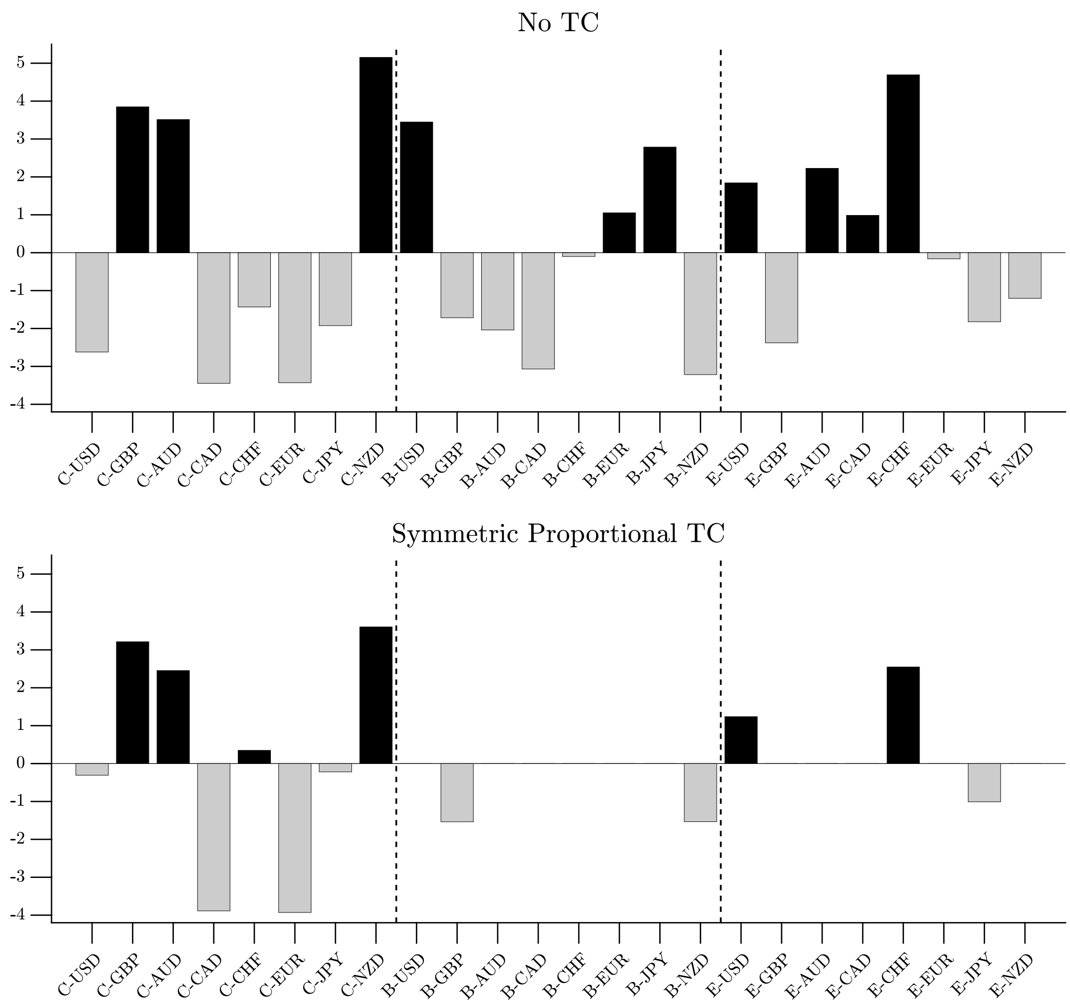

Figure 1. **Optimal Weights: Symmetric Markets.** The upper panel plots the portfolio weights in each asset denominated in USD assuming that investors face no trade barriers. The lower panel plots the portfolio weights in each asset denominated in USD assuming that investors face symmetric proportional transaction costs. C-XXX is the short-term bond for currency XXX, i.e. currency returns,, B-XXX is the long-term bond for currency XXX, E-XXX is the equity index for currency XXX. Data is monthly and runs from January 1988 to December 2015.

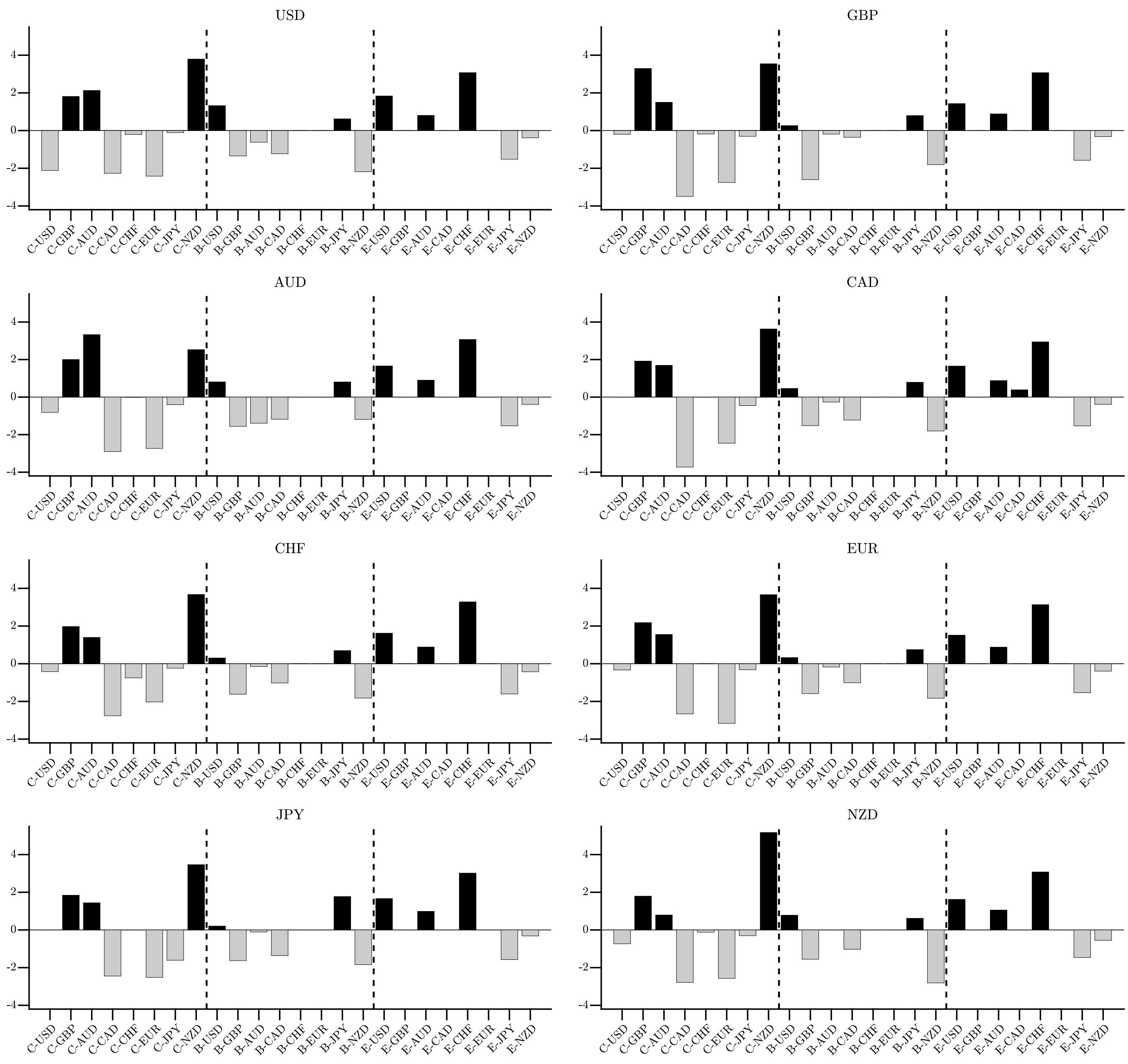

Figure 2. **Optimal Weights: Bid-Ask Spreads.** This figure plots the portfolio weights in each asset for all currency denominations assuming that investors face bid-ask spreads. C-XXX is the short-term bond for currency XXX, i.e. currency returns,, B-XXX is the long-term bond for currency XXX, E-XXX is the equity index for currency XXX. Data is monthly and runs from January 1988 to December 2015.

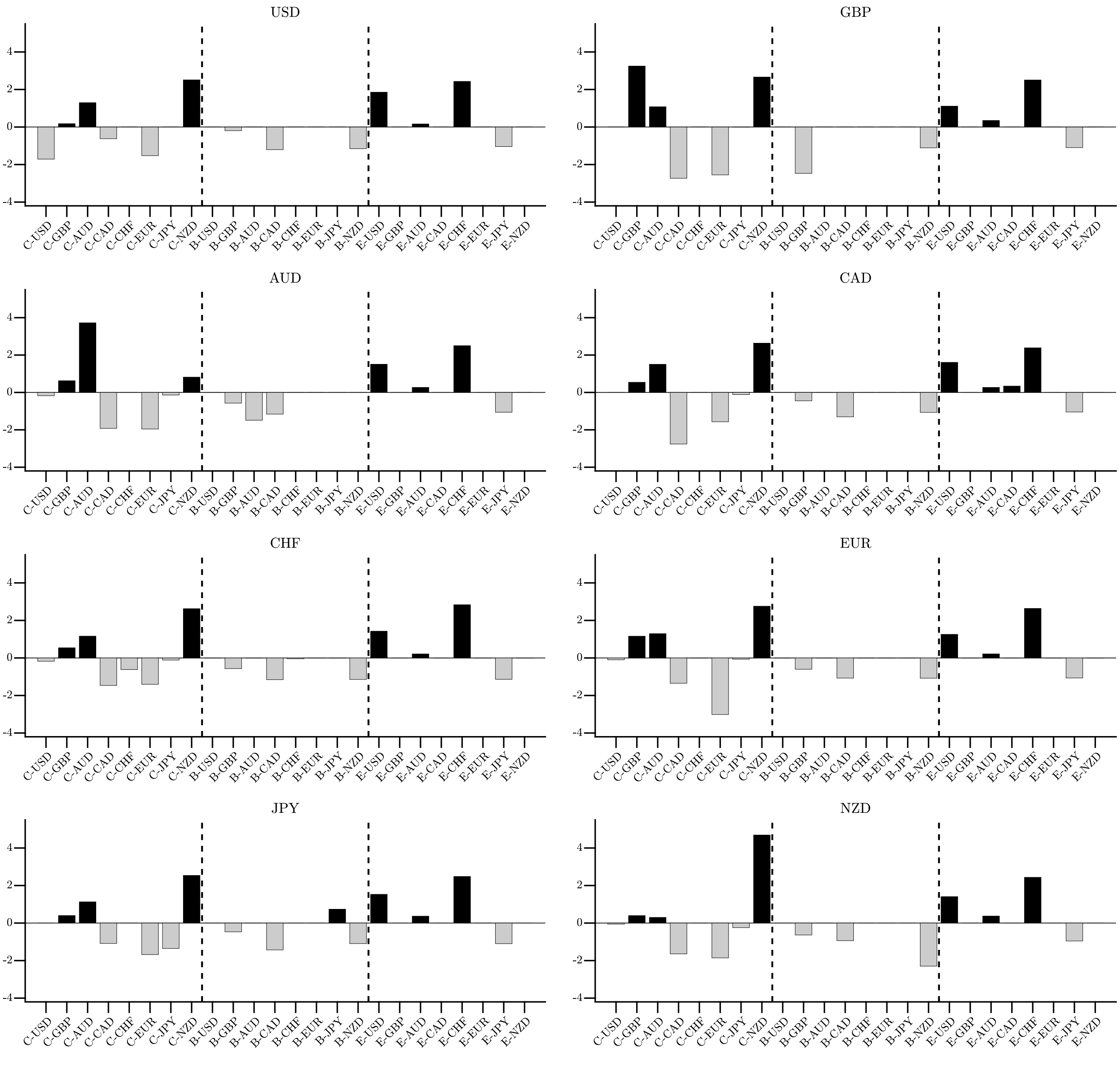

Figure 3. **Optimal Weights: Asymmetric Proportional Transaction Costs.** This figure plots the portfolio weights in each asset for all currency denominations assuming that investors face proportional transaction costs. C-XXX is the short-term bond for currency XXX, i.e. currency returns,, B-XXX is the long-term bond for currency XXX, E-XXX is the equity index for currency XXX. Data is monthly and runs from January 1988 to December 2015.

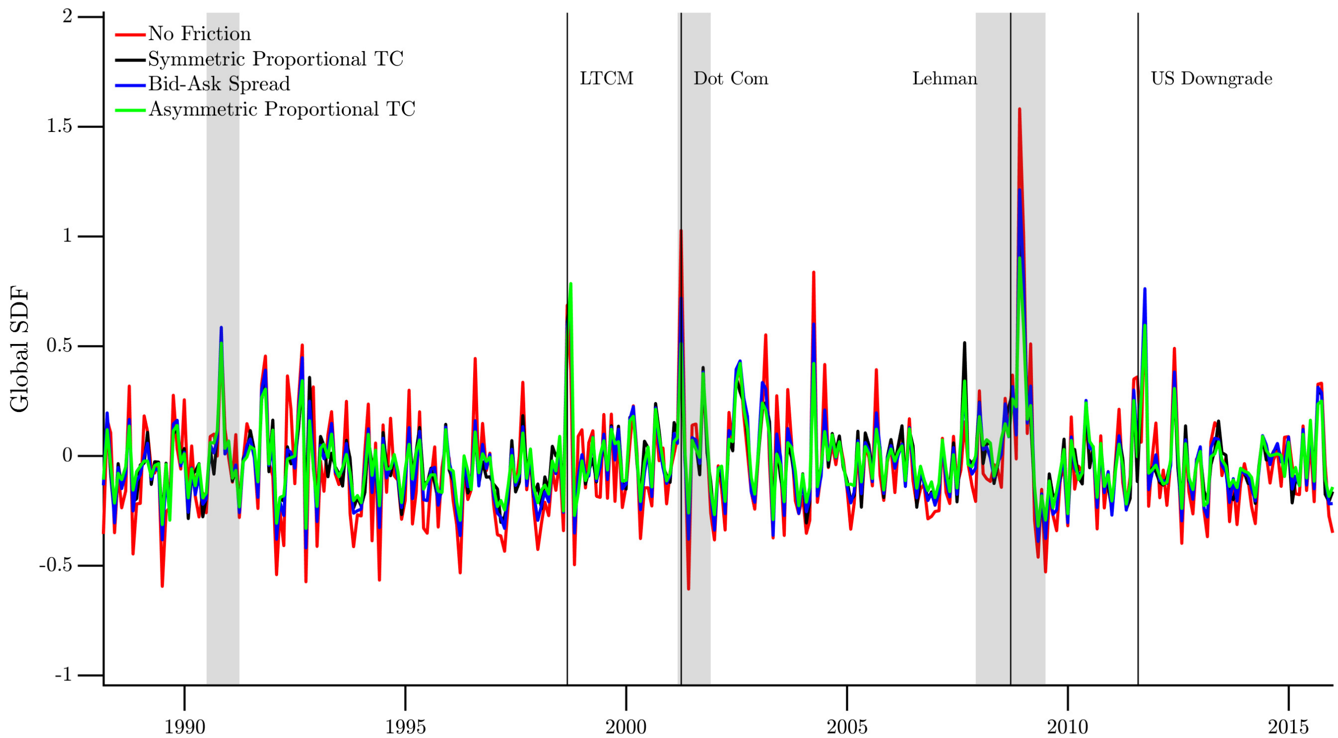

Figure 4. **Global SDF factor.** This figure plots the currency-independent global SDF risk factor, i.e., $\overline{M}_0 = \exp(G)$, where $G = -\frac{1}{M}\sum_{j=1}^{M} \ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)}$, estimated with no frictions (red), with symmetric proportional transaction costs (black), with bid-ask spreads (blue) and asymmetric proportional transaction cost (green). Gray bars indicate recessions according to NBER. Data is monthly and runs from January 1988 to December 2015.

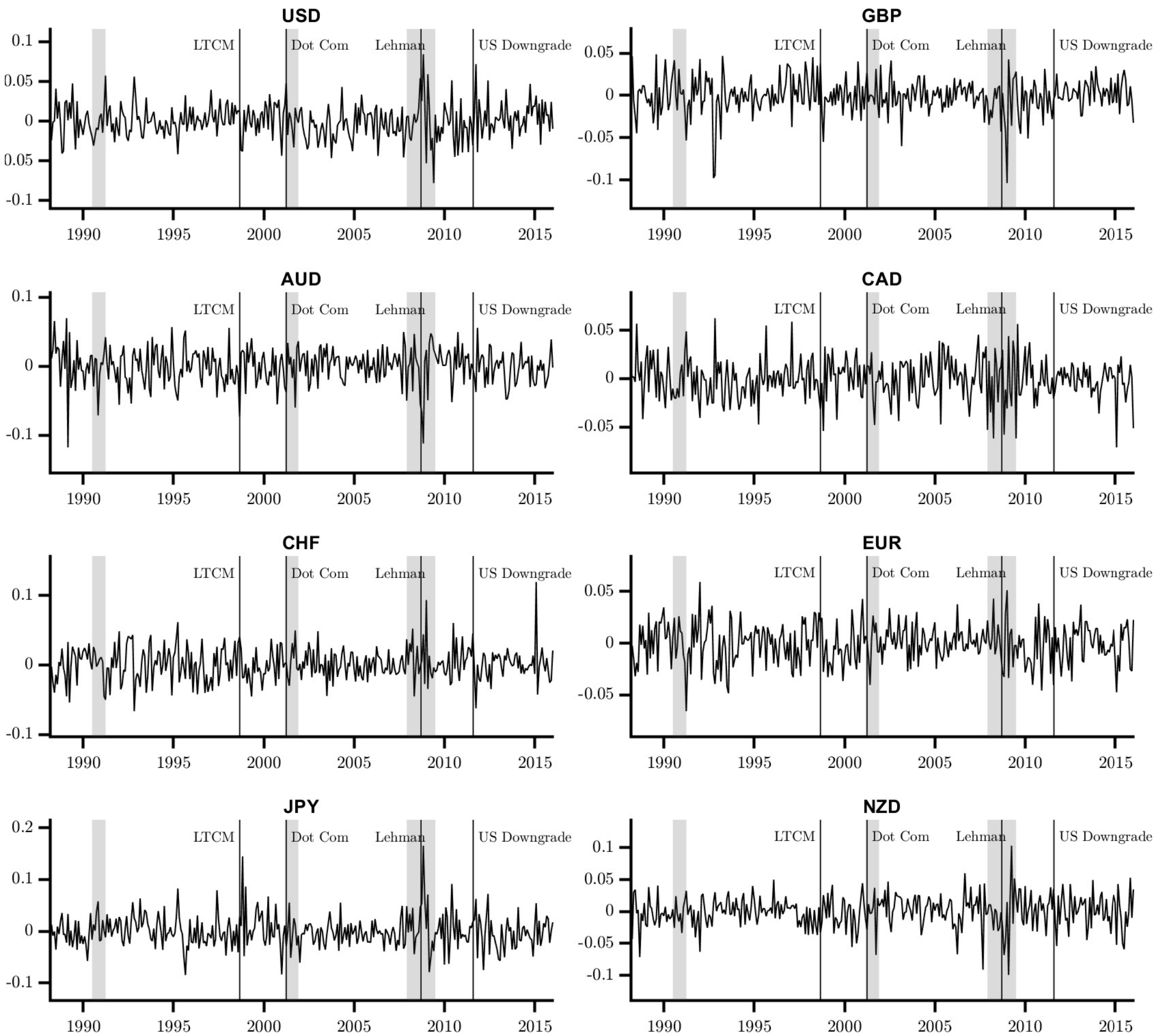

Figure 5. **Local currency baskets.** This figure plots the country-specific currency basket, i.e., the average appreciation of each local currency with respect to the remaining currencies. Gray bars indicate recessions according to NBER. Data is monthly and runs from January 1988 to December 2015.

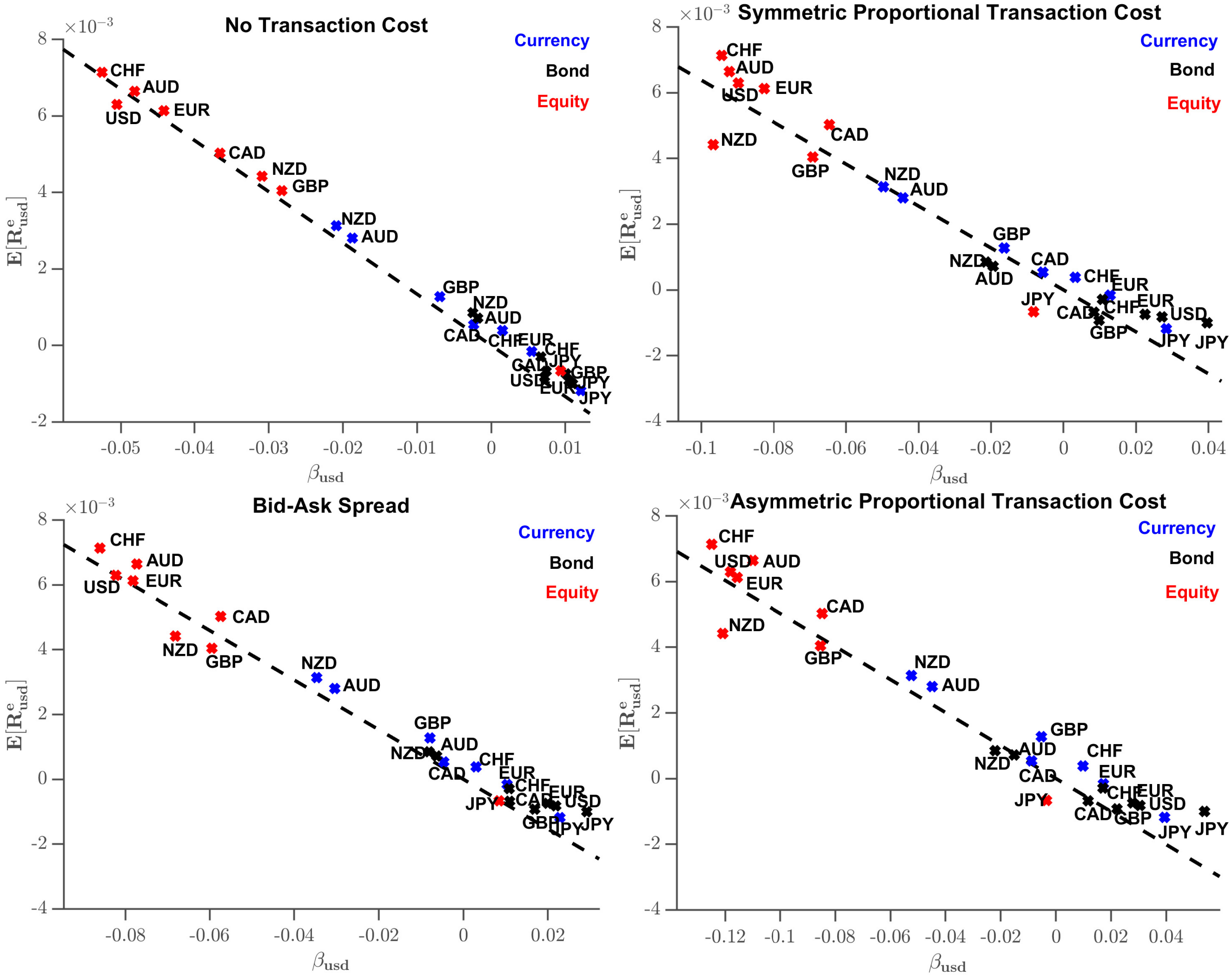

Figure 6. **In-Sample Risk-Return(USD) estimation with global SDF.** The upper panel reports the in-sample risk-return relation for USD-denominated currencies, bonds and equities in symmetric market settings, no transaction costs (left) and symmetric proportional transaction costs (right). The lower panel corresponds to asymmetric market settings, Bid-Ask spreads (left) and asymmetric proportional transaction costs (right). The figures report the relation between the expected excess returns (y-axis) and the risk factor exposures (x-axis), where the factor depends on only the global SDF, i.e., $\overline{M}_0 := \exp(G)$, where $G = -\frac{1}{M}\sum_{j=1}^{M} \ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)}$. The dashed line corresponds to the coefficient, λ, in the cross-sectional regression $E[\boldsymbol{R}^e_{USD}] = \lambda \beta_{USD} + \xi$, where the factor loading β_{USD} is instead the coefficient in the time series regression $\boldsymbol{R}^e_{USD} = \beta_0 + \beta_{USD}\overline{M}_0 + \epsilon$. Data is monthly and runs from January 1988 to December 2015.

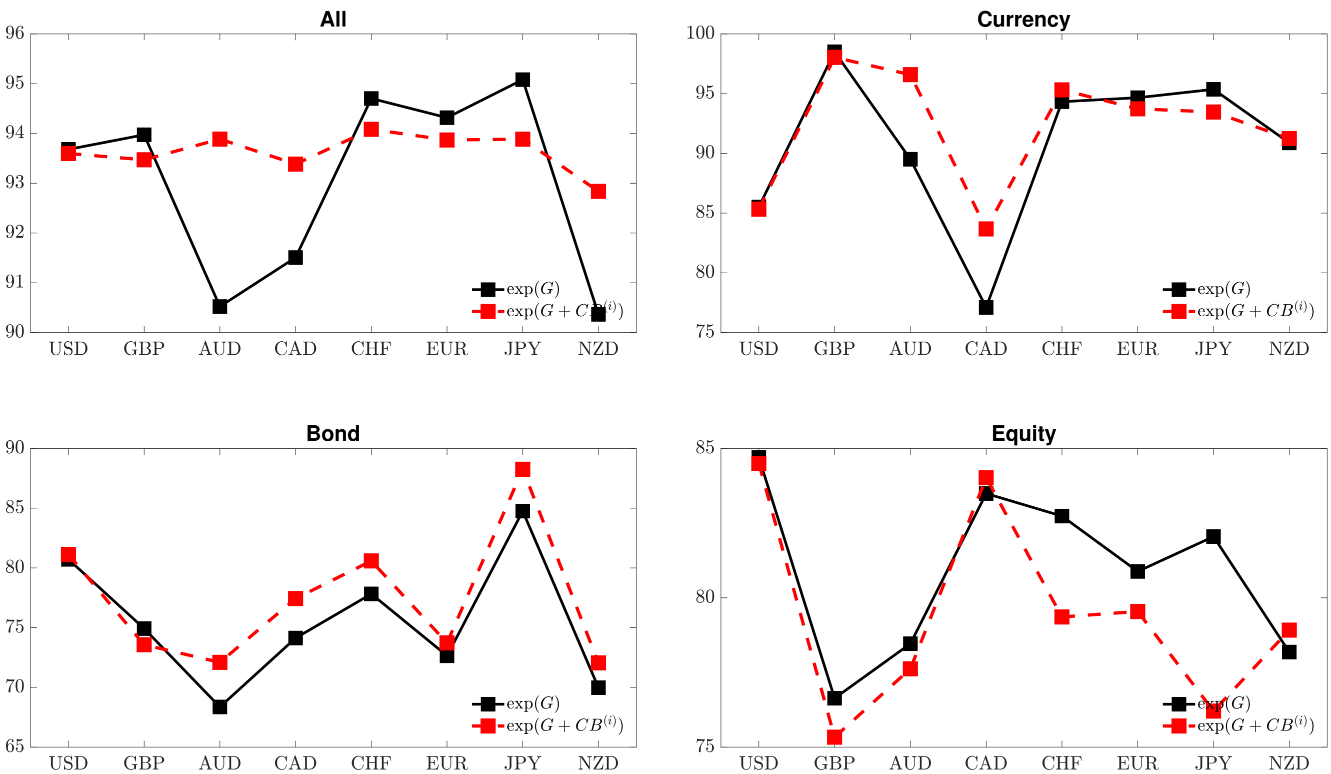

Figure 7. **In-sample cross sectional pricing across all denominations.** This figure reports the cross-sectional variation explained in asymmetric proportional transaction cost setting by (1) a factor model using only the global factor, $\overline{M}_0$ in continuous black line and (2) a factor model using both the global factor and the local currency basket factor, $\widetilde{M}_0^{(i)}$, in dashed red line. On y-axis the cross-sectional R^2 is reported when pricing all assets (top-left), currency returns (top-right), long-term bonds (bottom-left) and international equity indices (borrom-right). Data is monthly and runs from January 1988 to December 2015.

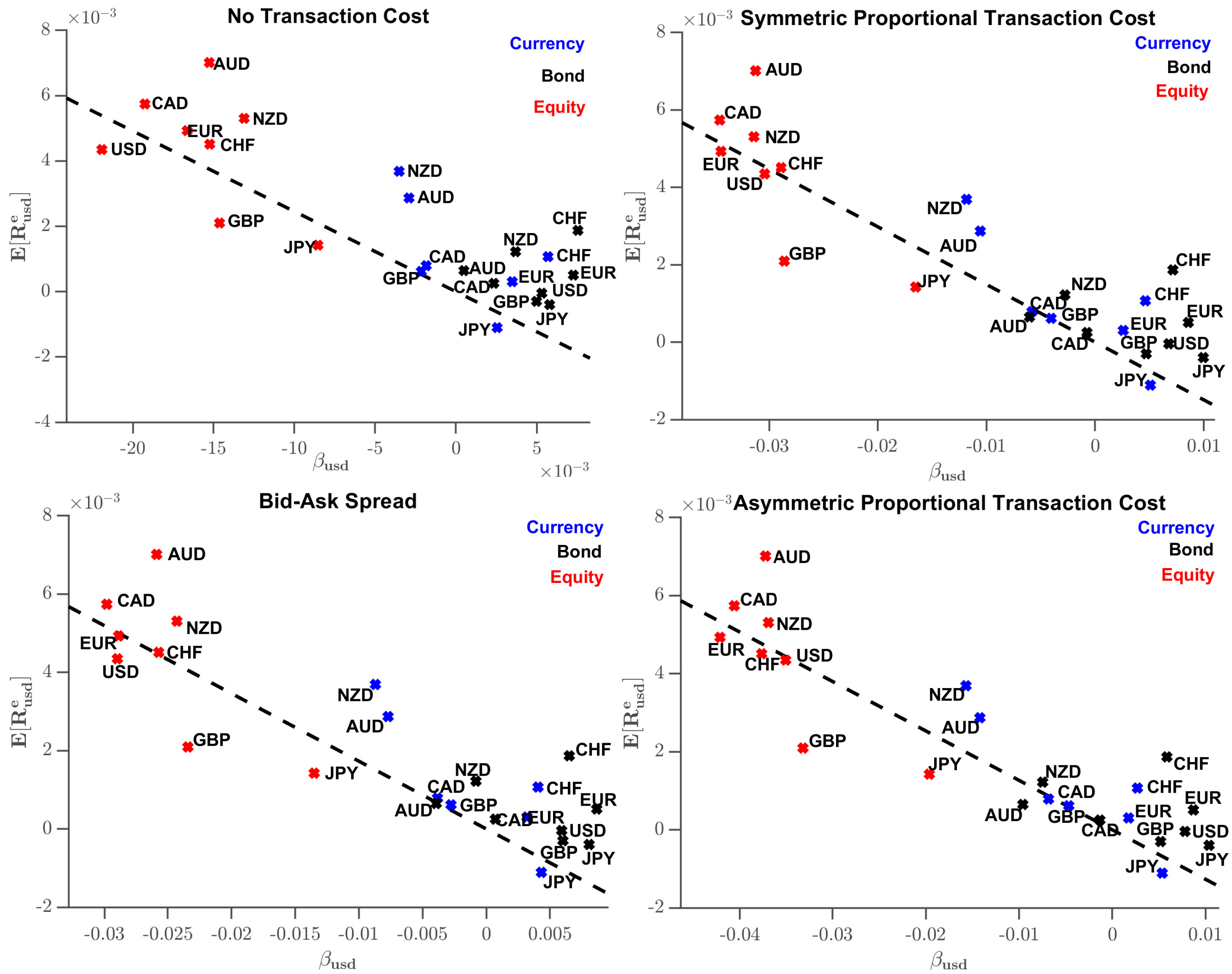

Figure 8. **Out-of-Sample Risk-Return(USD) estimation with global SDF.** The upper panel reports the out-of-sample risk-return relation for USD-denominated currencies, bonds and equities in symmetric market settings, no transaction costs (left) and symmetric proportional transaction costs (right). The lower panel corresponds to asymmetric market settings, Bid-Ask spreads (left) and asymmetric proportional transaction costs (right). The figures report the relation between the expected excess returns (y-axis) and the risk factor exposures (x-axis), where the factor depends on only the global SDF, i.e., $\overline{M}_0 := \exp(G)$, where $G = -\frac{1}{M}\sum_{j=1}^{M} \ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)}$. The dashed line corresponds to the coefficient, λ, in the cross-sectional regression $E[\boldsymbol{R}^e_{USD}] = \lambda \beta_{USD} + \xi$, where the factor loading β_{USD} is instead the coefficient in the time series regression $\boldsymbol{R}^e_{USD} = \beta_0 + \beta_{USD}\overline{M}_0 + \epsilon$. Data is monthly and runs from January 1988 to December 2015.

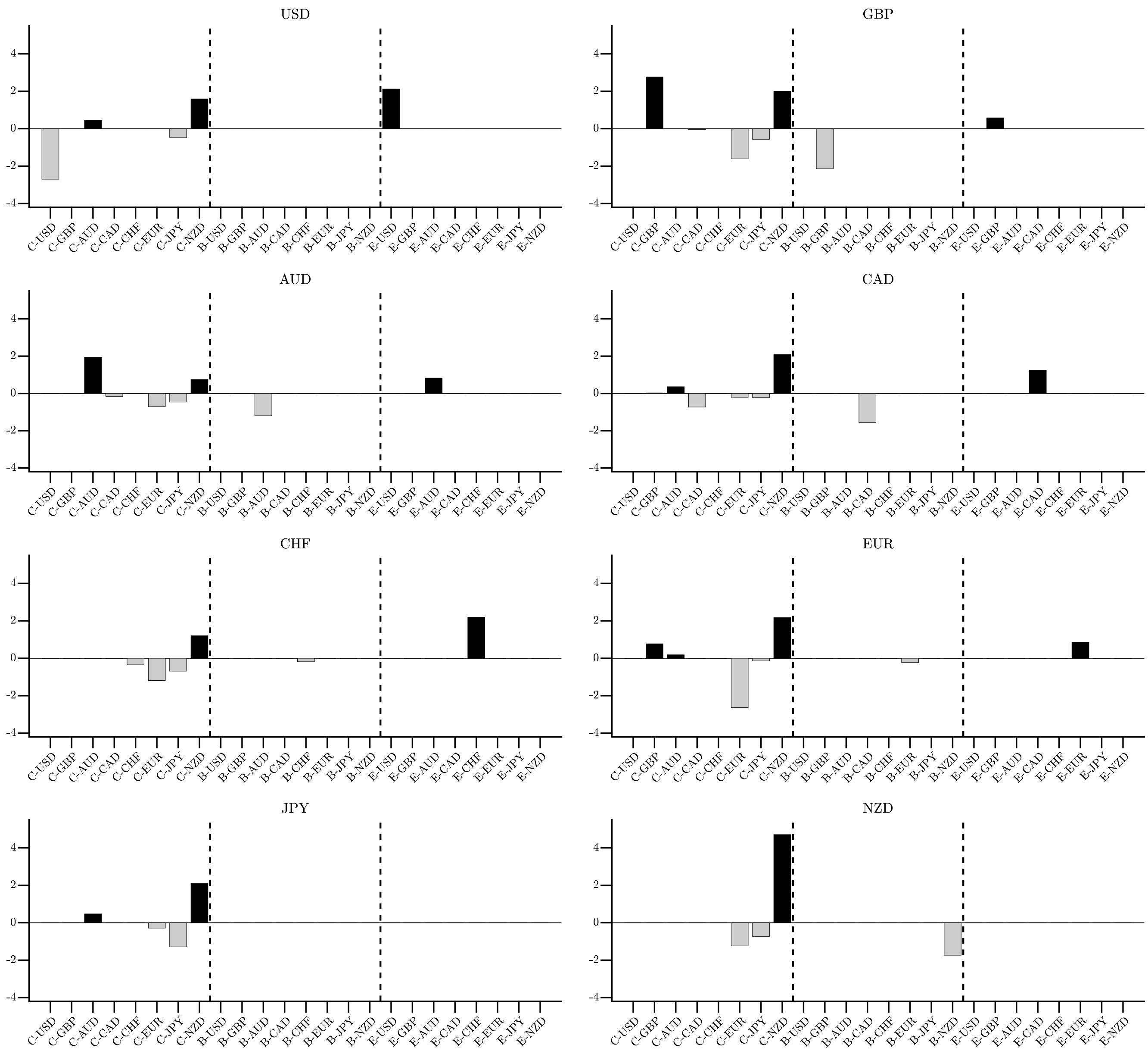

Figure 9. **Home Bias.** This figure plots the optimal portfolio weights in an asymmetric market setting where transaction costs on foreign assets are six times larger than in local markets. Data is monthly and runs from January 1988 to December 2015.

Tables

Table 1. **Summary Statistics Global SDFs**

This table reports summary statistics for minimum entropy SDFs denominated in different currencies assuming four different market structures: no market frictions, symmetric transaction costs, bid-ask spreads, and asymmetric transaction costs. Data is monthly and runs from January 1988 to December 2015.

	Symmetric Markets								**Asymmetric Markets**							
	USD	GBP	AUD	CAD	CHF	EUR	JPY	NZD	USD	GBP	AUD	CAD	CHF	EUR	JPY	NZD
	Panel A: No Frictions								*Panel C: Asymmetric Bid-Ask Spreads*							
mean	0.997	0.996	0.995	0.996	0.998	0.997	0.999	0.995	0.997	0.996	0.995	0.996	0.998	0.997	0.999	0.995
stdev	0.359	0.347	0.348	0.352	0.355	0.361	0.369	0.341	0.265	0.256	0.248	0.255	0.257	0.264	0.270	0.246
	Panel B: Symmetric Proportional Transaction Cost								*Panel D: Asymmetric Proportional Transaction Cost*							
mean	0.997	0.996	0.995	0.996	0.998	0.997	0.999	0.995	0.997	0.996	0.995	0.996	0.998	0.997	0.999	0.995
stdev	0.234	0.225	0.221	0.230	0.233	0.237	0.245	0.217	0.200	0.194	0.184	0.196	0.194	0.206	0.207	0.185
	SDF Correlations No Frictions								*SDF Correlation Asymmetric Bid-Ask Spreads*							
	USD	GBP	AUD	CAD	CHF	EUR	JPY	NZD	USD	GBP	AUD	CAD	CHF	EUR	JPY	NZD
AUD	1.00	1.00	0.99	1.00	0.99	1.00	1.00	0.99	1.00	0.98	0.98	0.98	0.97	0.98	0.97	0.98
CAD	1.00	1.00	0.99	1.00	1.00	1.00	0.99	0.99	0.98	1.00	0.99	0.99	0.98	0.99	0.98	0.99
CHF	0.99	0.99	1.00	1.00	0.99	1.00	0.99	1.00	0.98	0.99	1.00	0.99	0.97	0.99	0.98	0.99
EUR	1.00	1.00	1.00	1.00	0.99	1.00	0.99	1.00	0.98	0.99	0.99	1.00	0.98	0.98	0.98	0.99
GBP	0.99	1.00	0.99	0.99	1.00	1.00	0.99	1.00	0.97	0.98	0.97	0.98	1.00	0.99	0.97	0.98
JPY	1.00	1.00	1.00	1.00	1.00	1.00	0.99	1.00	0.98	0.99	0.99	0.98	0.99	1.00	0.98	0.99
NZD	1.00	0.99	0.99	0.99	0.99	0.99	1.00	0.99	0.97	0.98	0.98	0.98	0.97	0.98	1.00	0.98
USD	0.99	0.99	1.00	1.00	1.00	1.00	0.99	1.00	0.98	0.99	0.99	0.99	0.98	0.99	0.98	1.00
	SDF Correlation Proportional Transaction Cost								*SDF Correlation Proportional Transaction Cost*							
AUD	1.00	0.99	0.99	0.99	0.99	0.99	0.99	0.99	1.00	0.97	0.96	0.95	0.95	0.95	0.96	0.96
CAD	0.99	1.00	0.98	0.99	0.99	0.99	0.99	0.99	0.97	1.00	0.98	0.97	0.97	0.98	0.97	0.98
CHF	0.99	0.98	1.00	0.99	0.99	0.99	0.98	0.99	0.96	0.98	1.00	0.97	0.97	0.97	0.96	0.99
EUR	0.99	0.99	0.99	1.00	0.99	0.99	0.99	0.99	0.95	0.97	0.97	1.00	0.95	0.95	0.94	0.97
GBP	0.99	0.99	0.99	0.99	1.00	1.00	0.99	0.99	0.95	0.97	0.97	0.95	1.00	0.98	0.97	0.98
JPY	0.99	0.99	0.99	0.99	1.00	1.00	0.99	0.99	0.95	0.98	0.97	0.95	0.98	1.00	0.95	0.98
NZD	0.99	0.99	0.98	0.99	0.99	0.99	1.00	0.99	0.96	0.97	0.96	0.94	0.97	0.95	1.00	0.97
USD	0.99	0.99	0.99	0.99	0.99	0.99	0.99	1.00	0.96	0.98	0.99	0.97	0.98	0.98	0.97	1.00

Table 2. **Risk Prices FX: In-Sample**

This table reports estimated in-sample risk prices in the two-step Fama and MacBeth (1973) regressions of USD denominated currency excess returns on estimated factor loadings of (1) the corresponding local SDF (M_0^{usd}), (2) approximation of the latter with the global SDF $\left(G = -\frac{1}{M}\sum_{j=1}^{M} \ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)}\right)$, i.e., $\overline{M}_0 := \exp(\text{G})$, and (3) approximation with both the global SDF and the local currency basket factor, i.e., $\widetilde{M}_0^{usd} := \exp(\text{G} + CB^{(i)})$. The global SDF factor is the average SDF calculated from the cross-section of all local SDFs. The currency basket factor is the average appreciation of the local currency. Shanken (1992)-corrected standard errors are reported in brackets. Labels (**) and (*) denote significance at the 1% and 5% level, respectively. Data runs from January 1988 to December 2015.

Symmetric Markets									
Panel A: No Frictions					*Panel B: Proportional TC*				
M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)	M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)
-0.129^{**}			100.000	0.000	-0.055^{**}			100.000	0.000
[0.073]					[0.030]				
	-0.142^{**}		94.087	0.044		-0.060^{**}		94.252	0.043
	[0.073]					[0.029]			
		-0.126^{**}	99.883	0.005			-0.054^{**}	99.886	0.005
		[0.073]					[0.030]		
Asymmetric Markets									
Panel C: Bid-Ask spreads					*Panel D: Proportional TC*				
M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)	M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)
-0.075^{**}			98.512	0.018	-0.048^{**}			94.748	0.033
[0.044]					[0.029]				
	-0.083^{**}		91.020	0.056		-0.054^{**}		85.510	0.075
	[0.041]					[0.026]			
		-0.075^{**}	99.529	0.011			-0.053^{**}	85.332	0.075
		[0.042]					[0.026]		

Table 3. **Risk Prices All: In-Sample**

This table reports estimated in-sample risk prices in the two-step Fama and MacBeth (1973) regressions of USD denominated currency, bond and equity excess returns on estimated factor loadings of(1) the corresponding local SDF (M_0^{usd}), (2) approximation of the latter with the global SDF $\left(G = -\frac{1}{M}\sum_{j=1}^{M} \ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)}\right)$, i.e., $\overline{M}_0 :=$ $\exp(\mathrm{G})$, and (3) approximation with both the global SDF and the local currency basket factor, i.e., $\widetilde{M}_0^{usd} :=$ $\exp(\mathrm{G} + CB^{(i)})$. The global SDF factor is the average SDF calculated from the cross-section of all local SDFs. The currency basket factor is the average appreciation of the local currency. Shanken (1992)-corrected standard errors are reported in brackets. Labels (**) and (*) denote significance at the 1% and 5% level, respectively. Data runs from January 1988 to December 2015.

Symmetric Markets									
Panel A: No Frictions					Panel B: Proportional TC				
M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)	M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)
-0.129^{**}			100.000	0.000	-0.060^{**}			91.505	0.081
[0.050]					[0.025]				
	-0.134^{**}		98.096	0.043		-0.064^{**}		93.092	0.079
	[0.049]					[0.025]			
		-0.128^{**}	99.962	0.006			-0.059^{**}	90.931	0.083
		[0.050]					[0.025]		

Asymmetric Markets									
Panel C: Bid-Ask spreads					Panel D: Proportional TC				
M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)	M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)
-0.073^{**}			97.384	0.045	-0.047^{**}			91.962	0.076
[0.029]					[0.020]				
	-0.077^{**}		96.441	0.059		-0.052^{**}		93.680	0.078
	[0.028]					[0.019]			
		-0.073^{**}	97.594	0.044			-0.051^{**}	93.597	0.079
		[0.029]					[0.020]		

Table 4. **Risk Prices FX: Out-of-Sample**

This table reports estimated out-of-sample risk prices in the two-step Fama and MacBeth (1973) regressions of USD denominated currency excess returns on estimated factor loadings of (1) the corresponding local SDF (M_0^{usd}), (2) approximation of the latter with the global SDF $\left(G = -\frac{1}{M}\sum_{j=1}^{M} \ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)}\right)$, i.e., $\overline{M}_0 := \exp(\mathrm{G})$, and (3) approximation with both the global SDF and the local currency basket factor, i.e., $\widetilde{M}_0^{usd} := \exp(\mathrm{G} + CB^{(i)})$. The global SDF factor is the average SDF calculated from the cross-section of all local SDFs. The currency basket factor is the average appreciation of the local currency. Shanken (1992)-corrected standard errors are reported in brackets. Labels (**) and (*) denote significance at the 1% and 5% level, respectively. Data runs from January 1988 to December 2015.

Symmetric Markets									
Panel A: No Frictions					*Panel B: Proportional TC*				
M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)	M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)
-0.273			44.058	0.180	-0.229^{**}			73.499	0.099
[0.246]					[0.182]				
	-0.242		41.971	0.186		-0.232^{**}		71.440	0.108
	[0.241]					[0.180]			
		-0.369	53.986	0.149			-0.212^{**}	78.897	0.081
		[0.306]					[0.174]		
Asymmetric Markets									
Panel C: Bid-Ask spreads					*Panel D: Proportional TC*				
M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)	M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)
-0.286^{**}			74.165	0.099	-0.190^{**}			82.661	0.069
[0.225]					[0.158]				
	-0.294^{*}		68.524	0.120		-0.205^{**}		81.585	0.079
	[0.221]					[0.162]			
		-0.278^{**}	79.001	0.084			-0.177^{**}	86.019	0.061
		[0.223]					[0.147]		

Table 5. **Risk Prices All: Out-of-Sample**

This table reports estimated out-of-sample risk prices in the two-step Fama and MacBeth (1973) regressions of USD denominated currency, bond and equity excess returns on estimated factor loadings of (1) the corresponding local SDF (M_0^{usd}), (2) approximation of the latter with the global SDF $\left(G = -\frac{1}{M}\sum_{j=1}^{M} \ln \boldsymbol{\theta}_0' \boldsymbol{R}^{(j)}\right)$, i.e., $\overline{M}_0 := \exp(\mathrm{G})$, and (3) approximation with both the global SDF and the local currency basket factor, i.e., $\widetilde{M}_0^{usd} := \exp(\mathrm{G} + CB^{(i)})$. The global SDF factor is the average SDF calculated from the cross-section of all local SDFs. The currency basket factor is the average appreciation of the local currency. Shanken (1992)-corrected standard errors are reported in brackets. Labels $(^{**})$ and $(^{*})$ denote significance at the 1% and 5% level, respectively. Data runs from January 1988 to December 2015.

Symmetric Markets

Panel A: No Frictions					*Panel B: Proportional TC*				
M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)	M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)
-0.248^{**}			68.773	0.176	-0.149^{**}			79.110	0.125
[0.170]					[0.104]				
	-0.246^{**}		68.050	0.180		-0.150^{**}		78.444	0.129
	[0.168]					[0.104]			
		-0.245^{**}	72.933	0.155			-0.143^{**}	81.043	0.114
		[0.170]					[0.100]		

Asymmetric Markets

Panel C: Bid-Ask spreads					*Panel D: Proportional TC*				
M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)	M_0^{usd}	$\overline{M}_0$	$\widetilde{M}_0^{usd}$	R^2(%)	RMSE(%)
-0.170^{**}			78.603	0.127	-0.126^{**}			81.204	0.110
[0.116]					[0.088]				
	-0.174^{**}		76.942	0.138		-0.129^{**}		80.554	0.117
	[0.117]					[0.089]			
		-0.167^{**}	80.299	0.119			-0.122^{**}	82.459	0.105
		[0.114]					[0.085]		

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Appendix A Proofs

Proof of Proposition 1

Proof. Sublinearity of the transaction cost function h implies that of the price function, π, and that the set of payoffs, $\mathcal{Y}$, is a convex cone. Hence, by Chen (2001, Theorem 5) and Clark (1993, Theorem 6), the arbitrage-free condition is equivalent to the no-free-lunch condition (Harrison and Kreps (1979)). By Chen (2001, Theorem 1), there exists a strictly positive, continuous linear functional ψ defined on L^q, such that $\psi|_{\mathcal{Y}} \leq \pi$. Moreover, by the Riesz Representation Theorem, there exists a strictly positive element, M, in the dual space of L^q, i.e. L^p, such that $\psi(Y) = \mathbb{E}[MY]$. Hence, we have that $\mathbb{E}[MY] \leq \pi(Y)$ for all Y in $\mathcal{Y}$. The definition of the price function, π implies that

$$\boldsymbol{\theta}_S'\mathbb{E}[M\boldsymbol{Z}_S] + \boldsymbol{\theta}_D'\mathbb{E}[M\boldsymbol{Z}_D] \leq \boldsymbol{\theta}_S'\boldsymbol{P}_S + \boldsymbol{\theta}_D'\boldsymbol{P}_D + h(\boldsymbol{\theta}_D) \tag{A.1}$$

for any $\boldsymbol{\theta} \in \mathbb{R}^N$. Hence, for any $\boldsymbol{\theta}_S \in \boldsymbol{R}^{N_S} : \boldsymbol{\theta}_S'\mathbb{E}[M\boldsymbol{Z}_S] \leq \boldsymbol{\theta}_S'\boldsymbol{P}_S$. Thus, $\mathbb{E}[M\boldsymbol{Z}_S] - \boldsymbol{P}_S = 0$. Similarly, inequality (A.1) implies that $\boldsymbol{\theta}_S'\mathbb{E}[M\boldsymbol{Z}_D] \leq \boldsymbol{\theta}_D'\boldsymbol{P}_D + h(\boldsymbol{\theta}_D)$ for all $\boldsymbol{\theta}_D \in \mathbb{R}^{N_D}$. This together with Bauschke and Combettes (2011, Prop. 13.10 (i)) implies that $h^*(\mathbb{E}[M\boldsymbol{Z}_D - \boldsymbol{P}_D]) \leq 0$, where h^* is the convex conjugate of the transaction cost function, h. Since h is closed and sublinear, by Hiriart-Urruty and Lemaréchal (2012, Theorem 3.1.1), it is a support function of C_h. The convex conjugate of a support function of C_h is given by an indicator function of the set C_h, i.e., $h^* = \delta_{C_h}$. This concludes the proof. □

Proof of Proposition 2

Proof. We define the linear operator $A : L_q \to \mathbb{R}^N$, with $A(M) := \mathbb{E}[M\boldsymbol{R}]$, then the primal problem (6) can be written as:

$$\Pi_h = \inf_{M \in L^p} \{\mathbb{E}[\phi_+(M)] \, : \, A(M) \in P\} \ , \tag{A.2}$$

where the closed convex set P is defined as $P := \{\mathbf{1}_{N_S}\} \times (\{\mathbf{1}_{N_D}\} + C_h)$ and ϕ_+ is the restriction to the nonnegative real line of ϕ. Assuming no arbitrage, we know from Proposition 1 that there exists a strictly positive element, $\bar{M}$, of $\mathcal{M}_h$. Hence, $A(\bar{M})$ is an element of the set $ri(A(\text{dom}(\phi_+))) \cap ri(P)$, where ri denotes the relative interior.[24] By Borwein and Lewis (1992, Theorem 4.2), the following duality relation then holds:

$$\Pi = -\min_{\boldsymbol{\theta} \in \mathbb{R}^N} \left\{\mathbb{E}[\phi_+^*(A^T\boldsymbol{\theta})] + \delta_P^*(-\boldsymbol{\theta})\right\} \ , \tag{A.3}$$

where A^T is the adjoint map relative to the linear operator$A$.[25] Since $\delta_P(A(M)) = \delta_{\{\mathbf{1}_{N_S}\}}(A(M)|_S) + \delta_{\{1_{N_D}\}+C_h}(A(M)|_D)$, its convex conjugate by Hiriart-Urruty and Lemaréchal (2012, Theorem 3.1.1) reads:

$$\delta_P^*(-\boldsymbol{\theta}) = -\boldsymbol{\theta}_S'\mathbf{1}_{N_S} - \boldsymbol{\theta}_D'\mathbf{1}_{N_D} + h(-\boldsymbol{\theta}_D) \, .$$

Hence, we obtain the first result. Now, under the condition that $-\boldsymbol{\theta}_0'\boldsymbol{R} \in \text{dom}(\phi_+^*)$, the link between the solutions of the primal and the dual problem becomes a direct result of Korsaye, Quaini, and Trojani (2018, Proposition 2). This concludes the proof. □

[24] The relative interior of a set A is defined as the interior of the affine hull of set A.
[25] $A^T : \mathbb{R}^N \to L^p$ such that $\mathbb{E}[A^T(\theta)M] = \theta' A(M)$.

Proof of Proposition 3

Proof. Let $M_0^{(i)}$ be the minimum entropy SDF in economy (i). It then follows:

$$\begin{aligned}
M_0^{(i)} &= \arg\min_{M\in L_+^p}\{\mathbb{E}[-\ln M] : \mathbb{E}[M\boldsymbol{R}^{(i)} - \mathbf{1}] \in C\} \\
&= \arg\min_{M\in L_+^p}\{\mathbb{E}[-\ln(M) - \ln(X_j^{(i)})] : \mathbb{E}[MX_j^{(i)}\boldsymbol{R}^{(j)} - \mathbf{1}] \in C\} \\
&= \arg\min_{M\in L_+^p}\{\mathbb{E}[-\ln(MX_j^{(i)})] : \mathbb{E}[MX_j^{(i)}\boldsymbol{R}^{(j)} - \mathbf{1}] \in C\}
\end{aligned}$$

This implies that

$$\begin{aligned}
M_0^{(i)}X_j^{(i)} &= \arg\min_{M\in X_j^{(i)}L_+^p}\{\mathbb{E}[-\ln(M)] : \mathbb{E}[M\boldsymbol{R}^{(j)} - \mathbf{1}] \in C\} \\
&= M_0^{(j)}\,.
\end{aligned}$$

□