Group-Managed Real Options: Voting, Polarization, and Investment Dynamics*

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Abstract

We analyze a dynamic investment problem where decisions are made through voting within a group of agents with heterogeneous beliefs. We show that disagreement generates inefficient *under-investment*—the group rejects projects that are unanimously deemed profitable by each member—and *inertia*—investment is delayed relative to a single-agent case. When facing both investment and abandonment timing decisions, the group behavior cannot be replicated by that of a representative or "median" member. These coordination frictions hold in groups of any size, for general voting protocols and are exacerbated by polarization, investment reversibility, and more stringent voting rules.

Keywords: group decisions, dynamic voting, real investment.

1 Introduction

Many decisions of interest in economics and finance are dynamic *group* decisions. The determination of investment and financing by a corporation, the creation and management of a startup, and the portfolio decisions of households, are just a few examples of the pervasiveness of dynamic group decisions. Despite this pervasiveness, the vast majority of dynamic models in finance abstract from the multi-agent nature of decisions, resorting instead to a "representative agent" approach. While this approach has proved useful to develop economic insights and empirical predictions, it ignores an important aspect of real-world decisions: disagreement within groups. Because heterogeneous groups can behave in a starkly different way from individuals, a deeper understanding of observed group behavior calls for theories of dynamic group decisions.

In this paper, we answer this call by studying a canonical dynamic real option problem managed by a *group* of agents with heterogeneous beliefs. The group decides, by *voting*, on whether to acquire an option that gives the right to choose when to (i) invest in a cash-flow producing project and (ii) abandon and liquidate the project. Group members disagree on the expected growth of the project's cash flows. At any time, each member of the group can propose an action, but the group will only take that action if the proposal is supported by a majority. To consider the simplest form of conflict, we ignore asymmetric information, contractual differences and learning, and focus instead on the coordination frictions that emerge from the existence of *polarized* beliefs among group members.[1] This setup is (i) simple enough to provide a clear illustration of the main economic mechanism through which heterogeneous beliefs within a group affect option exercise timing decisions; and yet (ii) general enough to encompass a large class of problems studied by the dynamic corporate finance literature (e.g., dynamic investment, capital budgeting).

We show that, compared to the case in which a single agent manages the real option, heterogeneity of beliefs within the group and voting lead to *inefficient underinvestment*, in that the group refuses to acquire the option to subsequently invest, despite the fact that each member individually would find the option worth undertaking. The presence of conflicting beliefs among group members also leads to *group inertia*: all else being equal, investment is delayed when managed by a group instead of a representative agent. These *coordination frictions* stem from the fact that, in a sequen-

[1] In finance, the information-aggregation role of voting has been studied widely. See, e.g., Warther (1998), Gillette, Noe, and Rebello (2003), Harris and Raviv (2008), Baranchuk and Dybvig (2008), Maug and Rydqvist (2008), Levit and Malenko (2011), Malenko (2014), and Chemmanur and Fedaseyeu (2017).

tial voting context like ours, the "pivotal voter," that is, the group member whose vote determines the group's decision, changes over time. When group members cannot commit to future policies, future pivotal voters may impose negative externalities on current pivotal voters and, as a result, distort group choices. An important result of this paper is to demonstrate how these distortions can lead the group to inefficient outcomes.

To provide intuition for the nature of these coordination frictions, consider first a two-member group where an optimist O (she), and a pessimist P (he) face a take-it-or-leave-it decision to invest in a cash-flow producing project. Once the project is undertaken the group can decide to abandon it. Both the take-it-or-leave-it investment decision and the abandonment *timing* decisions are determined by majority rule (unanimity with two members). Consider this problem recursively, starting with the abandonment decision. Because O sees a higher growth rate for the cash flow than P, she is more reluctant to abandon the project. Unanimity implies that the abandonment time is *de-facto* determined by O: the group abandons if and only if O decides to abandon, making her the "pivotal voter" for the abandonment decision. In contrast, at the initial investment stage, P, being more pessimistic, values the project less than O and hence is more reluctant to invest. Since P anticipates that the abandonment decision dictated by O is suboptimal from his viewpoint, he may vote against investment in the first place. In sum, the disagreement between the current pivotal voter for investment (P) and the future pivotal voter for abandonment (O) can induce P to veto investment.

The tension between pivotal voters illustrated in this example is central to our results. However, in this simple set up, the inefficient underinvestment can be easily resolved by adding a third member M (median) who holds intermediate beliefs and will be pivotal for both decisions. This is essentially an application of the "median voter theorem" (Black (1958)): the group behavior can be subsumed by the behavior of the member with median beliefs, who can be thought of as the representative agent in classic corporate finance models.

We show, however, that the median voter representation result of the group's behaviour does not carry out in more realistic settings. In our full model we add an investment timing decision and consider the more general setting in which the group faces (i) a take-it-or-leave-it decision to acquire an option (license) to choose (ii) a future investment time and (iii) a future abandonment time. In a two members group (P and O), we show that P remains pivotal for the licensing decision and O remains pivotal for the abandonment decision. However, depending on the characteristic of the project, we show that *any* member can be pivotal for the investment timing decision. In particular, although O places a higher value on the investment option than P, she can be less eager to exercise the call than P and therefore be pivotal for the investment timing decision. Hence, there is no single median voter for all decisions faced by the group.

The reason for this result is that the investment option is a *compound* call option which, upon exercise, generates the project's cash flows as well as the (put) option to abandon. At any given time, the incentive for immediate exercise of the call option depends therefore on the value of cash flows relative to that of the put option. O places a higher value than P on the cash flow component of the call's intrinsic value while P places a higher value than O on the put component. Depending on the strength of each determinant, O may prefer to exercise the option earlier or later than P. As a result, any member can be pivotal on the investment timing decision and, even if one were to add a member with median beliefs to the two-member group, the group behavior cannot be represented by the behavior of the median member.

The identity of the investment timing pivotal member has important economic implications. In this two-member group, inertia or delayed investment obtains when P is pivotal for the investment timing decision: Knowing that the group will suboptimally abandon the project, P delays investment relative to his autarkic optimum. On the other hand, when O is pivotal for the investment timing decision, underinvestment is more severe since, in this case, O is pivotal for both the investment and abandonment timing decisions, making the initial license even less attractive to P. As a result, P will have an even stronger incentive to veto the initial option acquisition.

Harnessing the tractability of our continuous-time real option setup, we obtain closed-form solutions for the value of the group-managed real option and the corresponding investment and abandonment timing decisions. We show that the coordination frictions that emerge in a group-managed real option are more severe in (i) highly polarized groups, and (ii) projects where investment is easily reversible. The first result obtains because more polarized groups produce more disagreement between pivotal voters. The second result obtains because, when investment is reversible, the put-option component of the project becomes relatively more important. As a result, pessimistic members are held up by more optimistic members for both investment and abandonment timing, making the project less attractive to pessimists. Volatility has an ambiguous effect on the investment decision: on the one hand, higher volatility delays investment as the option to wait is more valuable. However, higher volatility also increases the value of the abandonment option and this, in turn, increases the incentive to invest. The resulting effect depends on the relative moneyness of the (call) option to invest and of the (put) option to abandon. In our numerical analysis we find that the coordination friction is more severe for medium/low levels of cash flow volatility.

Our results rest on three key assumptions: (i) a group whose members hold polarized beliefs faces a sequence of decisions overtime; [2] (ii) the group settles disagreement through a voting rule such as majority, super-majority, or unanimity; (iii) group members cannot settle their disagreement by trading their decision rights or by pre-committing to future investment/abandonment policies. Sequential investment decisions, polarized beliefs, and voting fit well in many investment decisions. These assumptions accurately capture, for example, the environment of corporate boards, startups and young firms. The assumption that group members are not able to sell their decision rights to each other is descriptive of many important group decisions. For example, startups are often built on the complementary talents of founding partners who usually do not have the capital to buy each other out in case of disagreement. In the Online Appendix, however, we explore how, besides voting, the acquisition of control through a takeover or a targeted share purchase, can act as an alternative way to resolve disagreement within the group.

Finally, we show that underinvestment and inertia can occur in groups of *any* size, governed by a wide range of non-dictatorial voting rules, such as unanimity, super-majority, and majority with vetoers. Voting rules that are more stringent than majority produce more polarized pivotal voters and therefore exacerbate the group coordination frictions.

Our paper contributes to the real options literature. The early stages of this literature (Brennan and Schwartz (1985); McDonald and Siegel (1986) and Dixit and Pindyck (1994)) focus on a single decision maker who exclusively controls and benefits from the investment decision. More recent contributions to this literature extend the basic setting to investigate the effect of informational frictions on the exercise of real options such as moral hazard (Grenadier and Wang (2005)), signalling and adverse selection (Morellec and Schürhoff (2011) and Grenadier and Malenko (2011)). We add to this literature by studying the management of a real option by a group of equally-informed agents with heterogeneous beliefs. We show that the process through which a group makes decision leads to frictions that, as with informational frictions, distort timing relative to an autarkic decision maker. The sources of these frictions are, however, fundamentally different. On the one hand,

[2] Other sources of heterogeneity, such as risk aversion, discount rates, or contractual claims, will generate similar inefficiencies. Chen and Lambrecht (2019) discuss how inefficiencies due to heterogeneous risk aversion impact firm's payout policies, capital structure, and internal governance.

informational frictions result from differential access to and acquisition of information by different parties. On the other hand, the heterogeneity of beliefs at the heart of our analysis results from differences in the way that information is interpreted. To the best of our knowledge, ours is the first study that formally analyzes the exercise of real options by a group of agents with heterogeneous beliefs.

Our paper also contributes to the literature on disagreement and differences in beliefs in finance. This literature is motivated by the observation that while the "common prior" assumption is appropriate in a setting where information is plentiful and posterior beliefs have converged, it is less tenable in situations where agents lack information and/or experience (Morris (1995)). In these settings, agents would hold different views and people would agree to disagree. A large literature in finance studies the implications of disagreement on both asset prices and corporate decisions.[3] We contribute to this literature by developing a model to analyze the impact of differences in beliefs within a decision making group on the dynamics of corporate investment. Allen and Gale (1999) and Thakor and Whited (2011) also study the implication of differences in beliefs on investment. However, while they focus on the differences in beliefs between the shareholders and the manager, we explicitly model the *voting* game between members of a decision making group with heterogeneous beliefs in the absence of any other agency or information frictions. Importantly, we add to this literature by studying the implications of disagreement for investment dynamics. To the best of our knowledge, our paper is the first to address optimal investment timing by a decision making group with heterogeneous beliefs. As emphasized by Li, Maug, and Schwartz-Ziv (2020), the friction originating from disagreement among shareholders has relevant implications for corporate governance. While agency and information frictions can be addressed with, respectively, alignment of incentives and disclosure, coordination frictions due to differences in beliefs may be better addressed through trading. In the Online Appendix we extend the model to include the possibility of trade among group members and show that the possibility of tender offers among group members can lead to a more homogeneous group and mitigate underinvestment and group inertia.

[3]This literature is too vast to be reviewed here. Lintner (1965), Miller (1977), Harrison and Kreps (1978), Ross (1976) and, Scheinkman and Xiong (2003) are important contribution to the asset pricing literature that allow for differences in prior beliefs. Theoretical contributions to the corporate finance literature include Harris and Raviv (1993), Kandel and Pearson (1995), Boot, Gopalan, and Thakor (2006), and more recently, Kakhbod, Loginova, Malenko, and Malenko (2019). Empirically, earlier evidence of revealed differences in beliefs includes Kandel and Pearson (1995), Dittmar and Thakor (2007) among others. More recently, Li, Maug, and Schwartz-Ziv (2020) find empirical support for the difference in beliefs in the voting behavior of mutual funds.

Finally, at a broader level, our paper emphasizes that changes of the pivotal voter over time is a source of coordination friction that distorts current choices. The prior literature on dynamic collective decisions emphasized this point in different contexts. In a political economy model, Roberts (2015) demonstrates that group time inconsistency can arise when group members can endogenously include new members in the group.[4] In a model of corporate investment, Garlappi, Giammarino, and Lazrak (2017) show that time inconsistency can arise when group members learn in a Bayesian way from a public signal. In a model of corporate boards, Donaldson, Malenko, and Piacentino (2019) assume that board members have a status quo bias and show that the coordination friction takes the form of an inefficient deadlock for the board. Our novel insight is to show that important coordination frictions can emerge even in the absence of endogenous group size, learning, or status quo bias. In fact, they emerge rather naturally in any investment decision involving the acquisition of a compound option that involves the timing of both investment and abandonment. The interaction between the long- and short-position embedded in such an option creates coordination frictions and, implicitly, a time inconsistency in the group. Given the pervasiveness of dynamic investment/abandonment decisions, and the importance of timing decisions that we identify, our theoretical results are relevant for a wide range of problems in corporate finance.

The rest of the paper proceeds as follows. Section 2 presents the basic continuous-time model of real option exercise by a group of two members that makes decisions through a unanimity voting rule. Section 3 defines group inertia and underinvestment, illustrates the conditions under which these coordination frictions occur and, based on a numerical experiment, discusses the economic determinants of these frictions. Section 4 extends our results to groups of any size and to alternative voting mechanisms. Section 5 discusses some empirical predictions of our model. Section 6 concludes. Appendix A contains proofs of all propositions. The online appendix proposes trading solutions to the underinvestment problem.

[4] Acemoglu, Egorov, and Sonin (2018) also analyze a voting model with a changing electorate where the time inconsistency issue emerges. More broadly, the change of pivotal voters over time is present in many dynamic bargaining models of political economy, e.g., Strulovici (2010), Dziuda and Loeper (2016), and Chan, Lizzeri, Suen, and Yariv (2017).

2 Model

Consider a group of two members, P (for pessimist, he) and O (for optimist, she) who face a canonical real option problem. Specifically, at time zero the group must decide whether to acquire the right, or license, to invest in a project generating an uncertain cash flow stream. Failure to acquire the license implies the dissolution of the group. In this sense, the licensing decision can be thought of as essential to the group existence as an economic entity. At any time after licensing, the group must decide whether and when to invest and, subsequently, whether and when to abandon the project and liquidate it for fixed redeployment value.

2.1 Setup

Technology. At time 0 the group faces a take-it-or-leave-it decision of whether to acquire a license for a cost $L > 0$ (*licensing, or group formation decision*). The acquisition of the license gives the group the right, at any future time, to pay an investment cost $I > 0$ in order to acquire a project that will generate a random cash flow X_t per unit of time, starting from the investment date (*investment decision*). If the group invests in the project, it subsequently faces the choice of abandoning it at any time and liquidating it for a certain cash recovery amount $0 < A < I$ (*abandonment decision*).

Beliefs. P and O disagree on the growth rate of the cash flow X_t. Specifically, we assume that each member $n \in \{P, O\}$ of the group believes that the cash flow X_t of the project is governed by a geometric Brownian motion with drift μ_n and volatility σ, that is,

$$dX_t = \mu_n X_t dt + \sigma X_t dB_{n,t}, \quad X_0 = x > 0, \quad n \in \{P, O\} \tag{1}$$

where $\mu_P < \mu_O$ and $B_{n,t}$ is a standard Brownian motion under agent n's belief.

Group members do not update their beliefs about the growth rate of cash flows after observing a realization of X_t. In this sense, we refer to their beliefs as being "polarized." Allowing for learning about the expected growth rate in our framework complicates the timing policies as they would depend on additional state variables representing posterior beliefs about the expected growth rate of return (e.g. Décamps, Mariotti, and Villeneuve (2005)). Ultimately, if there is enough learning, the disagreement will dissipate over time and the coordination friction will vanish. Conceptually however, our underinvestment and group inertia results still hold if we allow agents to learn about the growth rate, as long as posterior beliefs remain sufficiently different during the lifetime of the project. In some realistic settings, such as the case of investment in startups and new technologies, learning about the mean growth rate of a project cash flow can take a long time. Therefore, our assumption of agents with polarized beliefs about cash flow growth rates captures the persistence of disagreement in these settings.

Governance. Group members share equally the license cost, L, the investment cost I, the project cash flows X_t, and the abandonment value A. In the subsequent analysis, to simplify the notation, we interpret the variables L, I, A, and X_t as representing per-capita quantities. Interpreting these as total quantities would not change the interpretation of our results.

Each member has one vote in subsequent decisions. The licensing, investment, and abandonment decisions are each determined by separate votes, each subject to strict majority rule (unanimity in a group with two members). The licensing vote takes place at time 0 while the investment vote can happen at *any* time afterwards, and the abandonment vote can be triggered at any time after investment. In particular, following the acquisition of the initial license, any member of the group can, at any time, propose investment, thereby triggering a group vote. Investment will take place at that point in time only if both parties agree. Similarly, at any time after investment is undertaken each group member can propose abandonment. The group will abandon the project if such a proposal receives unanimous support. When voting on any of the three decisions, each group member is self interested and correctly anticipates how the group would make subsequent decisions.

Individual valuations. We assume that agents are risk-neutral and discount the cash flow X_t at the risk free rate r. Ignoring for now the option to abandon at a future time, member n's subjective valuation of the cash flow stream X_t, conditional on investing at time 0 and observing the initial cash flow $X_0 = x$ is

$$\mathbb{E}_n\left[\int_0^\infty e^{-rt} X_t dt \,\middle|\, X_0 = x\right] = \frac{x}{r - \mu_n}, \quad n \in \{P, O\}, \tag{2}$$

where $\mathbb{E}_n$ denotes the expectation under member n's belief. Since $\mu_P < \mu_O$, to insure that subjective valuations are properly defined for each agent, we impose that $\mu_O < r$.

2.2 Individual investment and abandonment decisions

Before studying the group investment and abandonment decisions, we examine the optimal *individual* investment and abandonment decisions of a group member. Individually, each group member faces a simple optimal stopping problem involving the choice of the investment/abandonment times (τ_n, ν_n) that maximize the present value of cash flow X_t, given the investment cost I and abandonment value A. Formally, member n solves the following optimal stopping problem

$$V_n^*(x) = \sup_{\tau_n \le \nu_n} \mathbb{E}_n \left[-Ie^{-r\tau_n} + \int_{\tau_n}^{\nu_n} X_t e^{-rt} dt + Ae^{-r\nu_n} \middle| X_0 = x \right], \tag{3}$$

where $V_n^*(x)$ is the value of the license, that is, the value of the compound option to invest and subsequently abandon. The maximization in problem (3) takes place over the set of all stopping times $\tau_n \le \nu_n$, reflecting the optimization over a large class of timing policies. The constraint $\tau_n \le \nu_n$ captures the assumption that abandonment can only occur after investment has been undertaken. Member n will buy the license if and only if $V_n^*(x) \ge L$.

The solution of problem (3) is standard (see, e.g., Dixit and Pindyck (1994)) and involves finding two cash flow thresholds $X_n^{A,*} < X_n^{I,*}$ such that, assuming $X_0 = x < X_n^{I,*}$, the optimal strategy for member n is to (i) invest the first time X_t hits $X_n^{I,*}$ from below, and, after investment, (ii) abandon the first time X_t hits $X_n^{A,*}$ from above. One important insight from the theory of optimal stopping is that the threshold stopping rules we highlight are optimal among *all* stopping rules, including those that depend on the whole path of X_t.

To characterize the individual optimal investment and abandonment strategies, consider first an arbitrary pair of investment and abandonment thresholds, $X^A < X^I$, and define member n's time-0 subjective valuation associated with these thresholds as

$$V_n(x, X^I, X^A) \equiv \mathbb{E}_n \left[-Ie^{-r\tau_{X^I}} + \int_{\tau_{X^I}}^{\tau_{X^A}} X_t e^{-rt} dt + Ae^{-r\tau_{X^A}} \middle| X_0 = x \right], \tag{4}$$

where τ_{X^I} and τ_{X^A} are the hitting times associated to the thresholds X^I and X^A:

$$\tau_{X^I} = \inf\left\{t \ge 0 \ : \ X_t \ge X^I\right\}, \quad \tau_{X^A} = \inf\left\{t \ge \tau_{X^I} \ : \ X_t \le X^A\right\} \text{ with } X^A < X^I. \tag{5}$$

Let $W_n(x, X^A)$ be member n's subjective value of operating the project from the instant after investment is made through to the abandonment date:

$$W_n(x, X^A) \equiv \mathbb{E}_n \left[\int_0^{\tau_{X^A}} X_t e^{-rt} dt + A e^{-r\tau_{X^A}} \middle| X_0 = x \right], \quad x \geq X^A, \tag{6}$$

and $W_n(x, X^A) = A$ for $x < X^A$.

The following proposition provides closed-form expressions of the valuations in equation (4) and (6).

Proposition 1. *Let $X^I > 0$ be an arbitrary investment threshold and $0 < X^A < X^I$ an arbitrary abandonment threshold. Member n's subjective valuation of the operating project $W_n(x, X^A)$ defined in equation (6) is given by*

$$W_n(x, X^A) = \begin{cases} \frac{x}{r-\mu_n} + \left(A - \frac{X^A}{r-\mu_n}\right) \pi_n(x, X^A) & \text{if } x \geq X^A \\ A & \text{if } x \leq X^A \end{cases}, \tag{7}$$

where

$$\pi_n(x, X) = \left(\frac{x}{X}\right)^{m_n} \mathbf{1}_{x \geq X} + \left(\frac{x}{X}\right)^{q_n} \mathbf{1}_{x \leq X}, \tag{8}$$

with

$$m_n = \frac{-(\mu_n - \frac{\sigma^2}{2}) - \sqrt{(\mu_n - \frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2} < 0 \text{ and } q_n = \frac{-(\mu_n - \frac{\sigma^2}{2}) + \sqrt{(\mu_n - \frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2} > 1. \tag{9}$$

Member n's subjective valuation of the option to invest in the project, defined in equation (4) is given by

$$V_n(x, X^I, X^A) = \begin{cases} W_n(x, X^A) - I, & \text{if } x \geq X^I \\ \left(W_n(X^I, X^A) - I\right) \pi_n(x, X^I), & \text{if } x \leq X^I \end{cases}, \tag{10}$$

where $W_n(x, X^A)$ is given in equation (7).

The value $V_n(x, X^I, X^A)$ can be thought of as the value of (a) an option to pay the investment cost I at time τ_{X^I} and receive X_t over the time interval (τ_{X^I}, ∞) plus (b) an option to exchange the value of the cash flow stream X over (τ_{X^A}, ∞) with the certain lump sum payment A at time τ_{X^A}. Intuitively, the term $\pi_n(x, X)$ represents the value to agent n of a security that pays \$1 the first time X_t hits X, starting at $X_0 = x$. Formally, $\pi_n(x, X) = \mathbb{E}_n\left[e^{-r\tau_X} \,|\, X_0 = x\right]$ where τ_X is the first hitting time of the barrier X, that is, $\tau_X = \inf\{t \geq 0 \,:\, X_t = X\}$.

By standard results from optimal stopping theory (e.g., Peskir and Shiryaev (2006), Pham (2009), and Øksendal (2013)), the solution of the optimal stopping problem (3) can then be obtained by finding the thresholds $X^A = X_n^{A,*}$, and, $X^I = X_n^{I,*}$ that maximize equation (10). The following proposition provides a closed-form solution of the optimal individual investment/abandonment decisions.

Proposition 2. *Consider member n's problem of determining the optimal investment and abandonment time, described in equation (3).*

1. *The optimal abandonment threshold $X_n^{A,*}$ is given by*

$$X_n^{A,*} = A\left(r + m_n\frac{\sigma^2}{2}\right) \equiv -\frac{m_n}{1-m_n}A(r-\mu_n), \quad n \in \{P, O\}, \tag{11}$$

   with m_n given in equation (9).

2. *The optimal investment threshold $X_n^{I,*}$ is the largest root of the following equation*

$$\frac{X_n^{I,*}}{r-\mu_n}(q_n - 1) + \left(A - \frac{X_n^{A,*}}{r-\mu_n}\right)\left(\frac{X_n^{I,*}}{X_n^{A,*}}\right)^{m_n}(q_n - m_n) = q_n I, \quad n \in \{P, O\}, \tag{12}$$

   with $X_n^{A,*}$ given in equation (11) and m_n, q_n given in equation (9).

Agent n's subjective time-0 valuation under the optimal policy is then

$$V_n^*(x) = V_n(x, X_n^{I,*}, X_n^{A,*}), \quad n \in \{P, O\}, \tag{13}$$

where the function $V_n(\cdot)$ is defined in equation (10).

Using equations (9) and (11) we have that $\mu_P < \mu_O$ implies $m_O < m_P$. Therefore, from equation (12) in Proposition 2, we obtain that $X_O^{A,*} < X_P^{A,*}$, that is, the pessimist has a strictly higher abandonment threshold than the optimist. Therefore, when acting individually, P will always abandon the project earlier than O.

2.3 Group investment and abandonment decisions

Based on the individual timing strategies and project values derived in the previous section, we now examine how individual voting decisions translate into the group's licensing, investment, and abandonment choices.

We examine equilibrium voting recursively, starting with the abandonment decision. Assume that the level of cash flow x is larger than $X_P^{A,*}$—the level of cash flow at which the pessimist would optimally abandon. The pessimist understands that, if X_t declines from the current level to $X_P^{A,*}$ the optimist would *not* support an abandonment proposal, since, by Proposition 2, $X_O^{A,*} < X_P^{A,*}$. Furthermore, the optimist knows that, if she rejects abandonment at $X_P^{A,*}$ and waits until the cash flow falls to the lower level $X_O^{A,*}$, the pessimist will support her abandonment proposal because $X_O^{A,*} < X_P^{A,*}$. Therefore, even though the pessimist would prefer to abandon sooner, he is still better off supporting abandonment as soon as the optimist is willing to do so. Because the expected abandonment policy of the group is the same as the optimist's optimal abandonment policy, the group abandonment policy can be characterized by the threshold $X_{\mathrm{G}}^A = X_O^{A,*}$ and the group will abandon at $\tau_{X_O^{A,*}} := \inf_{t\geq 0}\{X_t \leq X_O^{A,*}\}$. We refer to O as the *pivotal voter* for the abandonment decision.

Consider next the investment decision. The pivotal member for the investment timing decision is the member whose preferred investment timing is latest, since they will block any proposal to invest earlier. With the common expectation that O will be pivotal for the abandonment decision, both P and O chose an optimal investment strategy conditional on O's abandonment decision. For O, the optimal investment threshold is her autarkic threshold $X_O^{I,*}$ defined in Equation (12) with $n = O$. In contrast, because P does not get to choose the group abandonment threshold, his (constrained) optimal investment threshold $X_P^{I,\mathrm{SB}}$ is *second best* and is given by the largest root of the following equation:

$$\frac{X_P^{I,\mathrm{SB}}}{r-\mu_P}(q_P - 1) + \left(A - \frac{X_O^{A,*}}{r-\mu_P}\right)\left(\frac{X_P^{I,\mathrm{SB}}}{X_O^{A,*}}\right)^{m_P}(q_P - m_P) = q_P I. \tag{14}$$

Suppose that $X_O^{I,*} < X_P^{I,\mathrm{SB}}$ and that the initial level of cash flow satisfies $x < X_O^{I,*}$. Under these conditions O wants to invest at the earliest time $\tau_{X_O^{I,*}}$ but P will vote against investment at that time. Additionally P knows that if he proposes investment at the later time $\tau_{X_P^{I,\mathrm{SB}}}$, O

will support the proposal because $X_O^{I,*} < X_P^{I,\mathrm{SB}}$. Therefore the group's investment timing can be characterized by a threshold given by $X_{\mathrm{G}}^I = X_P^{I,\mathrm{SB}}$ and P is pivotal for the investment decision. Similarly, when $X_P^{I,\mathrm{SB}} < X_O^{I,*}$, the group's investment threshold is $X_{\mathrm{G}}^I = X_O^{I,*}$ and O is pivotal for the investment decision. Therefore, the group threshold for investment and abandonment emerging as an equilibrium outcome is

$$X_{\mathrm{G}}^I = \max\{X_P^{I,\mathrm{SB}}, X_O^{I,*}\}, \quad \text{and} \quad X_{\mathrm{G}}^A = X_O^{A,*}.$$

Internalizing the future investment/abandonment threshold of the group, member n's project valuation at time 0 when the cash flow is equal to x, is given by

$$V_{n,\mathrm{G}}(x) = V_n(x, X_{\mathrm{G}}^I, X_{\mathrm{G}}^A) \equiv V_n(x, X_{\mathrm{G}}^I, X_O^{A,*}), \; n \in \{P, O\}, \tag{15}$$

where the function $V_n(\cdot)$ is defined in equation (10). Member n will vote for the licensing decision if and only if $V_{n,\mathrm{G}}(x) \geq L$. In the next section we show that $V_{P,\mathrm{G}}(x) < V_{O,\mathrm{G}}(x)$ for all x, implying that the group acquires the license if and only if member P's valuation under the anticipated group investment/abandonment strategy is larger than the licensing fee, that is, $V_{P,\mathrm{G}}(x) \geq L$. This makes P the pivotal member for the licensing decision. To summarize, (i) O is the pivotal voter for abandonment; (ii) P is the pivotal voter for licensing; and (iii) either P or O can be pivotal for the investment decision.

3 Group coordination frictions

In this section we study the licensing/investment/abandonment decisions of the group and contrast them to those of hypothetical members who act independently and retain full control at every stage of the project. We identify two main coordination frictions. First, a group of two members with heterogeneous beliefs tends to invest *later* than the pivotal member would on their own (*investment inertia*). Second, a group may decide to forgo the acquisition of a project license, despite it being a positive net present value decision from the point of view of each individual member (*inefficient licensing*). An inefficiency arises because a project that is perceived being worth pursuing by each member is foregone by the group.

3.1 Investment inertia

In Section 2 we showed that the group's abandonment decision is always aligned with O's optimal abandonment decision, that is, $X_{\mathrm{G}}^{A} = X_{O}^{A,*}$. The following proposition establishes that investment by a two-member group is (weakly) delayed relative to the optimal timing of either group member.

Proposition 3 (Group investment inertia). *The group investment threshold X_{G}^{I} satisfies*

$$X_{G}^{I} \geq \max\{X_{O}^{I,*}, X_{P}^{I,*}\}, \tag{16}$$

where $X_{O}^{I,*}$ and $X_{P}^{I,*}$ denote the optimal individual investment thresholds for O and P, determined from equation (12). Therefore, the group always invests at a later time than each of the individual members, that is,

$$\tau_{X_{G}^{I}} \geq \max\{\tau_{X_{O}^{I,*}}, \tau_{X_{P}^{I,*}}\}, \quad a.s.$$

Proposition 3 shows that when control rights over a project are shared within a group, investment inertia occurs, compared to the case in which control rights are fully retained by either one of the group members acting as a "dictator". When O is pivotal for the investment decision, $X_{\mathrm{G}}^{I} = X_{O}^{I,*}$. Member O acts as a dictator of the group and therefore there is no inertia. Group inertia occurs, however, when P is the pivotal voter for investment. Knowing that O will be pivotal for the abandonment decision, P realizes that investment is less attractive to him and responds by using his pivotal power to delay investment relative to his investment timing under autarky ($X_{\mathrm{G}}^{I} = X_{P}^{I,\mathrm{SB}} > X_{P}^{I,*}$).

The result highlighted in Proposition 3 provides a novel and alternative mechanism for investment inertia within a standard real option setting. Existing work (see, e.g., Grenadier and Wang (2005)) shows that, in a dynamic model where investment decisions are delegated to managers, moral hazard leads to greater inertia, as the manager holds a more valuable option to wait than the owner. In our setting investment inertia obtains without asymmetric information and moral hazard, but emerges purely as a consequence of a coordination friction brought about by the heterogeneity of beliefs within a group.

3.2 Inefficient licensing

A second dimension in which coordination frictions within a group may matter relates to the initial licensing decision that determines the existence of a group as an economic entity. The following proposition compares the subjective project valuation of each group member when decisions are made according to (i) the group policy and (ii) the individual members' optimal policies.

Proposition 4. *Let $V_n^*(x)$ and $V_{n,G}(x)$, $n \in \{O, P\}$ denote member n's individual and group project valuation, as defined in equations (13) and (15), respectively. Then, if $\mu_P < \mu_O < r$, we have that, for all $x > 0$,*

$$V_{P,G}(x) < \min\{V_{O,G}(x), V_P^*(x)\} < \max\{V_{O,G}(x), V_P^*(x)\} \leq V_O^*(x). \tag{17}$$

Condition (17) implies that, when the project will be managed by the group, P's valuation of the project is smaller than O's valuation, $V_{P,G}(x) < V_{O,G}(x)$. While O is the pivotal voter for the abandonment decision, this inequality implies that P is the pivotal voter for the licensing decision: If the project is attractive to P, $L < V_{P,G}(x)$ and then it is also attractive to O. Condition (17) further implies that P's valuation of the project under the group management is lower than P's valuation if he retains full control, which, in turn, is lower than O's valuation under her full control, that is, $V_{P,G}(x) < V_P^*(x) < V_O^*(x)$. We refer to *underinvestment* as a situation in which the licensing fee L is larger than P's group valuation but smaller than P's individual valuation, that is,

$$V_{P,G}(x) < L \leq V_P^*(x) < V_O^*(x). \tag{18}$$

In such cases, although both P and O would invest if they were able to unilaterally pick the subsequent investment and abandonment timing, that is, $L \leq V_P^*(x) < V_O^*(x)$, the group does not acquire the investment license, that is, $V_{P,G}(x) < L$. The last inequality implies that P votes against licensing and, by the strict majority rule (unanimity in a group with two members), the group does not acquire the license. In practice, the underinvestment condition (18) implies that a group does not form to undertake a project that all group members find valuable.

Figure 1 illustrates the difference between P's valuation under individually optimal and group exercise strategies. The shaded area in Figure 1 illustrates the underinvestment region. For a given level of cash flow x ($x = 0.4$), there is an interval of values for the licensing cost $(\underline{L}(x); \overline{L}(x)]$ with $\underline{L}(x) = V_{P,G}(x) = 3.78$, and $\overline{L}(x) = V_P^*(x) = 3.90$, for which underinvestment occurs. When $L < \underline{L}(x)$, the group invests since, even though the pessimist perceives the group's abandonment threshold $X_O^{A,*}$ as suboptimal, he still finds the project attractive ($L \leq V_{P,G}(x)$) and supports investment. When $L > \overline{L}(x)$, the group does not invest because the pessimist would vote against investment even if he had full control of the abandonment decision, that is, $L > V_P^*(x)$. In both cases the group decision is efficient. In contrast, when $L \in (\underline{L}(x); \overline{L}(x)]$, underinvestment occurs since P votes against the acquisition of a license for a project that he would instead find profitable if single-managed.

Underinvestment is *inefficient* if there exists a Pareto improving governance arrangement under which each group member is better off. The following proposition shows that, in the presence of underinvestment, both group members would benefit from giving member P full control right over the project management.

Proposition 5 (**Inefficient Underinvestment**). *Assume that the underinvestment condition (18) holds and that both group members commit to give member P full control over the investment and abandonment timing decisions. Then the group will acquire the time-0 license, that is,*

$$L \leq V_P^*(x) \leq V_O(x, X_P^{I,*}, X_P^{A,*}), \text{ for all } x > 0, \tag{19}$$

with $V_O(\cdot)$ defined in equation (4). As a result, relative to a majority governance rule, the new governance rule Pareto improves the payoff from the project to both group members and solves the underinvestment problem.

Under the alternative governance arrangement of Proposition 5, the group invests and abandons at P's optimal thresholds. Therefore P's subjective project valuation is $V_P(x, X_P^{I,*}, X_P^{A,*}) = V_P^*(x)$ and O's valuation is $V_O(x, X_P^{I,*}, X_P^{A,*})$. Equation (19) states that the perceived valuation of both members are larger than the licensing fee L and therefore, the group will acquire the license under this alternative governance arrangement. Furthermore, giving P control of both decisions Pareto improves the equilibrium outcome of a strict majority decision rule. To see this, note that under strict majority, underinvestment occurs when condition (18) is satisfied. In this case the investment is not undertaken and both P and O end up keeping their initial cash endowment L. Hence, giving decision power to P eliminates underinvestment and generates payoffs that are greater than L to *both* agents, a Pareto improvement. This shows that the underinvestment arising under the majority rule is inefficient because there is another governance rule which enables investment and yields a Pareto improving allocation.

3.3 A measure of the coordination friction cost

When condition (18) holds, underinvestment occurs because P perceives the project as unattractive and hence he vetoes the licensing decision. The coordination friction emerges therefore from the wedge between P's project valuation under his control versus that under the group's control. One intuitive way to measure this coordination friction cost is to determine the level of initial cash flow x under which P is indifferent between group governance and individual control. Formally,

Definition 1. *Prior to investment, the coordination friction cost $\delta \geq 0$ is defined by*

$$V_P^*(x) = V_{P,G}(x\,(1+\delta)). \tag{20}$$

Using the expressions of $V_P^*(x)$ and $V_{P,G}(x)$ provided in equations (10) and (15), we can express the coordination friction cost δ as follows:

$$\delta = \Gamma \cdot \frac{X_G^I}{X_P^{I,*}} - 1, \quad \text{where} \quad \Gamma \equiv \left(\frac{W_P(X_P^{I,*}, X_P^{A,*}) - I}{W_P(X_G^I, X_O^{A,*}) - I} \right)^{1/q_P} \tag{21}$$

with $W_P(\cdot)$ defined in equation (7), and $m_P < 0$ and $q_P > 1$ given in equation (9).[5]

To see that the measure δ captures the coordination friction cost, notice that δ can be directly related to the range of license-fee values L for which underinvestment occurs. In fact, from (18), underinvestment occurs if

$$V_{P,G}(x) < L < V_P^*(x) \equiv V_{P,G}(x\,(1+\delta)). \tag{22}$$

The larger the value of δ, the larger is the wedge between P's individual and group valuation, and therefore the more likely is the occurrence of underinvestment.

3.4 Coordination friction cost: polarization, volatility, and investment irreversibility

In this section we study the effect of polarization, cash flow volatility and investment irreversibility on group decisions. We measure polarization as the difference between the group members' beliefs

[5] The cost δ is independent of the initial cash flow x when the cash flow level is in the investment option continuation region for both P, that is $x < X_P^{I,*}$, and for the group, that is $x(1+\delta) \leq X_G^I$. Using equation (21), it can be shown that the two conditions are satisfied when $x \leq \Gamma \cdot X_P^{I,*}$, with Γ defined in equation (21). When the cash flow is larger than this threshold, the investment option should be exercised by the group and the coordination cost becomes cash flow dependent because P's and O's abandonment options dependence in wealth are not scale independent anymore.

on cash flow growth, $\mu_O - \mu_P$, uncertainty as the volatility of cash flow σ, and investment irreversibility as the ratio I/A of the investment sunk cost I and the abandonment value A. To the extent that assets that are harder to redeploy have a low liquidation value (e.g., because of fire sales discounts), projects with intangible assets are characterized by a high value of the ratio I/A.

Figure 2 summarizes the effect of these three forces. Panel A reports the investment and abandonment strategies of the group (respectively, solid-red and solid-blue lines) as well as the individually optimal thresholds of each group member (dashed lines). Panel B reports the coordination friction cost δ, defined in equation (20).

Note that in all figures in Panel A, the group investment threshold X_{G}^{I} (solid-red line) is always larger than the individually optimal investment thresholds, $X_P^{I,*}$ and $X_O^{I,*}$. This is an illustration of the investment inertia property of Proposition 3. The group abandonment threshold X_{G}^{A} (solid-blue line) corresponds to O's abandonment threshold and is always below the abandonment threshold that is individually optimal for P (dashed-blue line). The severity of the discrepancy between individually optimal and group threshold is quantified through the coordination friction cost δ reported in Panel B. A larger value of the cost δ implies that P requires a higher level of initial cash flow x to be indifferent between joining a group and operating the same project individually.

Effect of polarization. In the left-most graph of Panel A we fix the pessimist belief μ_P and vary $\mu_O \in (\mu_P, r)$. A larger value of μ_O corresponds to a higher degree of polarization. As polarization increases, the group abandons at lower levels of cash flow. In the figure, the solid-blue line, representing the group abandonment threshold, further departs from P's optimal abandonment threshold (dashed-blue) as polarization increases. Therefore, as polarization increases, the group abandonment policy becomes more unappealing to P who responds by holding up his vote for investment and thereby delaying investment for the group. In the figure, we see that the group investment threshold X_{G}^{I} (solid-red line) diverges from P's individually optimal investment threshold (dashed-black line).

The left-most graph of Panel B reports the effect of polarization on the coordination friction cost δ. The cost is zero when there is no polarization ($\mu_P = \mu_O = 0.01$) and increases to about 4% when polarization is at its maximum level ($\mu_O \approx r$). When polarization is extreme, to be indifferent between operating a project in a group or as an individual, P would require an initial level of cash flow, or equivalently, a project size, that is 4% higher than that of a project individually managed.

Effect of volatility. In the middle graphs of Figure 2 we report the effect of cash flow volatility on investment/abandonment thresholds (Panel A) and coordination friction cost (Panel B). As is well-understood from standard real option theory, volatility leads to investment/abandonment delays, that is, the investment (resp. abandonment) thresholds increase (resp. decrease) with σ. In our model, this happens for both the individual and group thresholds. As can be seen in Panel A all investment (resp. abandonment) thresholds are increasing (resp. decreasing) in σ. Interestingly though, volatility impacts the *ranking* between the individually optimal investment thresholds. Comparing P's optimal investment threshold $X_P^{I,*}$ (dashed-red line) to O's threshold $X_O^{I,*}$ (dashed-black line), we note that $X_O^{I,*} < X_P^{I,*}$ when σ is low while the opposite is true when σ is high.

To understand the ranking of investment threshold, it is useful to think of the option to invest as a *compound* (call) option: upon exercise it gives the right to future cash flow from the project plus the (put) option to abandon. From equation (10), the intrinsic value of the investment option is given by

$$\text{Intrinsic value} = \underbrace{\frac{x}{r-\mu_n}}_{\text{Value of ``assets in place''}} + \underbrace{\left(A - \frac{X^A}{r-\mu_n}\right)\pi_n(x, X^A)}_{\text{Value of option to abandon}} - \underbrace{I}_{\text{Exercise price}} \tag{23}$$

The first term represents the value of the perpetual cash flow stream from the project, the second term is the value of the option to abandon and the third term is the investment cost. All else being equal, a higher intrinsic value means an earlier investment exercise, or equivalently, a lower investment threshold.

An increase in volatility affects the option to abandon component of the intrinsic value defined in equation (23). Because P is more pessimistic than O, he values the abandonment option more, that is, all else being equal, the price $\pi_P(x, X^A) > \pi_O(x, X^A)$. For sufficiently high σ the attractiveness of the abandonment option to the pessimist can overcome the attractiveness of a higher cash flow growth μ_O for the optimist thus leading to a *lower* threshold for P than for O. When σ is low, the option to abandon is not as valuable and therefore the attractiveness of cash flow growth μ_O dominates the desirability of the abandonment option. In this case O's investment threshold is lower than P's.

The effect of volatility on the coordination cost is the combined effect from the discrepancy between the group and P's investment and the discrepancy between O's and P abandonment thresholds. As the middle graph in Panel B shows, the effect is non-monotonic. At low levels of volatility, the first discrepancy dominates. Starting from the lowest level of volatility of 15%, higher volatility initially increases the coordination cost, as the group investment threshold (red line in Panel A) moves away from P's individually optimal investment threshold (dashed-red line in Panel A). When the volatility is larger (above 20%), the second discrepancy dominates. As volatility increases in the region of high volatility, P's abandonment threshold (dashed-blue line in Panel A) moves closer to O' abandonment threshold (blue line in Panel A) thereby reducing the importance of disagreement on abandonment time and mitigating the coordination friction cost δ. This results in a hump-shaped coordination cost.

Effect of investment irreversibility. The ratio of investment cost to abandonment value I/A captures, in a reduced form, the degree of investment irreversibility. Intuitively, high investment irreversibility results in a low liquidation value A and therefore in a high I/A ratio. Similarly, a low I/A ratio reflects low investment irreversibility. In the right-most, graphs of Figure 2, we fix the abandonment value A and change the investment cost I without loss of generality. The top right figure in Panel A shows that the investment thresholds for both the individual and the group are, increasing in I/A: When I/A increases, investment irreversibility increases and the option to abandon is less valuable, which delays investment. Note, also that, when investment is more easily reversible (low I/A), P's individually optimal investment threshold $X_P^{I,*}$ (dashed-red line) is *lower* than O's individual threshold, rendering O pivotal for the investment decision. To understand this fact it is useful to consider again the intrinsic value of the investment option introduced in equation (23). When A is high, the project is easily reversible and the option to abandon is relatively more valuable. Being pessimist, P value the option to abandon more than O and, for sufficiently high A, he will be more eager to invest than O. However, because the group does not abandon at P's desired threshold choice but at O's threshold, note also that the group investment threshold (solid-red line) is larger than *both* P and O's optimal investment thresholds, as predicted by Proposition 3. Because in the figure we keep A constant, the abandonment thresholds for both individual and group (dashed- and solid-blue lines) are obviously unaffected by the ratio I/A.

Panel B reports the effect of investment irreversibility on the coordination cost. As investment irreversibility increases, that is, I/A increases, P's individually optimal investment threshold tend to get closer to the group investment threshold (Panel A) thus reducing the discrepancy between individually optimal and group behavior. As a consequence, the coordination cost δ decreases in the ratio I/A: inefficient underinvestment is more important for projects where investment is more easily reversible. To the extent that firms with more easily reversible project are less financially constrained, this result is consistent with Thakor and Whited (2011) who, in a different context, document that the effect of disagreement between shareholder and management on investment and Tobin's q is stronger for firms with a greater degree of financial flexibility.

Note finally that, by comparing the level of the coordination frictions in the three graphs of panel B, we see that the coordination friction cost is more sensitive to group polarization than to volatility and investment irreversibility.

4 Inertia and underinvestment in a general group setting

In this section, we broaden the scope of our results by considering groups of more than two members and more general voting protocols, including super-majority and majority rules, both with and without vetoers. In Section 4.1, we show that, under this general framework, the group behaviour cannot be represented by a single group member or even a subgroup of members. In Section 4.2, we characterize underinvestment and inertia in this general setting and discuss whether different voting protocols can mitigate or exacerbate the underinvestment problem.

4.1 Larger groups and more general voting rules

Consider a group of N agents, $\mathcal{N} = \{1, \ldots, N\}$, with N an arbitrary integer larger than 2. Each member n believes that the cash flow process X_t follows the dynamics described in equation (1), with μ_n such that $\mu_1 < \mu_2 < \ldots < \mu_N < r$. For each member n, we denote by $X_n^{A,*}$ and $X_n^{I,*}$ the optimal abandonment and investment threshold defined in equations (12) and (11), respecively.

A governance, or voting rule refers to a set of instructions that dictates how a group makes decisions based on a voting outcome. Given a group $\mathcal{N}$, any voting rule can be broadly defined with a set of *decisive coalitions* $\mathcal{D} \subseteq 2^{\mathcal{N}}/\{\emptyset\}$. A decisive coalition $\mathcal{C}$ is a subset of $\mathcal{N}$ such that, if any proposal is made and that all members of $\mathcal{C}$ vote to accept (reject) it, then the group accepts (rejects) the proposal. Any voting rule is thus defined by the set of all decisive coalitions (see, e.g., Austen-Smith and Banks (1999))). The following definition formally characterizes the class of voting rules we consider.

Definition 2 (**Governance rule**). *A governance, or voting, rule with quota* $k \in (N/2, N]$ *and vetoers* $\mathcal{V} = \{v_1, v_2, \ldots, v_M\} \subseteq \mathcal{N}$ *with* $M \leq N$ *and* $v_1 < v_2 < \ldots < v_M$*, is defined by the set of decisive coalitions*

$$\mathcal{D} = \{\mathcal{C} \subseteq \mathcal{N} :\mid \mathcal{C} \mid \geq k \text{ and } \mathcal{V} \subseteq \mathcal{C}\},$$

with $\mid \mathcal{C} \mid$ *denoting the number of members in the coalition* $\mathcal{C}$*.*

According to the definition, a group governed by a rule with quota k and a set of vetoers $\mathcal{V}$ will accept a proposal if at least k members including all the vetoers vote for the proposal. Note that the set of vetoers $\mathcal{V}$ is arranged according the beliefs of its members, that is $\mu_{v_1} < \mu_{v_2} < \ldots < \mu_{v_M}$.

Definition 2 nests unanimity as the special case in which every agent has veto power, that is $\mathcal{V} = \mathcal{N}$, and hence the set of decisive coalitions is the singleton $\mathcal{D} = \{\mathcal{N}\}$.[6] Simple majority rule, is also a special case of Definition 2 that is obtained when the set of vetoers is empty, $\mathcal{V} = \varnothing$, and the quota k is $k = N/2 + 1$, if N is even, and $k = (N + 1)/2$, if N is odd. If a majority group gives special veto power to a single member, such as the chair of a board of directors or a founder of a company, then these members need to be included in the set of vetoers $\mathcal{V}$. Finally, Definition 2 also nests the super-majority rule, according to which no member has veto power $\mathcal{V} = \varnothing$, and the quota k satisfies $N/2 + 1 < k < N$.

Any voting rule identifies two key group members which we will refer to as *optimistic pivot*, n_O and the *pessimistic pivot*, n_P. The following definition formally defines these two pivots for an N-member group governed by any voting rule from the class described in Definition 2.

Definition 3 (**Pivotal members**). *In a N-member group governed by a rule with quota* $k \in (N/2, N]$ *and vetoers set* $\mathcal{V} = \{v_1, v_2 \ldots, v_M\}$*, described in Definition 2, the optimistic pivot is the member* $n_O = \max\{k, v_M\}$*, and the pessimistic pivot is* $n_P = \min\{N - k + 1, v_1\}$*.*

In the context of the 2-members group from Section 2, the optimistic and pessimistic pivots are trivially the two members, that is, $n_O = O$ and $n_P = P$. In an N-member group governed by a majority rule with no vetoers, if N is even, the two pivotal members are those with more moderate beliefs, that is, $n_P = N/2$ and $n_O = N/2 + 1$, while, if N is odd, they coincide with the member

[6] Unanimity can alternatively be defined by setting the full quota $k = N$, in which case the set of vetoers is irrelevant since all members are vetoers by definition.

holding median beliefs, that is, $n_O = n_P = (N+1)/2$. Under a unanimity rule, the two pivots are the members with the most extreme beliefs, $n_P = 1$ and $n_O = N$. Under a supermajority rule and no vetoers, the beliefs of the pivots are between (i) the extreme beliefs of the pivotal members under the unanimity rule and (ii) the moderate beliefs of the pivotal member under the majority rule.

The main result of this section is to characterize the pivotal agents for an N-member group governed by any of the voting rules described in Definition 2 and facing the licensing, investment, and abandonment decision discussed in Section 2. As a preliminary step, in the following proposition, we provide sufficient conditions insuring that the value of the individual investment option for each group member is non-negative and increasing in the beliefs about cash flow growth rate μ.

Proposition 6 (**Non-negativity and monotonicity of investment option**). *Consider a N-member group ruled by a governance rule within the class described in Definition 2. Assume that:*

(i) The intrinsic value of the investment option to the most pessimistic group member is non-negative, that is

$$I < W_1(X_G^I, X_G^A). \tag{24}$$

where X_G^I and X_G^A denote the group's investment and abandonment threshold.

(ii) The cross sectional distribution of cash flow growth rate beliefs μ_n satisfies the following restriction

$$m_1\left(2r + m_1\frac{\sigma^2}{2}\right) < r\, m_{n_O}, \tag{25}$$

with n_O denoting the optimistic pivot from Definition 3 and m_1, m_{n_O} given in equation (9).

Then, the individual valuations of the investment option $V_n(x, X_G^I, X_G^A)$, are such that

$$0 < V_1(x, X_G^I, X_G^A) \leq V_2(x, X_G^I, X_G^A) \leq \cdots \leq V_N(x, X_G^I, X_G^A), \quad \text{for all } x > 0. \tag{26}$$

The condition in equation (24) states that the NPV to the most pessimistic member upon group investment, that is, when $X_t = X_G^I$, is positive. Because all agents have perfect information on each other's beliefs, if this assumption were violated, member $n = 1$ would not be willing to join the group in the first place even if the license fee L is equal to 0.[7] The condition in equation (25)

[7] Unlike in zero-sum games where the self interest of one group member is negatively associated with the self interest of other members, our group members share the cash flow from the project. The sharing of cash flows

restricts the dispersion of beliefs of the most pessimistic member, $n = 1$ and the optimistic pivot $n = n_O$. As the analysis in the proof of Proposition 6 shows, condition (25) is satisfied when the distribution of beliefs about growth rates $(\mu_1, \ldots, \mu_{n_O})$ is not too dispersed. This condition is sufficient to guarantee that group members' abandonment option values $W_n(x, X_G^A)$ are uniformly ranked according to beliefs, that is,

$$W_1(x, X_G^A) \leq W_2(x, X_G^A) \leq \cdots \leq W_N(x, X_G^A), \quad \text{for all } x > 0. \tag{27}$$

The ranking of the abandonment $W_n(x, X_G^A)$ and the positive NPV condition (24) imply that the values of the investment options $V_n(X_G^I, X_G^A)$ are ranked according to member's beliefs as shown in equation (26).[8]

Proposition 6 allows us to identify the pivotal voters for a group with an arbitrary number of members, using any of the governance rules given in Definition 2 and facing the licensing, investment, and abandonment decisions described in Section 2. This is done in the next proposition.

Proposition 7 (Pivotal voters in group-managed real option). *Consider an N-member group ruled by a governance rule within the class described in Definition 2 and assume that conditions (i) and (ii) of Proposition 6 hold. Then, when facing the licensing, investment and abandonment problem of Section 2*

1. *Only two group members determine the group licensing and abandonment timing decisions: the pessimistic pivot, n_P, is pivotal for the licensing decision and, the optimistic pivot, n_O, is pivotal for the abandonment decision.*

2. *Any group member $n \in \{1, 2, \ldots, N\}$ can be pivotal for the investment timing decision.*

Proposition 7 states that the group behavior cannot be represented by the behavior of a *unique* member. This result shows that, in our dynamic setting, the prescriptions of the "median voter theorem" of static voting models (e..g., Downs (1957)) are violated. According to this celebrated

mitigates the impact of disagreement. Assumption (24) constrains the negative externality that each group member can impose on each other since it requires that no member be forced to exercise an option with a negative intrinsic value.

[8] Although condition (ii) in Proposition 6 is sufficient to prove the inequalities (26), it is not necessary. In our numerical analysis we find that the valuation rankings (26) hold even when condition (ii) in Proposition 6 does not hold. Moreover, we could not find a combination of parameters for which the valuation ranking in equation (26) is violated. Hence, we suspect that the ranking in equation (26) of Proposition 6 holds more generally.

result[9] with the exception of a "pivotal member", every other group member has no real voting power, despite having explicit voting rights. An implication of the median voter theorem is that small changes in the beliefs of non-pivotal members would not affect the group's behavior, provided that these small changes do not affect the ranking of the pivotal members in the distribution of group beliefs. In our *dynamic* voting real option problem, these conditions are clearly not satisfied. To the extent that any member can be pivotal for the investment timing decision, individual preferences *do* influence the group behavior and the median voter theorem fails. In our setting, the group behavior can be represented by three, possibly distinct, group members. However, there is no obvious time-invariant governance rule that can reproduce the group's behavior with a fictitious three-member group.

To illustrate the result in Proposition 7, we consider a three-agent group with members P, M, O with beliefs, $\mu_P < \mu_M < \mu_O < r$. Agent M has "median" beliefs. We assume that the group is governed by strict majority and therefore group member M is both the optimistic and pessimistic pivot: $n_O = n_P = M$. The abandonment thresholds from equation (11) satisfy the ranking $X_O^{A,*} < X_M^{A,*} < X_P^{A,*}$ and by the majority rule, member M is pivotal for the abandonment decision, i.e., $X_G^A = X_M^{A,*}$. Condition (26) becomes $0 \le V_P(x, X_G^I, X_G^A) \le V_M(x, X_G^I, X_G^A) \le V_O(x, X_G^I, X_G^A)$ and therefore M is also pivotal for the licensing decision. Absent the investment timing decision, that is, when the group can only invest at time $t = 0$, the group's real option problem will trivially reduce to the optimal exercise problem of the median member M and there will be no coordination frictions. However, when the group also faces an investment timing decision, *any member* can be pivotal for that decision. Panel A of figure 3 reports, as a function of the project cash flow volatility, the preferred investment threshold $X_n^{I,\text{SB}}$ of each member $n \in \{P, M, O\}$, rationally anticipating that the abandonment decision will be determined by member M. If the group acts according to the majority rule, the group invests at the second highest threshold, which identifies the pivotal member for the investment decision. As the figure shows, depending on the level of volatility, *any* of the three members can be pivotal in the investment decision. The dark line highlighted with stars represents the resulting *group* investment threshold as a function of volatility. Panel B further clarifies that the identity of the pivotal agent for the investment decision in this example varies

[9] The conditions for the theorem to hold require that agents vote along a single dimension and preferences are single-peaked. Both conditions are satisfied in our model for each single vote. However, in our setting voting is sequential and it is the interaction between consecutive votes that creates the change of pivotal voters over time and ultimately precludes a representation of the group behaviour with a single member behaviour or even with a subset of group members.

depending on the level of volatility. While M is pivotal for investment when volatility is high or low, both P and O can be pivotal for investment for the intermediate level of volatility. In the case of groups with three members, or more generally of a group with an odd number of members, the median voter theorem fails if the group faces both an investment timing and an abandonment timing decisions and holds in the absence of investment timing.[10]

4.2 Underinvestment and inertia in a general framework

We now characterize inertia and underinvestment under the general voting framework of this section. We consider a group of N members operating under the general governance rule of Definition 2.

In an N-member group, we define investment inertia as a case in which the group investment threshold is larger than the individually optimal threshold of the pivotal member for the investment decision. The next proposition characterizes inertia in a general N-member group.

Proposition 8 (Group investment inertia). *Consider a N-member group governed by any of the rules of Definition 2. The group investment threshold X_{G}^{I} satisfies*

$$X_{n_I}^{I,*} \leq X_{G}^{I}, \tag{28}$$

where $X_{n_I}^{I,*}$ is the optimal individual investment thresholds for member n_I determined from equation (12) with $n = n_I$ and where n_I denotes the pivotal member for the investment decision. Therefore, the group always invests at a later time than the optimal investment time of pivotal member n_I, that is,

$$\tau_{X_G^I} \geq \tau_{X_{n_n}^{I,*}}, \quad a.s.$$

The intuition of this result is that the group investment threshold is the second best investment threshold of member n_I, $X_{\mathrm{G}}^{I} = X_{n_I}^{I,\mathrm{SB}}$ and, as a result, member n_I responds to the suboptimal abandonment policy imposed by the group by delaying investment.

[10] More generally, in the absence of timing decision, it can be shown that the decisions of any N-member group governed by a voting rule described in Definition 2 can be replicated by those of a fictitious group with only two members, n_P and n_O, and governed by the unanimity rule. This result is similar to Compte and Jehiel (2010) who show that a group facing a search problem behaves as a fictitious group formed by two pivots, ruled by unanimity. In the presence of investment timing, such a representation does not appear to be possible for two reasons. First, Proposition 7 shows that any member can be pivotal for the investment timing and therefore that the identity of the pivotal voter changes with the project characteristics, making the potential duplication less appealing. Second, Proposition 7 shows that up to three distinct pivotal members emerge, but we are unable to find a stationary voting rule that duplicates the group behaviour with a fictitious group formed with the three pivotal members.

In the context of a three-member group, Figure 4 compares the group investment threshold X_G^I from panel A of Figure 3 to the individually optimal investment threshold $X_{n_I}^{I,*}$ of the pivotal agent, as reported in Panel B of Figure 3, for different levels of cash flow volatility. As the figure shows, the group always invests at a higher threshold than any of the pivotal members. Note that for low and high levels of volatility, there is no inertia because the group's investment threshold corresponds to the individually optimal investment threshold of the pivotal member M. When the volatility level is intermediate, inertia occurs because the group threshold (Black line) is above the pivot's optimal threshold (red line). So, for this range of volatility levels, the group behavior in the investment decision is *not* subsumed by any of the behavior of its constituent members. When the three-member group is ruled by unanimity, we recover the result of Proposition 3: because the group investment threshold is the largest of all second best investment threshold, it is also larger than any individual optimal threshold, $X_{\mathrm{G}}^I \geq X_n^{I,*}$ for $n = P, M, O$.

Our next proposition provides a characterization of inefficient underinvestment in an N-member group.

Proposition 9 (**Inefficient Underinvestment**). *Consider a N-member group governed by any of the rules of Definition 2 and assume that conditions (i) and (ii) of Proposition 6 hold. Inefficient underinvestment occurs when the following condition holds:*

$$V_{n_P}(x, X_G^I, X_G^A) < L \leq V_1^*(x), \tag{29}$$

where the group's abandonment threshold is given by $X_G^A = X_{n_O}^A$, the group's investment threshold is given by $X_G^I = X_{n_I}^{I,SB}$ and x is the current cash flow level.

Proposition 9 generalizes the concept of inefficient licensing introduced in Section 3 to a group with an arbitrary number of members and more general voting rule. It also shows how an increase in majority requirement of a governance rule exacerbates the underinvestment problem. Intuitively, more stringent voting rules produce more polarized optimistic pivots and pessimistic pivots which creates a wider *underinvestment gap* defined by the interval $[V_{n_P}(x, X_G^I, X_G^A), V_1^*(x)]$. To illustrate the effect of different voting rules on underinvestment, we consider again a group of three members: P, M, and O, whose beliefs about cash flow growth satisfy the ranking $\mu_P < \mu_M < \mu_O < r$. When the group is governed by the majority rule, M is pivotal in both the licensing and abandonment decisions. In contrast, in a group governed by unanimity, $n_P = P$ is pivotal for the licensing decision and $n_O = O$ is pivotal for the abandonment decision. Relative to the majority governance rule, the unanimity rule creates more polarized pivotal voters and this intuitively implies a stronger coordination friction under unanimity than under majority. To quantify the severity of the coordination friction, following the analysis in Section 3.3, we define the coordination friction cost $\delta(x)$ in a three-member group governed by strict majority as

$$V_P^*(x) = V_M(x\,(1+\delta^{\text{Majority}}(x)), X_G^I, X_M^{A,*}) \tag{30}$$

where X_G^I is the group's investment threshold. The term $\delta^{\text{Majority}}(x)$ measures the severity of underinvestment as it increases in the gap $[V_M(x, X_G^I, X_G^A), V_P^*(x)]$ between the value to M of the group managed option and the value to P of the individually-managed option.

Similarly we define the coordination friction cost for a three-agent group governed by unanimity as

$$V_P^*(x) = V_P(x\,(1+\delta^{\text{Unanimity}}(x)), X_G^I, X_O^{A,*}). \tag{31}$$

By definition, $\delta^{\text{Unanimity}}(x) > 0$.[11]

Figure 5 illustrates the effect of different voting rules on underinvestment by reporting the coordination friction costs $\delta^{\text{Majority}}(x)$ (red line) and $\delta^{\text{Unanimity}}(x)$ (blue line), obtained, respectively, from equations (30) and (31). The figure shows that, in contrast to the case of a two-member group, underinvestment does not always occur with majority voting. In particular, for low or high levels of project cash flow x, the coordination cost $\delta^{\text{Majority}}(x)$ is negative, implying that underinvestment is impossible, that is, Condition (29) is violated. Importantly, the coordination cost $\delta^{\text{Unanimity}}(x)$ under unanimity is always positive and larger than that under majority. The intuition for this result is that when the majority requirement of a voting rule is more stringent, like in the unanimity case, it produces more polarized pivotal voters and as a result it creates more underinvestment.

We finally observe that when the underinvestment condition (29) holds for an N-member group, investment is inefficient as in the two-member group case discussed in Proposition 5. It can be shown that if all group members commit to give the most pessimist member $n = 1$ full control over the investment and abandonment timing decisions (i.e., $X_G^I = X_1^{I,*}$ and $X_G^A = X_1^{A,*}$), the group would acquire the licence and the resulting valuation would represent a Pareto improvement to *all* group

[11] Note that when the group is governed by the majority rule, the definition in equation (30) involves interpersonal projects evaluations, and therefore the cost $\delta^{\text{Majority}}(x)$ is state dependent. Similarly to the two members case, $\delta^{\text{Unanimity}}(x)$ is independent of x for low value of x but changes with x for large values of x.

members, that is,

$$L \leq V_1^*(x) \leq V_2(x, X_1^{I,*}, X_1^{A,*}) \leq \cdots \leq V_N(x, X_1^{I,*}, X_1^{A,*}), \text{ for all } x > 0. \tag{32}$$

5 Empirical implications

The key economic insight of our model is that, in a setting where a group of heterogeneous agents faces sequential decisions and disagreements are resolved through voting, coordination frictions emerges that can distort both the quantity and timing of investments. Our theory shows that the effect of group decision making on investment distortions depends on (a) the nature of the group and (b) the nature of the investment opportunity. In turn, these provide natural areas of focus for empirical studies.

With respect to the nature of the group, our model shows that the more polarized are the beliefs in a group, the more reluctant the group is to invest, both initially and in subsequent expansions. Directly measuring polarization is difficult but indirect measures have already been applied in other contexts. For instance, Adams, Akyol, and Verwijmeren (2018) show that more diverse boards are associated with poorer corporate performance. They suggest that the diversity of skill that they document may proxy for diversity of beliefs. In light of our results, therefore, empirical explorations can go further to relate board diversity to investment quantity and timing. Similarly, Balsmeier, Fleming, and Manso (2017) find that firms with boards having more independent directors tend to be less innovative. If we accept that independent directors bring different beliefs to the table, then our analysis suggests that investment levels and timing are related to the degree of director independence. More broadly, the implication of our model are consistent with the existing literature on the composition of the shareholder base that documents a positive effect of shareholder base cohesiveness on firm valuation (e.g., Kandel, Massa, and Simonov (2011),Schwartz-Ziv and Volkova (2020) and Brav, Jiang, Li, and Pinnington (2018)).

While a corporate board fits our description of a group, venture capital syndicates are also closely aligned with our theoretical construct. Existing empirical evidence on VCs is consistent with our assumption that heterogeneous groups make investment decisions. For instance, according to Nanda and Rhodes-Kropf (2018), over a recent 15-year period the average startup that received VC funding had 3 investors. Furthermore, Guler (2007) finds that VCs vary considerably in the degree to which they exercise their abandonment option. In addition, when VCs differ in their exit strategy, they are also more likely to have conflicting views on whether to extend financing.

In the context of venture capital syndicates, then, our framework provides a new alternative perspective on decision-making. As emphasized byNanda and Rhodes-Kropf (2018), while the academic literature has extensively studied the frictions emerging from the asymmetric information between the venture capitalist (VC) and the entrepreneur, relatively little work has been devoted to the study of the coordination frictions emerging when multiple investors come together to finance a new venture.[12] In the context of our theory, the design of syndicates —e.g., the pervasiveness of "relational contracts," (Baker, Gibbons, and Murphy (2002))—and the use of contractual features observed in practice—e.g., the pervasiveness of dual-class shares in VC-backed companies (Gornall and Strebulaev (2020))—can represent a response to the inefficiencies driven by the dynamic voting structure we study. Similarly, Gompers, Mukharlyamov, and Xuan (2016) show that individual venture capitalists tend to associate with other venture capitalists having common characteristics and backgrounds (e.g., ethnicity, education, past employer, and degree from a top university). Consistent with our theory, this evidence that VC syndicates tend to be formed by investors with similar characteristics indicates a desire to mitigate the cost of disagreement via repeated game interaction.

6 Conclusion

We examine the acquisition and subsequent management of a real option by a group of agents with heterogeneous beliefs that make licensing and investment/abandoment timing decisions through voting. We show that while the optimal abandonment timing decisions are ranked according to agents' beliefs, the same is not necessarily true for optimal investment timing decisions. Specifically, depending on beliefs polarization or project characteristics such as cash flow volatility and reversibility, more pessimistic members can be more eager to invest even though they place a lower value the project than optimists. Importantly, therefore, the member with median beliefs is not the same as the member with median investment timing. This result implies that the group behaviour cannot be subsumed by the behaviour of a representative member even in a group with odd number of members.

[12]This literature is too vast to be reviewed here. Seminal contributions are, for example, Gompers (1995), Kaplan and Strömberg (2003, 2004), Hellmann (1998), and Cornelli and Yosha (2003).

Our analysis implies that in such a setting the group may reject an investment opportunity even though each member of the group sees the opportunity as valuable. Moreover, we show that all else being equal, a group exhibits investment inertia, relative to optimal investment timing of the member who act as a pivotal member for the investment decision of the group. The results hold for group of any size and with general voting rules including majority, supermajority or unanimity. Underinvestment is more likely if the views of pivotal group members are polarized, if the investment is more easily reversible, if the voting mechanism requires a more stringent majority, or if members with extreme beliefs have veto powers in the decision. In the Online Appendix we show that contractual solutions to the underinvestment problem can take the form of tender offers with shotgun provisions in which the optimist group member acquires the shares of the pessimist.

Our results have implications for future empirical and theoretical work. Empirically, our analysis identifies characteristics of the investment opportunities that relate to coordination frictions. In particular, we show how the volatility of the underlying cash flows, the heterogeneity of decision making groups and the degree of irreversibilty of a project (book cost versus recovery value) affect the severity of coordination frictions within a group and the resulting investment dynamics. Future empirical work could take the challenge to provide a quantitative assessment of the coordination frictions we highlight in this study.

Theoretically, our results can be generalized in several directions. First, our analysis of different voting rules suggests that majority is the "best rule" in that it minimizes underinvestment. Evaluating voting rules in our context would require however a broader perspective of the benefits and costs of each rule. A normative assessment of voting rules would call for a welfare analysis in the presence of heterogeneous beliefs (see, e.g., Brunnermeier, Simsek, and Xiong (2014) and Gilboa, Samuelson, and Schmeidler (2014)), a challenging task that can be tackled in future research.

Second, the results of our paper apply generally to groups that mediate conflicts through majority voting, and to groups in which agents have veto power. While our model is a stylized description of the final decision-making process of groups, in reality votes are cast in the context of dynamic pre-vote interactions. Developing theories that realistically capture the political economy of corporate decisions is a fascinating subject of future research.

A Appendix: Proofs

Proof of Proposition 1

When $x \geq X^I$, we have $\tau^I = 0$ and $V_n(x, X^I, X^A) = W_n(x, X^A) - I$ where we recall that W_n is defined in equation (A.1). Note that

$$\begin{aligned} W_n(x, X^A) &= \mathbb{E}_n\left[\int_0^\infty X_t e^{-rt}dt - \int_{\tau_A}^\infty X_t e^{-rt}dt + Ae^{-r\tau_A}\middle| X_0 = x\right] \\ &= \frac{x}{r-\mu_n} + \left(A - \frac{X^A}{r-\mu_n}\right)\mathbb{E}_n[e^{-r\tau_A}\,|\,X_0 = x],\ \text{for } x \geq X^A, \end{aligned} \tag{A.1}$$

and $W_n(x, X^A) = A$ for $x < X^A$. Let $f(x) = \mathbb{E}_n[e^{-r(\tau_A - t)}|X_t = x]$, where $x \geq X^A$. The process $(e^{-rt}f(X_t))_{t\geq 0}$ is a martingale and therefore its drift is null. Using the dynamics (1) and Itô's formula, we obtain that f must satisfy the ODE

$$\mu_n x f'(x) + \frac{1}{2}\sigma^2 x^2 f''(x) = rf(x), \quad x \geq X^A,$$

whose general solution is of the form $f(x) = ax^{m_n} + bx^{q_n}$, with $m_n < 0$ and $q_n > 1$ are given in Equation (8) and solve the quadratic equation

$$\mu_n\beta + \frac{1}{2}\sigma^2\beta(\beta - 1) - r = 0. \tag{A.2}$$

Imposing the boundary conditions $f(X^A) = 1$ and $\lim_{x\to\infty} f(x) = 0$, we obtain that $b = 0$ and $a = (X^A)^{-m_n}$, thus yielding $f(x) = \left(\frac{x}{X^A}\right)^{m_n}$ and hence

$$\pi_n(x, X^A) = \mathbb{E}_n[e^{r\tau_A}\,|\,X_0 = x] = \left(\frac{x}{X^A}\right)^{m_n}, \quad x > X^A. \tag{A.3}$$

This proves equation (10) when $x > X^I$. When $x < X^I$, the law of iterated expectation together with the strong Markov property of the process X_t shows that

$$V_n(x, X^I, X^A) = \mathbb{E}_n\left[e^{-r\tau^I}(W_n(X_{\tau^I}, X^A) - I)\middle| X_0 = x\right].$$

Using the fact that $X_{\tau^I} = X^I$ yields

$$V_n(x, X^I, X^A) = (W_n(X^I, X^A) - I)\mathbb{E}_n\left[e^{-r\tau^I}\middle| X_0 = x\right] \equiv (W_n(X^I, X^A) - I)\pi_n(x, X^I),$$

where, by a similar argument as above,

$$\pi_n(x, X^I) = \mathbb{E}_n[e^{r\tau^I} \big| X_0 = x] = \left(\frac{x}{X^I}\right)^{q_n}, \quad x < X^I. \tag{A.4}$$

which in turn proves equation (10) when $x \leq X^I$.

Proof of Proposition 2

To find the optimal abandonment threshold $X^A = X_n^{A,*}$, we differentiate (10) with respect to X^A, when $x \geq X^I$. This leads to the following optimal abandonment threshold for member n

$$X_n^{A,*} = A\frac{m_n}{m_n - 1}(r - \mu_n), \quad n \in \{P, O\}. \tag{A.5}$$

Using the fact that m_n is a root of the quadratic equation (A.2), we can re-express the threshold in equation (A.5) as follows:

$$X_n^{A,*} = A\left(r + m_n\frac{\sigma^2}{2}\right), \quad n \in \{P, O\}. \tag{A.6}$$

From Equations (9), it can be shown that $\mu_P < \mu_O$ implies $m_O < m_P$. Therefore, equation (A.6) implies that $X_O^{A,*} < X_P^{A,*}$, that is, the pessimist has a strictly higher abandonment threshold than the optimist. Therefore, when acting individually, P will abandon the project earlier than O, $\tau_{X_P^{A,*}} < \tau_{X_O^{A,*}}$, a.s.

To find the optimal investment threshold $X^I = X_n^{I,*}$ for member n, we maximize the option value before investment is undertaken. Differentiating equation (10) with respect to X^I (taking $X^A = X_n^{A,*}$ as given) when $x \leq X^I$ we obtain the following necessary condition for optimality of the investment threshold $X_n^{I,*} > X_n^{A,*}$:

$$\frac{X_n^{I,*}}{r - \mu_n}(q_n - 1) + \left(A - \frac{X_n^{A,*}}{r - \mu_n}\right)\left(\frac{X_n^{I,*}}{X_n^{A,*}}\right)^{m_n}(q_n - m_n) = q_n I, \quad n \in \{P, O\}. \tag{A.7}$$

Direct inspection shows that equation (A.7) has at most two roots and that the optimal investment threshold corresponds to the largest of the two. The largest root is the only one that satisfies the smooth pasting condition and is larger than $X_n^{A,*}$. The individual optimal valuations of the firm at time 0 are then given by

$$V_n^*(x) = V_n(x, X_n^{I,*}, X_n^{A,*}), \quad n \in \{P, O\},$$

where we recall that V_n, $X_n^{A,*}$, and $X_n^{I,*}$ are defined in equations (10), (11), and (12). ∎

Proof of Proposition 3

When O is pivotal for the investment decision, $X_P^{I,\mathrm{SB}} < X_O^{I,*}$ and therefore $X_{\mathrm{G}}^I = X_O^{I,*}$. In this case, if $X_P^{I,*} < X_P^{I,\mathrm{SB}}$ we will have $X_{\mathrm{G}}^I \equiv X_O^{I,*} > \max\{X_P^{I,*}, X_O^{I,*}\}$ which proves Proposition 3. When P is pivotal for the investment decision, $X_O^{I,*} < X_P^{I,\mathrm{SB}}$ and therefore $X_{\mathrm{G}}^I = X_P^{I,\mathrm{SB}}$. In this case, if $X_P^{I,*} < X_P^{I,\mathrm{SB}}$ we will have $X_{\mathrm{G}}^I \equiv X_P^{I,\mathrm{SB}} > \max\{X_P^{I,*}, X_O^{I,*}\}$ which proves Proposition 3.

Therefore, in both cases, to prove Proposition 3, we just need to prove that $X_P^{I,*} < X_P^{I,\mathrm{SB}}$. Recall that $X_P^{I,*}$ is P's optimal investment threshold when the abandonment threshold is $X_P^{A,*}$ and that $X_P^{I,\mathrm{SB}}$ is P's optimal investment threshold (second best) when the abandonment threshold is $X_O^{A,*}$. Recall also that, by Proposition 2, $X_O^{A,*} < X_P^{A,*}$.

The threshold $X_P^{I,*}$ is the optimal investment threshold for the optimal stopping problem (3) with $n = P$. Using the dynamic programming principle the optimal stopping problem (3) can alternatively be reformulated as

$$V_P^*(x) = \sup_\tau \mathbb{E}_P \left[e^{-r\tau} \left(W_P(X_\tau, X_P^{A,*}) - I \right) \middle| X_0 = x \right] \tag{A.8}$$

where the function W_P represents the value of the post-investment project when it is optimally abandoned and is defined in equation (A.1) with $n = P$ and, $y = X_P^{A,*}$. Both $V_P^*(x)$ and $W_P(x, X_P^{A,*})$ are known in closed form. We know that $V_P^*(x) = V_P(x, X_P^{I,*}, X_P^{A,*})$ and is given in equations (10). This implies that, when $x \leq X_P^{I,*}$, $V_P^*(x)$ admits the representation $V_P^*(x) = Cx^{q_P}$ for some positive constant C. Problem (A.8) is standard and is characterized by a twice differentiable value function that satisfies the value-matching and smooth-pasting conditions[13] at the optimal investment

[13] Problem (A.8) is a standard optimal stopping problem where the state process is one dimensional with non-degenerate diffusion term $\sigma > 0$. Therefore value-matching and smooth-pasting hold at the boundary of the optimal continuation region as shown, for example, in Proposition 5.2.1 in Pham (2009).

threshold $X_P^{I,*}$:

$$V_P^*(X_P^{I,*}) = W_P(X_P^{I,*}, X_P^{A,*}) - I, \qquad \left(\frac{\partial V_P^*(x)}{\partial x}\right)_{x=X_P^{I,*}} = \left(\frac{\partial W_P(x, X_P^{A,*})}{\partial x}\right)_{x=X_P^{I,*}}$$

Taking the ratio of each side of the value matching and smooth pasting conditions gives

$$\frac{V_P^*(X_P^{I,*})}{\left(\frac{\partial V_P^*(x)}{\partial x}\right)_{x=X_P^{I,*}}} = \frac{W_P(X_P^{I,*}, X_P^{A,*}) - I}{\left(\frac{\partial W_P(x,X_P^{A,*})}{\partial x}\right)_{x=X_P^{I,*}}}. \tag{A.9}$$

Using the closed-form expressions of $V_P^*(x)$ and $W_P(x, X_P^{A,*})$, it can be shown that condition (A.9) is equivalent to

$$\frac{X_P^{I,*}}{q_P} = \frac{W_P(X_P^{I,*}, X_P^{A,*}) - I}{\frac{1-m_P}{r-\mu_P} + m_P \frac{1}{X_P^{I,*}} W_P(X_P^{I,*}, X_P^{A,*}))}$$

and rearranging the terms gives

$$f(X_P^{I,*}) = W_P(X_P^{I,*}, X_P^{A,*}) \text{ where } f(x) := \frac{1-m_P}{q_P - m_P}\frac{1}{r-\mu_P}x + \frac{q_P}{q_P - m_P} I \tag{A.10}$$

We therefore see that the optimal investment threshold is geometrically characterized by the intersection of the graph of $W_P(\cdot, X_P^{A,*})$ and the line represented by the function $f(\cdot)$. We now show that this intersection is unique and occurs at a point larger than $X_P^{A,*}$. To prove this, notice that

$$f(X_P^{A,*}) = A + \frac{q_P}{q_P - m_P}(I - A) > A = W_P(X_P^{A,*}, X_P^{A,*})$$

and we know that $W_P(\cdot, X_P^{A,*})$ is non decreasing with an asymptote of slope $\frac{1}{r-\mu_P}$ at ∞. On the other hand, because $q_P > 1$, the function f has the slope $f'(x) = \frac{1-m_P}{q_P-m_P}\frac{1}{r-\mu_P}$ that is smaller than that of the function $W_P(\cdot, X_P^{A,*})$ at ∞ (i.e. $\frac{1}{r-\mu_P}$). Therefore, there is a unique intersection point of the curves $W_P(\cdot, X_P^{A,*})$ and $f(\cdot)$ and it characterizes the investment threshold $X_P^{I,*}$.

We now turn to the second best problem defined by

$$V_P^{\text{SB}}(x) = \sup_\tau \mathbb{E}_P\left[e^{-r\tau}\left(W_P(X_\tau, X_O^{A,*}) - I\right)\middle| X_0 = x\right]. \tag{A.11}$$

Both $V_P^{\text{SB}}(x)$ and $W_P(x, X_O^{A,*})$ are known in closed form. We know that $V_P^{\text{SB}}(x) = V_P(x, X_P^{I,\text{SB}}, X_O^{A,*})$ and from the expressions (10), we deduce that, when $x \leq X_P^{I,\text{SB}}$, $V_P^{\text{SB}}(x)$ admits the representation $V_P^{\text{SB}}(x) = Cx^{q_P}$ for some positive constant C. The function $W_P(\cdot, X_O^{A,*})$ also admits a closed form expression given in equation (7). The optimal control problem (A.11) is also standard and obeys value matching and smooth pasting. Following the same steps as those used to analyze Problem (A.8) gives a the following characterization of the second best investment threshold

$$f(X_P^{I,\text{SB}}) = W_P(X_P^{I,\text{SB}}, X_O^{A,*}), \tag{A.12}$$

where the function $f(\cdot)$ is defined in equation (A.10). Therefore, the second best investment threshold is also characterized geometrically as the intersection point of the function $f(x)$ with the function $W_P(\cdot, X_O^{A,*})$. Notice that the function $W_P(\cdot, X_O^{A,*})$ has also an asymptote of slope $\frac{1}{r-\mu_P}$ at ∞. Furthermore, because $X_P^{A,*}$ is the optimal abandonment threshold for member P, we have

$$W_P(x, X_O^{A,*}) \leq W_P(x, X_P^{A,*}) \text{ for all } x > 0.$$

Therefore the curve of the function $W_P(\cdot, X_O^{A,*})$ is always below that of the function $W_P(\cdot, X_P^{A,*})$. The function $f(x)$ intersects $W_P(\cdot, X_P^{A,*})$ at a smaller value of cash flow x than that at which it intersects $W_P(\cdot, X_O^{A,*})$. This means that $X_P^{I,*} \leq X_P^{I,\text{SB}}$ which concludes the proof. ∎

Proof of Proposition 4

Because V_P^* and V_O^* solve the optimal stopping problem (4), they must dominate the value derived from any other stopping policy. Therefore, we have, $V_{P,G}(x) \equiv V_P(x, X_G^I, X_O^A) < V_P^*(x)$ and, $V_{O,G}(x) \equiv V_O(x, X_G^I, X_O^A) \leq V_O^*(x)$ for all $x > 0$.

We now show that $V_P^*(x) < V_O^*(x)$ for all $x > 0$. For $n = O, P$, we denote the optimal stopping times $\tau_n^{I,*} = \inf\left\{t \geq 0 \ : \ X_t \geq X_n^{I,*}\right\}$ and, $\tau_n^{A,*} = \inf\left\{t \geq \tau_n^{I,*} \ : \ X_t \leq X_n^{A,*}\right\}$. Then,

$$\begin{aligned}
V_P^*(x) &= \mathbb{E}_P\left[-Ie^{-r\tau_P^{I,*}} + \int_{\tau_P^{I,*}}^{\tau_P^{A,*}} X_t e^{-rt} dt + Ae^{-r\tau_P^{A,*}} \,\middle|\, X_0 = x\right] \\
&\leq \mathbb{E}_P\left[-Ie^{-r\tau_P^{I,*}} + \int_{\tau_P^{I,*}}^{\tau_P^{A,*}} X_t e^{(\mu_O-\mu_P)t} e^{-rt} dt + Ae^{-r\tau_P^{A,*}} \,\middle|\, X_0 = x\right] \\
&= \mathbb{E}_P\left[-Ie^{-r\tau_P^{I,*}} + \int_{\tau_P^{I,*}}^{\tau_P^{A,*}} Y_t e^{-rt} dt + Ae^{-r\tau_P^{A,*}} \,\middle|\, X_0 = x\right]
\end{aligned}$$

where the above inequality follows from $\mu_O > \mu_P$ and where we denote $Y_t \equiv X_t e^{(\mu_O - \mu_P)t}$. Under P's subjective probability measure, denoted by $\mathbb{Q}_P$, the process Y is a geometric Brownian motion satisfying

$$dY_t = \mu_O Y_t dt + \sigma Y_t dB_{P,t}, \quad Y_0 = x$$

and furthermore, the stopping times $(\tau_P^{I,*}, \tau_P^{A,*})$ can be written as functionals of the path of the process Y as follow:

$$\tau_P^{I,*} = \inf\left\{t \geq 0 \ : \ Y_t \geq e^{(\mu_O - \mu_P)t} X_P^{I,*}\right\} \quad \text{and} \quad \tau_P^{A,*} = \inf\left\{t \geq \tau_P^{I,*} \ : \ Y_t \leq e^{(\mu_O - \mu_P)t} X_P^{A,*}\right\}.$$

Notice that the law of the triplet $(Y_t, \tau_P^{I,*}, \tau_P^{A,*})$ under P's subjective beliefs $\mathbb{Q}_P$ is identical to the law of the triplet $(X_t, \nu_P^{I,*}, \nu_P^{A,*})$ under O's subjective beliefs $\mathbb{Q}_O$, where we define the stopping times $(\nu_P^{I,*}, \nu_P^{A,*})$ as follows:

$$\nu_P^{I,*} = \inf\left\{t \geq 0 \ : \ X_t \geq e^{(\mu_O - \mu_P)t} X_P^{I,*}\right\} \quad \text{and} \quad \nu_P^{A,*} = \inf\left\{t \geq \nu_P^{I,*} \ : \ X_t \leq e^{(\mu_O - \mu_P)t} X_P^{A,*}\right\}.$$

Therefore, for any $x > 0$, we have

$$\begin{aligned}
V_P^*(x) &\leq \mathbb{E}_P\left[-Ie^{-r\tau_P^{I,*}} + \int_{\tau_P^{I,*}}^{\tau_P^{A,*}} Y_t e^{-rt} dt + Ae^{-r\tau_P^{A,*}} \middle| X_0 = x\right] \\
&= \mathbb{E}_O\left[-Ie^{-r\nu_P^{I,*}} + \int_{\nu_P^{I,*}}^{\nu_P^{A,*}} X_t e^{-rt} dt + Ae^{-r\nu_P^{A,*}} \middle| X_0 = x\right] \\
&\leq \sup_{\tau \leq \nu} \mathbb{E}_n\left[-Ie^{-r\tau} + \int_{\tau}^{\nu} X_t e^{-rt} dt + Ae^{-r\nu} \middle| X_0 = x\right] \\
&= V_O^*(x)
\end{aligned}$$

We now prove that $V_{P,G}(x) < V_{O,G}(x)$ for all $x > 0$. When the initial cash flow satisfies $x > X_G^I$, the investment option is already exercised. Because the group's abandonment corresponds to the optimist's optimal threshold, $X_G^I = X_O^{I,*}$, we have

$$V_{P,G}(x) = W_P(x, X_O^{A,*}), \quad V_{O,G}(x) = W_O(x, X_O^{A,*}).$$

Because $X_O^{A,*}$ is the optimal abandonment threshold for the optimist, we have

$$W_O(x, X_O^{A,*}) = \sup_{\tau \geq 0} \mathbb{E}_O \left[\int_0^\tau X_t e^{-rt} dt + A e^{-r\tau} \middle| X_0 = x \right]$$

and the optimum is attained at $\tau_O^{A,*} = \inf\{t \geq 0 : X_t \leq X_O^{A,*}\}$. Using the fact that $\mu_O > \mu_P$, for any $x > 0$, we have

$$\begin{aligned} W_P(x, X_O^{A,*}) &= \mathbb{E}_P \left[\int_0^{\tau_O^{A,*}} X_t e^{-rt} dt + A e^{-r\tau_O^{A,*}} \middle| X_0 = x \right] \\ &\leq \mathbb{E}_P \left[\int_0^{\tau_O^{A,*}} X_t e^{(\mu_O - \mu_P)t} e^{-rt} dt + A e^{-r\tau_O^{A,*}} \middle| X_0 = x \right] \end{aligned}$$

Defining again the process $Y_t = X_t e^{(\mu_O - \mu_P)t}$, we observe that the law of the couple $(Y_t, \tau_O^{A,*})$ under $\mathbb{Q}_P$ is identical to the law of the couple $(X_t, \nu_O^{A,*})$ under the probability $\mathbb{Q}_O$ where $\nu_O^{A,*} = \{t \geq 0 : X_t \leq X_O^{A,*}\ e^{(\mu_O - \mu_P)t}\}$. Therefore

$$\begin{aligned} W_P(x, X_O^{A,*}) &\leq \mathbb{E}_P \left[\int_0^{\tau_O^{A,*}} Y_t e^{-rt} dt + A e^{-r\tau_O^{A,*}} \middle| X_0 = x \right] \\ &= \mathbb{E}_O \left[\int_0^{\nu_O^{A,*}} X_t e^{-rt} dt + A e^{-r\nu_O^{A,*}} \middle| X_0 = x \right] \\ &\leq \sup_{\tau \geq 0} \mathbb{E}_O \left[\int_0^\tau X_t e^{-rt} dt + A e^{-r\tau} \middle| X_0 = x \right] \\ &= W_O(x, X_O^{A,*}) = V_{O,G}(x) \end{aligned}$$

for any $x > 0$. In particular, for $x > X_G^I$, we have

$$V_{P,G}(x) = W_P(x, X_O^{A,*}) \leq W_O(x, X_O^{A,*}) = V_{O,G}(x) \tag{A.13}$$

We now prove $V_{P,G}(x) < V_{O,G}(x)$ for $x \leq X_G^I$:

$$\begin{aligned} V_{P,G}(x) &= \mathbb{E}_P\left[\left(W_P(X_{\tau_{X_G^I}}, X_O^{A,*}) - I\right) e^{-r\tau_{X_G^I}} \middle| X_0 = x\right] \\ &= \left(W_P(X_G^I, X_O^{A,*}) - I\right)\left(\frac{x}{X_G^I}\right)^{q_P} \\ &\leq \left(W_O(X_G^I, X_O^{A,*}) - I\right)\left(\frac{x}{X_G^I}\right)^{q_P} \\ &\leq \left(W_O(X_G^I, X_O^{A,*}) - I\right)\left(\frac{x}{X_G^I}\right)^{q_O} \equiv V_{O,G}(x) \end{aligned}$$

where the first equality follows from the strong Markov property of the process X, the second equality follows from the fact that $X_{\tau_{X_G^I}} = X_G^I$, the first inequality follows from equations (A.13)–(A.13), and the last inequality holds because $x/X_G^I < 1$ and $q_O < q_P$.[14]

■

Proof of Proposition 5

Before we start the proof of Proposition 5, we establish a preliminary lemma showing that, when P controls the abandonment decision, the project valuation is larger for O than for P.

Lemma 1. *The values of the operating projet to members P and O, when P decides when to abandon and, when the current cash flow is $x > 0$, satisfy*

$$W_P(x, X_P^{A,*}) \leq W_O(x, X_P^{A,*}) \tag{A.14}$$

where $X_P^{A,*}$ is P's optimal abandonment threshold defined in Proposition 2 and where we recall that the functions W_n are given by equation (7) with $n = P, O$ and $X^A = X_P^{A,*}$.

Proof of Lemma 1: For ease of notation, denote by $W_n'(x, X_P^{A,*})$ and $W_n''(x, X_P^{A,*})$ the first and second derivative of W_n with respect to x

[14] Differentiating q_P with respect to μ_P we obtain

$$\frac{\partial q_P}{\partial \mu_P} = -\frac{q_P}{\sqrt{(\mu_n - \frac{\sigma^2}{2})^2 + 2\sigma^2 r}} < 0.$$

Therefore, since $\mu_O > \mu_P$ it follows that $q_O < q_P$.

Using the closed-form expression of W_n given in equation (7) and the closed form expression (11) of the optimal abandonment times $X_P^{A,*}$ and $X_O^{A,*}$ gives

$$W_n'(X_P^{A,*}, X_P^{A,*}) = \frac{1 - m_n}{X_P^{A,*}(r - \mu_n)} \left(X_P^{A,*} - X_n^{A,*} \right) \text{ for } n = P, O. \tag{A.15}$$

Therefore we have $W_P'(X_P^{A,*}, X_P^{A,*}) = 0$ and, $W_O'(X_P^{A,*}, X_P^{A,*}) > 0$. Using the closed-form expression (10) of W_O, we can also calculate the second derivative of W_O

$$W_O''(x, X_P^{A,*}) = m_O(m_O - 1) \left(A - \frac{X_P^{A,*}}{r - \mu_O} \right) \left(\frac{x}{X_P^{A,*}} \right)^{m_O} \text{ for } x \geq X_P^{A,*}. \tag{A.16}$$

Therefore when μ_O is such that $A - \frac{X_P^{A,*}}{r-\mu_O} > 0$, the function $W_O(\cdot, X_P^{A,*})$ is convex in x and when $A - \frac{X_P^{A,*}}{r-\mu_O} < 0$, the function $W_O(\cdot, X_P^{A,*})$ is concave in x. When the function $W_O(\cdot, X_P^{A,*})$ is convex in x, we have for $x \geq X_P^{A,*}$,

$$0 < W_O'(X_P^{A,*}, X_P^{A,*}) \leq W_O'(x, X_P^{A,*})$$

where the inequality follows from (A.15).

When the function $W_O(\cdot, X_P^{A,*})$ is concave in x, that is when μ_O is such that $A - \frac{X_P^{A,*}}{r-\mu_O} < 0$, we have for $x \geq X_P^{A,*}$

$$W_O'(x, X_P^{A,*}) \geq W_O'(\infty, X_P^{A,*}) = \frac{1}{r - \mu_O} > 0.$$

Therefore, for all values of μ_O in the interval (μ_P, r), we have $W_O'(x, X_P^{A,*}) > 0$ for all $x \geq X_P^{A,*}$.

Let us define the operator

$$\mathcal{L}_n = \mu_n x \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \quad \text{for} \quad n = O, P.$$

Standard arguments show that, for $n = O, P$, the function $W_n(\cdot, X_P^{A,*})$ is a solution to

$$\begin{aligned} rW_n(x, X_P^{A,*}) &= \mathcal{L}_n W_n(x, X_P^{A,*}) + x, & x &\geq X_P^{A,*}, \\ W_n(x, X_P^{A,*}) &= A, & x &\leq X_P^{A,*} \end{aligned}$$

Since $\mathcal{L}_O = \mathcal{L}_P + \xi x \frac{\partial}{\partial x}$ where $\xi = \mu_O - \mu_P > 0$, we have that the function $w(x) = W_O(x, X_P^{A,*}) - W_P(x, X_P^{A,*})$ is in turn a solution to

$$rw(x) = \mathcal{L}_P w(x) + \xi x W_O'(x, X_P^{A,*})(x), \qquad x \geq X_P^{A,*}, \tag{A.17}$$

$$w(x) = 0, \qquad x \leq X_P^{A,*}. \tag{A.18}$$

Denoting by $\nu = \tau_{X_P^{A,*}}$ the hitting time of the barrier $X_P^{A,*}$ from above, and by $t \wedge \nu = \inf\{t, \nu\}$, equation (A.17) implies that the process

$$M_t = e^{-rt\wedge\nu} w(X_{t\wedge\nu}) + \int_0^{t\wedge\nu} e^{-rs} \xi X_s W_O'(X_s, X_P^{A,*}) ds$$

is a local martingale under $\mathbb{Q}_P$ and it follows that there exists a sequence of stopping times $\theta_n \uparrow \infty$ such that

$$w(x) = M_0 = \mathbb{E}_P M_{\theta_n} = \mathbb{E}_P \left[e^{-r\nu\wedge\theta_n} w(X_{\nu\wedge\theta_n}) + \int_0^{\nu\wedge\theta_n} e^{-rs} \xi X_s W_O'(X_s, X_P^{A,*}) ds \right]. \tag{A.19}$$

We let now n go to infinity and determine the limit of the right hand side of (A.19).

Let us start with the integral term. Observe that the integrand is non-negative because $W_O'(X_s, X_P^{A,*}) > 0$ and therefore the integral increases as the upper limit of integration increases. By the monotone convergence theorem we therefore have

$$\lim_{n\to\infty} \mathbb{E}_P \left[\int_0^{\nu\wedge\theta_n} e^{-rs} \xi X_s W_O'(X_s, X_P^{A,*}) ds \right] = \mathbb{E}_P \left[\int_0^{\nu} e^{-rs} \xi X_s W_O'(X_s, X_P^{A,*}) ds \right] > 0. \tag{A.20}$$

Let us now turn to the first term inside the expectation in (A.19). Using the closed-form expression of W_n given in the first equation of (10), we can show that both W_O and W_P are such that $|W_n(x, X_P^{A,*})| \leq a_n + b_n|x|$ for some $a_n, b_n > 0$ and for $n = P, O$. Therefore, we also have that $|w(x)| \leq a + b|x|$ for some $a, b > 0$ and hence

$$e^{-r\nu\wedge\theta_n} w(X_{\nu\wedge\theta_n}) \leq a + b e^{-r\nu\wedge\theta_n} X_{\nu\wedge\theta_n} \leq a + b \sup_{t\geq 0} |e^{-rt} X_t|.$$

It is known (see equation (4.2) in Doob (1949)) that the distribution of the lifetime maximum of a Brownian motion with negative drift $\left(B_{P,t} - \alpha t\right)_{t\geq 0}$ with $\alpha > 0$ is given by

$$\mathbb{Q}_P\left(\sup_{t\geq 0}\left(B_{P,t} - \alpha t\right) \geq \beta\right) = e^{-2\alpha\beta} \text{ for all } \beta \geq 0.$$

Because $r > \mu_P$, using the fact that X_t follows a geometric Brownian motion given by equation (1) with $n = P$, we can choose $\alpha = \frac{r-\mu_P}{\sigma} + \frac{\sigma}{2} > 0$ and $\beta = \sigma^{-1}\ln\left(\frac{y}{x}\right) \geq 0$ for $y \geq x$ to obtain

$$\mathbb{Q}_P\left(\sup_{t\geq 0}\left(B_{P,t} - \alpha t\right) \geq \beta\right) = \mathbb{Q}_P\left(\sup_{t\geq 0}\left(e^{-rt}X_t\right) \geq y\right) = \left(\frac{y}{x}\right)^{-2\left(\frac{r-\mu_P}{\sigma^2}+\frac{1}{2}\right)}.$$

It follows that the density f of the random variable $\sup_{t\geq 0}\left(e^{-rt}X_t\right)$ is given by

$$f(y) = 2\left(\frac{r-\mu_P}{\sigma^2} + \frac{1}{2}\right)y^{-1}\left(\frac{y}{x}\right)^{-2\left(\frac{r-\mu_P}{\sigma^2}+\frac{1}{2}\right)}, \text{ for all } y \geq x.$$

Using the closed form of the density f, we now calculate by direct integration the expectation

$$\mathbb{E}_P\left[\sup_{t\geq 0}|e^{-rt}X_t|\right] = \int_x^\infty yf(y)dy = x\left(1 + \frac{\sigma^2}{2(r-\mu_P)}\right)$$

and the desired result now follows from the dominated convergence theorem by noting that

$$\lim_{n\to\infty}\mathbb{E}_P\left[e^{-r\nu\wedge\theta_n}w(X_{\nu\wedge\theta_n})\right] = \mathbb{E}_P\left[e^{-r\nu}w(X_\nu)\right] = 0. \tag{A.21}$$

Substituting equations (A.20) and (A.21) in equation (A.19), and noticing that $\nu = \tau_{X_P^{A,*}}$ gives

$$w(x) = \mathbb{E}_P\left[\int_0^{\tau_{X_P^{A,*}}} e^{-rs}\xi X_s W_O'(X_s, X_P^{A,*}))ds\right] > 0 \text{ for all } x > X_P^{A,*} \tag{A.22}$$

and, recalling the definition of w, this in turn implies (A.14) and concludes the proof.

∎

We can now complete the proof of Proposition 5. When the underinvestment condition (18) holds, we need to prove the right inequality of (19), that is, $V_P^*(x) \leq V_O(x, X_P^{I,*}, X_P^{A,*})$, for all $x > 0$ since the last inequality shows that both O and P vote for licensing when P acts as a dictator of the group.

When $x \geq X_P^{I,*}$, from equation (10) in Proposition 1, we have $V_P^*(x) = W_P(x, X_P^{A,*}) - I$ and $V_O(x, X_P^{I,*}, X_P^{A,*}) = W_O(x, X_P^{A,*}) - I$. Therefore the result follows from equation (A.14) in Lemma 1.

When $x \leq X_P^{I,*}$, we have

$$\begin{aligned}
V_P^*(x) &= \mathbb{E}_P\left[\left(W_P(X_{\tau_{X_P^{I,*}}}, X_P^{A,*}) - I\right) e^{-r\tau_{X_P^{I,*}}} \middle| X_0 = x\right] \\
&= \left(W_P(X_P^{I,*}, X_P^{A,*}) - I\right)\left(\frac{x}{X_P^{I,*}}\right)^{q_P} \\
&\leq \left(W_O(X_P^{I,*}, X_P^{A,*}) - I\right)\left(\frac{x}{X_P^{I,*}}\right)^{q_P} \\
&\leq \left(W_O(X_P^{I,*}, X_P^{A,*}) - I\right)\left(\frac{x}{X_P^{I,*}}\right)^{q_O} \\
&= V_O(x, X_P^{I,*}, X_P^{A,*})
\end{aligned}$$

Where the first inequality follows from equation (A.14) and the second inequality follows from the fact that $q_O < q_P$ and $x \leq X_P^{I,*}$. ∎

Proof of Proposition 6

Given any governance rule, the abandonment threshold are ranked according the beliefs: pessimistic members have higher abandonment thresholds than more optimistic members. As the cash flow falls, the group abandons the project when the cash flow process hits the optimistic pivot n_O's threshold. The group abandonment threshold coincides with n_O's abandonment threshold, member n_O is pivotal for the abandonment decision: $X_G^A = X_{n_O}^{A,*}$.

We now show that condition (25) implies that the abandonment options evaluations are uniformly ranked with beliefs, that is

$$W_1(x, X_G^A) \leq W_2(x, X_G^A) \leq \cdots \leq W_N(x, X_G^A) \text{ for all } x > 0. \tag{A.23}$$

For $x \leq X_G^A$ condition (A.23) is trivially satisfied because $W_n(x, X_G^A) = A$ for all $x \leq X_G^A$. For any $n = 1, \ldots, N$, using the identity $r - \mu_n = \frac{\sigma^2}{2}(1 - m_n)(q_n - 1)$ and, the closed-form expressions

(10)–(11) of W_n and $X_n^{A,*}$ gives

$$W'_n(X_G^A, X_G^A) \equiv W'_n(X_{n_O}^{A,*}, X_{n_O}^{A,*}) = \frac{m_{n_O} - m_n}{q_n - 1} \frac{A}{X_{n_O}^{A,*}}. \tag{A.24}$$

Recalling that both m_n and q_n are decreasing in n, equation (A.24) implies

$$W'_{n_O-1}(X_G^A, X_G^A) < 0 = W'_{n_O}(X_G^A, X_G^A) < W'_{n_O+1}(X_G^A, X_G^A) < \cdots < W'_N(X_G^A, X_G^A).$$

Using arguments similar to those in the proof of Lemma 1 shows that $W'_n(x, X_G^A) > 0$ for all $x > X_G^A$ and $n = n_O, n_O + 1, \ldots, N$ and that

$$W_{n+1}(x, X_G^A) - W_n(x, X_G^A) = \mathbb{E}_n \left[\int_0^{\tau_{X_O^{A,*}}} e^{-rs} \xi_n X_s W'_{n+1}(X_s, X_G^A)) ds \right] > 0 \text{ for } n = n_O - 1, \ldots, N-1$$

where $\xi_n = \mu_{n+1} - \mu_n > 0$. Therefore we have

$$W_{n_O-1}(x, X_G^A) \leq W_{n_O}(x, X_G^A) \leq \cdots \leq W_N(x, X_G^A) \text{ for all } x > 0. \tag{A.25}$$

We now prove the ranking of the abandonment option valuations of members $n = 1, \ldots, N_O - 1$. Differentiating directly equation (7) with respect to x, we can re-express equation (A.24) as follows

$$W'_n(X_G^A, X_G^A) \equiv W'_n(X_{n_O}^{A,*}, X_{n_O}^{A,*}) = A m_n \left[\frac{1}{X_G^A} - \frac{1}{X_n^{A,*}} \right]. \tag{A.26}$$

For $n = 1, \ldots, N_O - 1$, the sequence m_n is decreasing with respect to n and we have

$$-\frac{2r}{\sigma^2} < m_{n_O-1} < m_n < m_1 < \frac{\frac{\sigma^2}{2} - \sqrt{\left(\frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2} < 0$$

where the upper and lower bounds for this inequalities are given by

$$-\frac{2r}{\sigma^2} = \lim_{\mu_N \to r} m_N, \text{ and } \quad \frac{\frac{\sigma^2}{2} - \sqrt{\left(\frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2} = \lim_{\mu_1 \to 0} m_1.$$

We now prove that $W'_n(X_G^A, X_G^A)$ is increasing with n by showing that it is a decreasing function of m_n. Differentiating equation (A.26) with respect to m_n gives

$$\frac{\partial W'_n(X_G^A, X_G^A)}{\partial m_n} = A\left[\frac{1}{X_G^A} - \frac{1}{X_n^{A,*}}\right] + Am_n \frac{A\frac{\sigma^2}{2}}{\left(X_n^{A,*}\right)^2}. \tag{A.27}$$

Using the closed form expression of X_G^A and $X_n^{A,*}$ given by (11) to develop equation (A.27), it can be shown that $\frac{\partial W'_n(X_G^A, X_G^A)}{\partial m_n} < 0$ holds if and only if

$$m_n\left(2r + m_n\frac{\sigma^2}{2}\right) < rm_{n_O} \text{ for } n = 1, \ldots, N_O - 1. \tag{A.28}$$

Equation (A.28) can be expressed as $f(m_n) < g(m_{n_O})$ where f is the polynomial $f(y) = y(2r+y\frac{\sigma^2}{2})$ and g is the linear function $f(y) = ry$. Notice that $f(0) = g(0) = 0$ and $f\left(\frac{-2r}{\sigma^2}\right) = g\left(\frac{-2r}{\sigma^2}\right) = -\frac{2r^2}{\sigma^2}$ and that $g(y) \leq f(y)$ for $y \in [-\frac{2r^2}{\sigma^2}, 0]$. The minimum value of the polynomial f is reached at the point $y = -\frac{r}{\sigma^2}$.

- If $m_{n_O} < -\frac{r}{\sigma^2}$ and $m_1 \in (m_{n_O}, -\frac{r}{\sigma^2})$, and condition (25) is satisfied, then because the polynomial f is decreasing in the region $y \in (-2\frac{r}{\sigma^2}, -\frac{r}{\sigma^2})$ while the function g is increasing, we have $f(m_n) < g(m_{n_O})$ for $n = 1, \ldots, N_O$.

- If $m_{n_O} < -\frac{r}{\sigma^2}$ and $m_1 > -\frac{r}{\sigma^2}$, and condition (25) is satisfied, then for the same reason as in the previous step, condition (A.28) holds for all n for which $m_n \leq -\frac{r}{\sigma^2}$. For the members n for which $m_n \geq -\frac{r}{\sigma^2}$, we know that the polynomial f is increasing in the region $(-\frac{r}{\sigma^2}, 0)$ and therefore $f(m_n) \leq f(m_1)$ and, as a result, $f(m_n) < g(m_{n_O})$ holds for $n = 1, \ldots, N_O$.

- If $m_{n_O} \geq -\frac{r}{\sigma^2}$, then $m_n \in \left(-\frac{r}{\sigma^2}, \frac{\frac{\sigma^2}{2} - \sqrt{\left(\frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}}{\sigma^2}\right)$ which is a region where f is increasing. In this case, when condition (25) is satisfied we have $f(m_n) < g(m_{n_O})$ holds for $n = 1, \ldots, N_O$.

Therefore, in all subcases, when condition (25) holds, $f(m_n) < g(m_{n_O})$ holds for $n = 1, \ldots, N_O$ and hence $\frac{\partial W'_n(X_G^A, X_G^A)}{\partial m_n} < 0$ which in turn implies that the slope of W_n are ranked at the point $x = X_G^A$:

$$W'_1(X_G^A, X_G^A) < W'_2(X_G^A, X_G^A) < .. < W'_{n_O-1}(X_G^A, X_G^A) < 0.$$

This condition in turn implies that the curves $W_n(\cdot, X_G^A)$ are uniformly ranked with respect to n. To see this, fix $n \in \{1, , \ldots, n_O - 2\}$ and consider the functions $W_n(\cdot, X_G^A))$ and $W_{n+1}(\cdot, X_G^A))$. We have $W_n(X_G^A, X_G^A) = W_{n+1}(X_G^A, X_G^A) = A$ and $W'_n(X_G^A, X_G^A) < W'_{n+1}(X_G^A, X_G^A)$.

We now prove by contradiction that the curves $W_n(\cdot, X_G^A)$ and $W_{n+1}(\cdot, X_G^A)$ cannot cross. Notice that from the closed-form expression (10) of W_{n+1}, it can be checked that the function $W_{n+1}(\cdot, X_G^A)$ is convex with a first derivative growing from $W'_{n+1}(X_G^A, X_G^A) < 0$ at $x = X_G^A$ to $\frac{1}{r-\mu_{n+1}}$ at $x = \infty$. We denote by $\hat{x}$ the point at which $W'_{n+1}(\hat{x}, X_G^A) = 0$ and notice that the function $W_{n+1}(\cdot, X_G^A)$ is decreasing in the interval $(X_G^A, \hat{x})$ and increasing in the interval $(\hat{x}, \infty)$.

Let us assume that the curves $W_n(\cdot, X_G^A)$ and $W_{n+1}(\cdot, X_G^A)$ intersect and define $y \in (X_G^A, \infty)$ as the smallest scalar at which they intersect: $W_n(y, X_G^A) = W_{n+1}(y, X_G^A)$. Since $W'_n(X_G^A, X_G^A) < W'_{n+1}(X_G^A, X_G^A)$, we have $W_n(x, X_G^A) < W_{n+1}(x, X_G^A)$ for x in the right neighbourhood of X_G^A and also asymptotically, when x is large enough.

If $y \in (X_G^A, \hat{x}]$, then using arguments similar to those in the proof of Lemma 1 shows that

$$W_{n+1}(x, X_G^A) - W_n(x, X_G^A) = \mathbb{E}_n \left[\int_0^{\tau_{X_G^A} \wedge \tau_y} e^{-rs} \xi_n X_s W'_{n+1}(X_s, X_G^A)) ds \right] \quad \text{for} \quad X_G^A \leq x \leq y \tag{A.29}$$

where $\xi_n = \mu_{n+1} - \mu_n > 0$. Because the function $W_{n+1}(\cdot, X_G^A)$ is decreasing in the interval $(X_G^A, \hat{x})$, we have $W'_{n+1}(X_s, X_G^A)) < 0$ a.s. for $s \leq \tau_{X_G^A} \wedge \tau_y$. Therefore, equation (A.29) implies $W_{n+1}(x, X_G^A) < W_n(x, X_G^A)$ for all $x \in (X_G^A, y)$ contradicting the fact that $W_n(x, X_G^A) < W_{n+1}(x, X_G^A)$ for x in the right neighbourhood of X_G^A

Let us now instead assume that $y > \hat{x}$. Because $W_n(x, X_G^A) < W_{n+1}(x, X_G^A)$ for x large enough, there must exist $z > y$ such that the two curves cross again: $W_n(z, X_G^A) = W_{n+1}(z, X_G^A)$ and such that $W_n(x, X_G^A) > W_{n+1}(x, X_G^A)$ for $x \in (y, z)$. Using arguments similar to those in the proof of Lemma 1 shows that

$$W_{n+1}(x, X_G^A) - W_n(x, X_G^A) = \mathbb{E}_n \left[\int_0^{\tau_y \wedge \tau_z} e^{-rs} \xi_n X_s W'_{n+1}(X_s, X_G^A)) ds \right] \text{ for } y \leq x \leq z. \tag{A.30}$$

Because $y > \hat{x}$, the function $W_{n+1}(\cdot, X_G^A)$ is increasing in the interval (y, z) and hence $W'_{n+1}(X_s, X_G^A)) > 0$ a.s. for $s \leq \tau_y \wedge \tau_z$. Thus inequality (A.30) shows that $W_{n+1}(x, X_G^A) > W_n(x, X_G^A)$ for $y \leq x \leq z$ contradicting the fact that $W_n(x, X_G^A) > W_{n+1}(x, X_G^A)$ on this interval.

In all cases, we thus conclude that the two functions $W_n(\cdot, X_G^A)$ and $W_{n+1}(\cdot, X_G^A)$ cannot intersect and since the index n is arbitrary in the set $\{1, \ldots, N_O - 2\}$ we have

$$W_1(x, X_G^A) \leq W_2(x, X_G^A) \leq \cdots \leq W_{n_O - 1}(x, X_G^A) \text{ for all } x > 0. \tag{A.31}$$

Merging the rankings in (A.25) and (A.31), we conclude

$$W_1(x, X_G^A) \leq W_2(x, X_G^A) \leq \cdots \leq W_N(x, X_G^A) \text{ for all } x > 0. \tag{A.32}$$

Furthermore, inequality (24) implies

$$0 \leq W_1(x, X_G^A) - I \leq W_2(x, X_G^A) - I \leq \cdots \leq W_N(x, X_G^A) - I \text{ for all } x > 0. \tag{A.33}$$

We now prove the rankings of the compound option valuation (26). When $x \geq X_G^I$, from equation (10) in Proposition 1, we have $V_n^*(x) = W_n(x, X_G^A) - I$ and condition (A.33) implies the ranking (26).

When $x \leq X_G^I$, we have for $n = 1, \ldots, N-1$

$$\begin{aligned}
V_n(x, X_G^I, X_G^A) &= \mathbb{E}_n\left[\left(W_n(X_{\tau_{X_G^I}}, X_G^A) - I\right) e^{-r\tau_{X_G^I}} \,\middle|\, X_0 = x\right] \\
&= \left(W_n(X_G^I, X_G^A) - I\right)\left(\frac{x}{X_G^I}\right)^{q_n} \\
&\leq \left(W_{n+1}(X_G^I, X_G^A) - I\right)\left(\frac{x}{X_G^I}\right)^{q_n} \\
&\leq \left(W_{n+1}(X_G^I, X_G^A) - I\right)\left(\frac{x}{X_G^I}\right)^{q_{n+1}} \\
&= V_{n+1}(x, X_G^I, X_G^A)
\end{aligned}$$

where the first inequality follows from equation (24) and the second inequality follows from the fact that $q_{n+1} < q_n$ and $x \leq X_G^I$. Therefore the inequalities (26) are satisfied which concludes the proof. ∎

Proof of Proposition 7

Fix a voting rule within the class described in Definition 2 and assume that the current cash flow process is large enough that no group member votes for abandonment today. Recall that group members' optimal abandonment thresholds are ranked according to beliefs, $X_N^{A,*} < X_{N-1}^{A,*} < \cdots < X_1^{A,*}$. As the cash flow process decreases over time, more group members wish to abandon and the group will abandon at any time when all members of a decisive coalition votes for abandonment. The first time this happens is $\tau_{X_{n_O}^{A,*}}$ when the members of the decisive coalition $\mathcal{C}^A = \{1, 2, \ldots, k, v_M\}$ unanimously vote for abandonment. As the cash flow process decreases from large values to $X_{n_O}^{A,*}$, the last member to join the decisive coalition $\mathcal{C}^A$ is the optimistic pivot n_O and therefore n_O is the pivotal voter for abandonment.

For the licensing decision, recall from the inequalities (26) that the individual project valuations under group behaviour are ranked according to beliefs. The group will thus invest if the license fees is below the project valuation of a set of group members that form a decisive coalition. The smallest coalition that can be formed for licensing is $\mathcal{C}^L = \{N, N-1, .., N-k+1, v_1\}$. The member of that coalition who is the least eager to license is the pessimistic pivot n_P and therefore n_P is the pivotal member for licensing.

For the investment decision, the analysis of a group of two members in Subsection 2.3 shows that both group members can be pivotal for the investment timing decision. This happens in a group with two members because the second best investment thresholds are not ranked monotonically with believes. For the same reasons, this also happens with larger group and therefore any group member can be pivotal for the investment timing decision. ∎

Proof of Proposition 8

Following arguments similar to that of Proposition 3 and making use of the smooth pasting conditions for the first and second best optimal stopping problem for member n_I, it can be shown that

$$f(X_{n_I}^{I,*}) = W_{n_I}(X_{n_I}^{I,*}, X_{n_I}^{A,*}) \text{ and } f(X_{n_I}^{I,\text{SB}}) = W_{n_I}(X_{n_I}^{I,\text{SB}}, X_G^A) \tag{A.34}$$

where

$$f(x) := \frac{1 - m_{n_I}}{q_{n_I} - m_{n_I}} \frac{1}{r - \mu_{n_I}} x + \frac{q_{n_I}}{q_{n_I} - m_{n_I}} I. \tag{A.35}$$

Note that $f(X_{n_I}^{A,*}) = A + \frac{q_{n_I}}{q_{n_I} - m_{n_I}}(I - A) > A = W_{n_I}(X_{n_I}^{A,*}, X_{n_I}^{A,*}) \geq W_{n_I}(X_{n_I}^{A,*}, X_G^A)$ and that both functions $W_{n_I}(\cdot, X_G^A)$ and $W_{n_I}(\cdot, X_{n_I}^{A,*})$ are increasing for x large enough and have an asymptote of slope $\frac{1}{r-\mu_I}$ at infinity. Therefore, each equation in (A.34) has a unique root. Furthermore, because $X^{A,*}_{n_I}$ is the optimal abandonment time for member n_I, we have

$$W_{n_I}(x, X^A_G) \leq W_{n_I}(x, X^{A,*}_{n_I}) \text{ for all } x > 0,$$

and hence the line described by the linear equation $y = f(x)$ intersects the curve defined by the function $W_{n_I}(\cdot, X^A_G)$ at a larger cash flow level x than it intersects the curve defined by the function $W_{n_I}(\cdot, X^{A,*}_{n_I})$. This means that $X^{I,*}_{n_I} < X^{I,\text{SB}}_{n_I}$ which concludes the proof.

Proof of Proposition 9

Similar reasonaning to that in the proof of Proposition 4 shows that $V^*_1(x) < V^*_2(x) < \cdots < V^*_N(x)$ for all $x > 0$. Therefore, when $L < V^*_1(x)$, all group members would acquire the license if they have full control of the timing decisions of the project.

Recall from the inequalities (26) that the individual project valuations under group behaviour are ranked according to beliefs. Therefore, the left inequality of (29) implies

$$V_1(x, X^I_G, X^A_G) < V_2(x, X^I_G, X^A_G) < \cdots < V_{n_P}(x, X^I_G, X^A_G) < L \tag{A.36}$$

Therefore all group members $1, 2, .., n_P$ vote against licensing, and form a decisive coalition leading the group to reject the licence which concludes the proof. ∎

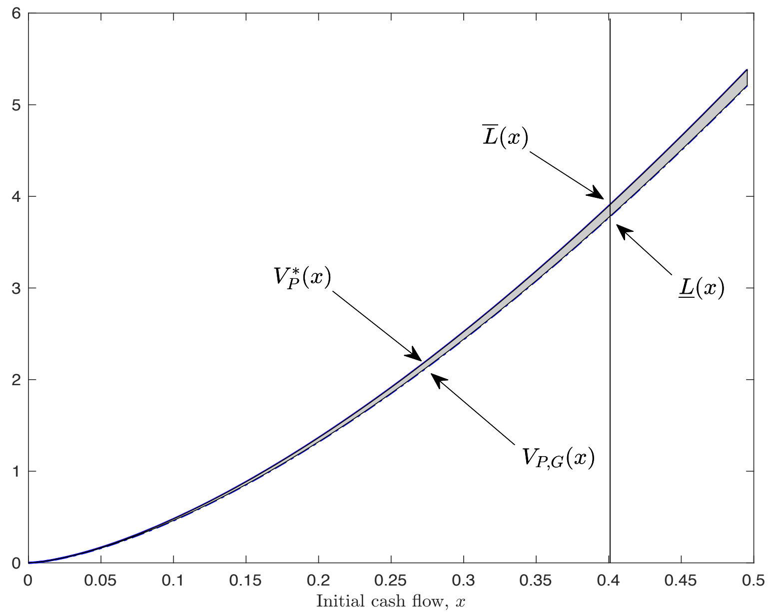

Figure 1: Pessimist's valuation under group and individually optimal policies

The figure reports member P's subjective valuation under the group policy, $V_{P,\mathrm{G}}(x)$ and under P's optimal policy, $V_P^*(x)$. Parameter values: $r = 0.05$, $\sigma = 0.3$, $A = 1$, $I = 1.1$, $\mu_P = 0.01$, $\mu_O = 0.04$.

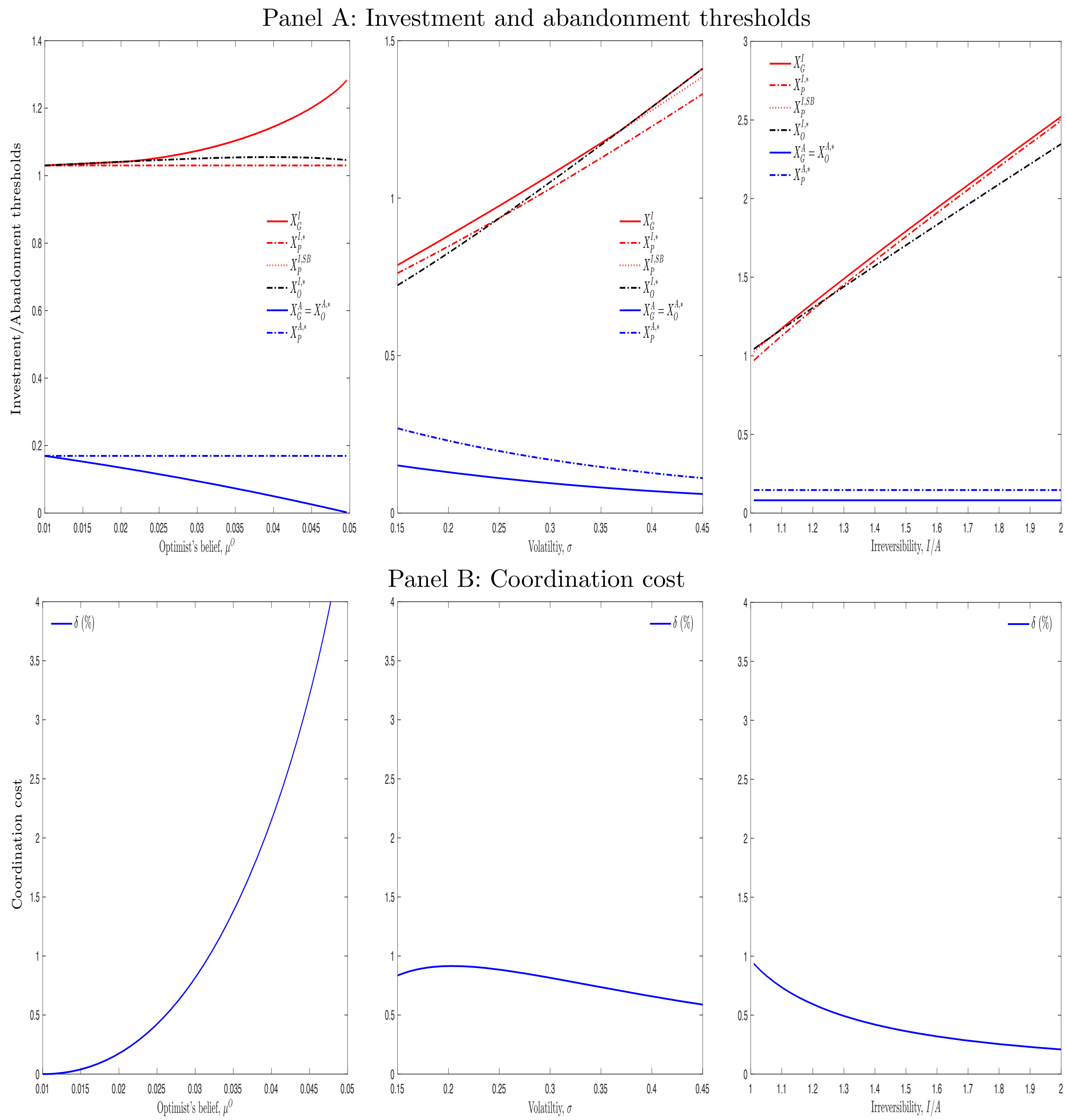

Figure 2: Coordination cost: the effect of polarization, volatility, and investment irreversibility.

Panel A reports the group (solid lines) and individual (dashed lines) investment and abandonment thresholds. The dashed-red and dashed-blue lines refer to P's individually optimal investment and abandonment threshold. The black-dashed line refers to O's individually optimal investment threshold. O's individually optimal abandonment threshold corresponds to the group abandonment threshold, X_{G}^{A} . Panel B reports the coordination cost δ, in percent, as defined in equation (3). Each figure reports comparative statics results around the following baseline configuration of parameters: $r = 0.05$, $\mu_O = 0.03$, $\mu_P = 0.01$, $\sigma = 0.3$, $A = 1$ and $I = 1.1$.

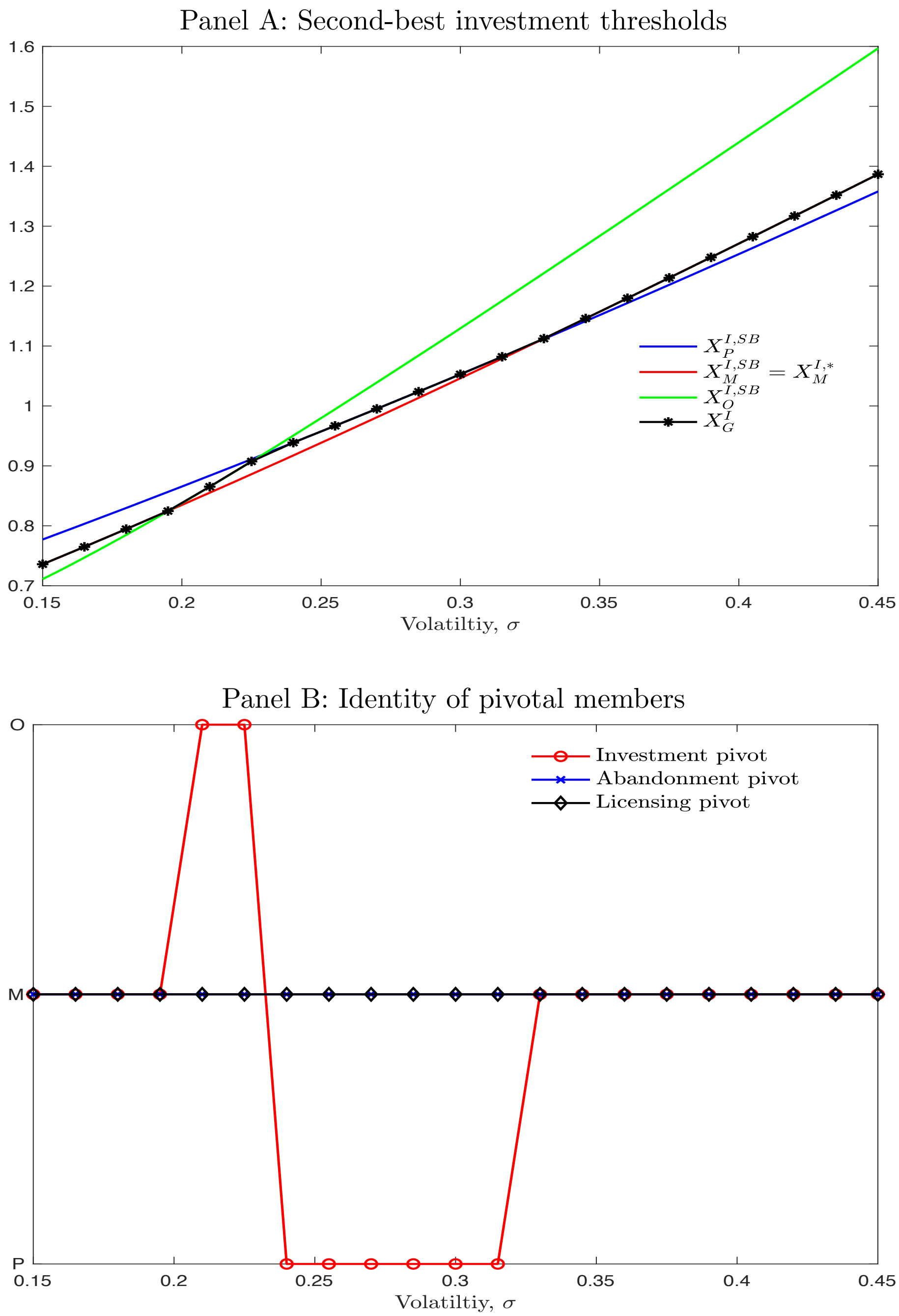

Figure 3: Three-agent group.

Panel A reports, as a function of project cash flow volatility, the individual investment thresholds of each group member, given the group abandonment threshold $X_G^A = X_O^{A,*}$. The line highlighted with stars, '$*$', represents the group investment threshold X_G^I, which, under strict majority, is the second highest threshold. Panel B reports the identity of the pivotal voter for the licensing, investment, and abandonment decisions, as a function of project cash flow volatility. Parameter values: $r = 0.05$, $A = 10$, $I = 11$, $\mu_P = 0.01$, $\mu_M = 0.025$, and $\mu_O = 0.04$.

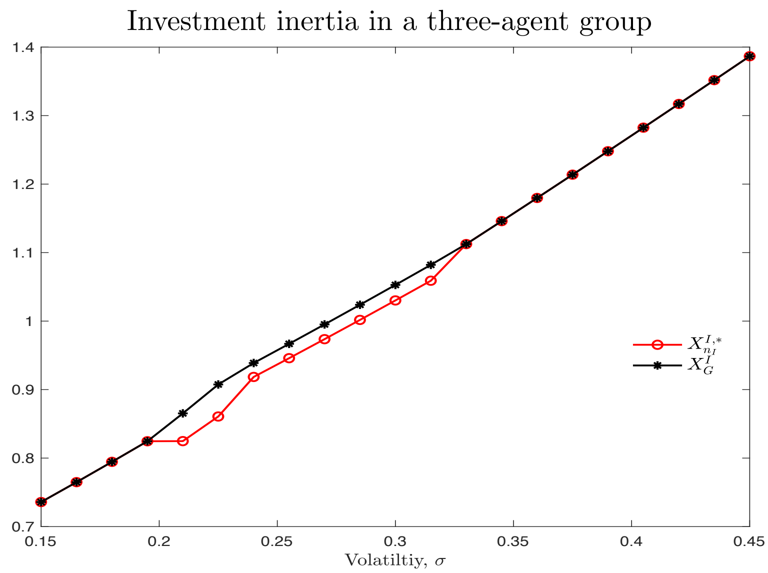

Figure 4: Investment inertia in a three-member group.

The figure reports, as a function of project cash flow volatility, the group investment threshold X_G^I and the individually optimal investment threshold of the agent that is pivotal $X_{\mathrm{n}_I}^{I,*}$. Panel A refers to a three-agent group and Panel B refers to a four-agent group. Parameter values: $r = 0.05$, $A = 10$, $I = 11$, $\mu_P = 0.01$, $\mu_M = 0.025$, and $\mu_O = 0.04$.

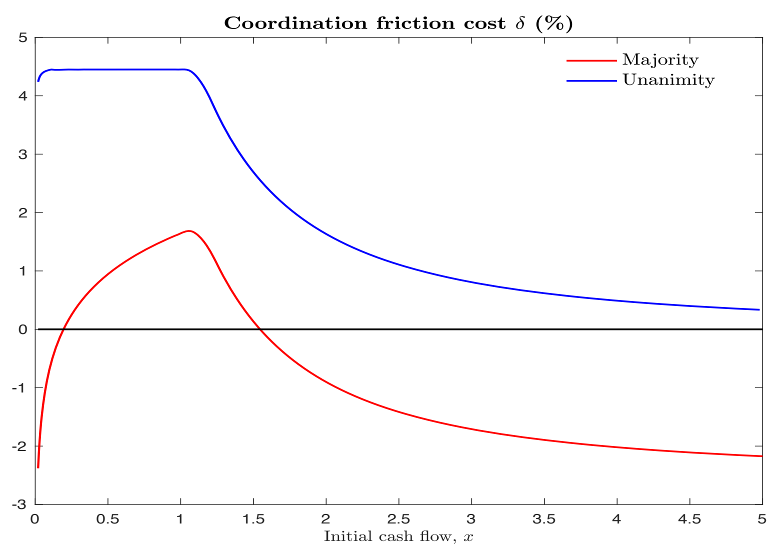

Figure 5: Underinvestment and voting protocols in a three-agent group.

The figure reports the coordination friction cost (in percent) defined in equations (30) and (31) in a three-agent group. The red line reports the cost under majority, $\delta^{\text{Majority}}(x)$, while the blue line reports the cost under unanimity, $\delta^{\text{Unanimity}}(x)$. Parameter values: $r = 0.05$, $A = 10$, $I = 11$, $\mu_P = 0.01$, $\mu_M = 0.011$, and $\mu_O = 0.049$.

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Group-Managed Real Options: Voting, Polarization, and Investment Dynamics

Online Appendix

Lorenzo Garlappi, **Ron Giammarino**, and **Ali Lazrak**

A Can trading among group members solve underinvestment?

In this Online Appendix we study whether allowing one member of the group to purchase the shares, and hence the votes, of other group members would eliminate the underinvestment problem. We consider a 2 members group in which each group member can trade their shares at a given post-investment date and show that a tender offer will typically *not* resolve the investment inefficiency. This happens because, after investment has taken place, O has full discretion over the remaining abandonment decision. Anticipating the possibility of facing an unfavorable tender offer, P responds by opposing the initial investment. Thus, the ability to trade does not alter the pessimist's incentives to oppose investment. Precommitting to a "shotgun provision," where a purchase offer by one member of the group must simultaneously contain an offer to sell at the same price reduces the bargaining power of O and makes investment more attractive to P. Our model highlights the novel insight of the shotgun option as a device to balance the future distribution of bargaining power among partners.[1]

[1]See Landeo and Spier (2014) for a legal treatment of shotgun provisions and a broad discussion of their role in the judicial management of business divorce.

Specifically, we consider the investment problem of Section 2 where the initial state is $X_0 = x$ and the underinvestment condition (18) is satisfied. If there is no trading between group members the analysis of Section 2 implies an equilibrium outcome where no investment takes place. In this equilibrium, O and P receive the time-0 allocation $\mathcal{A}_0^{\text{NT}} = (a_O^{\text{NT}}, a_P^{\text{NT}}) = (L, L)$.

We allow group members to trade their shares after initial license fees (L) and investment (I) are sunk. Allowing for trade *before* the investment cost I is sunk will trivially resolve the underinvestment problem because the optimist would buy out the pessimist and run the project alone. We do not consider this outcome to be economically interesting in situations when both group members have inalienable human capital that is required for the existence of the partnership. Instead, we assume that trading takes place when P and O first disagree about the abandonment decision, that is at time $\tau_{X_P^{A,*}} = \inf\{t \geq \tau_{X_G^I} : X_t \leq X_P^{A,*}\}$, and that P and O cannot commit to any price prior to time $\tau_{X_P^{A,*}}$. We investigate whether trading at this date can solve the underinvestment problem.

Tender offer at time $\tau_{X_P^{A,*}}$. We solve the problem by backward induction, which guarantees that the equilibrium strategy is subgame perfect. At time $\tau_{X_P^{A,*}}$, the cash flow level is $X_{\tau_{X_P^{A,*}}} = X_P^{A,*}$. In the absence of trading, since O is pivotal, the group will abandon the project at the stopping time $\tau_{X_O^{A,*}} > \tau_{X_P^{A,*}}$. At this time, the values of O and P's shares are, respectively, $W_O(X_P^{A*}, X_O^{A,*})$ and $W_P(X_P^{A,*}, X_O^{A,*})$ where the function W_n is defined for $n \in \{O, P\}$ by

$$W_n(x, y) = \mathbb{E}_n \left[\int_0^{\tau_y} X_t e^{-rt} dt + A e^{-r\tau_y} \middle| X_0 = x \right], \quad x, y > 0 \tag{A.1}$$

with τ_y denoting the hitting time of the threshold y. In the proof of Proposition 4, we show that $W_P(x, X_O^{A,*}) \leq W_O(x, X_O^{A,*})$ for all $x \geq 0$. Therefore, both O and P would agree on a tender offer in which O buys P's shares at any price q satisfying.

$$W_P(X_P^{A,*}, X_O^{A,*}) \leq q \leq W_O(X_P^{A,*}, X_O^{A,*}). \tag{A.2}$$

Because at time $\tau_{X_P^{A,*}}$ only P wants to abandon, O is *de facto* holding P up. It is then natural to assume that O has full bargaining power when trading with P. Under this assumption, the tender offer that O makes is no larger than $W_P(X_P^{A,*}, X_O^{A,*})$. There is therefore no gain from trade to P. Since P rationally anticipates this, his second-best investment threshold is not altered by the opportunity to trade at his preferred abandonment time. In contrast, O does gain from trading, but as we show in the proof of Proposition 10, the gain from trade does not alter her preferred investment timing. The investment timing of the group and P's initial valuation of the project are therefore not altered by trading. As a result P votes against licensing at time 0 and the underinvestment problem is not solved. The following proposition formalizes this result.

Proposition 10. *If O has full bargaining power when trading with P, then she will make a tender offer for P's share at the price $q = W_P(X_P^{A,*}, X_O^{A,*})$. At this price, the tender offer does not solve the underinvestment problem.*

Tender offer with shotgun provision at time $\tau_{X_P^{A,*}}$. Suppose instead that P and O sign a shotgun provision at time $t = 0$ whereby each can offer a price q to buy the shares of the other member at time $\tau_{X_P^{A,*}}$. If the offer is rejected, the recipient of the offer must buy the shares of the offerer at the price q. Intuitively, offers cannot be too low because, if the recipient rejects, the offerer is forced to sell at the stated price. It cannot be too high either because if accepted, the offerer acquires the recipient's shares at an expensive price. In sum, the shotgun provision generates a more balanced distribution of the gains from trade between O and P.

To illustrate how the presence of a shotgun provision solves the underinvestment problem and without loss of generality, we consider a case that is least favorable to P, that is, when only O can trigger a shotgun option by making a tender offer. The following proposition shows that, even in this conservative case, trading with a shotgun provision solves the underinvestment problem.

Proposition 11. *If O has full bargaining power when trading with P and tender offers have a shotgun provision, then there is a unique equilibrium in which O offers P the price $q = A$ to acquire his share. If O's individually optimal investment threshold is smaller than P's, that is, $X_O^{I,*} \leq X_P^{I,*}$, the shotgun provision solves the underinvestment problem.*

Intuitively, the gain from trade from the shogun provision at time $\tau_{X_P^{A,*}}$, makes the project more attractive to P and sways him to vote for licensing at time 0. The condition $X_O^{I,*} \leq X_P^{I,*}$ is sufficient for solving the underinvestment problem but not necessary. As we show in Section 3.4, this condition holds for projects with intangible assets (high I/A). [2]

This result shows the importance of shotgun options in our framework. Trading without a shotgun option is ineffective in resolving the underinvestment problem when O holds strong bargaining power in trading with P. In this case, all the gain from trade goes to O and, as a result, P opposes investment at the initial stage. The threat that P may reject a shotgun offer initiated by O, precludes O from offering an excessively low price, thus curbing O's strong bargaining power and making the investment more attractive to P.[3]

B Proofs

Proof of Proposition 10

Both group members anticipate that P will sell his shares to O at time $\tau_{X_P^{A,*}}$ at the price $q = W_P(X_P^{A*}, X_O^{A,*})$.

Let us consider P's investment timing decision first. Starting with $X_0 = x > 0$ at time 0, P's perceived payoff from investing at an arbitrary stopping time τ and trading at time $\tau_{X_P^{A,*}} = \inf\{t \geq \tau : X_t \leq X_P^{A,*}\}$ is

$$\mathbb{E}_P\left[e^{-r\tau}\left(\int_\tau^{\tau_{X_P^{A,*}}} X_t e^{-r(t-\tau)}dt + e^{-r(\tau_{X_P^{A,*}}-\tau)} q - I\right)\middle| X_0 = x\right]$$
$$\equiv \mathbb{E}_P\left[-e^{-r\tau}I + \int_\tau^{\tau_{X_P^{A,*}}} X_t e^{-rt}dt + e^{-r\tau_{X_P^{A,*}}} q \middle| X_0 = x\right]$$

[2] When $X_O^{I,*} > X_P^{I,*}$, the group investment threshold coincides with O's individually optimal investment threshold. O acts then as a dictator for both the investment and abandonment decisions. In this case, although P is compensated through trading for the group suboptimal abandonment policy, P's project valuation is negatively impacted by the fact that the group invests at a threshold $X_O^{I,*}$ that is suboptimal for P. As a result, even in the presence of a shotgun provision, a tender offer may or may not solve the underinvestment problem depending on the range of the licensing fee L.

[3] Notice that while in our framework only O can trigger a shotgun offer, the shotgun provision is in principle available to both O and P. Potentially, with more bargaining power, P could make a strategic tender offer for O's shares. Hence if the shotgun is also available to P he would place an even higher value on the initial investment. Regardless of who is the first to trigger the tender offer with shotgun clause, the underinvestment problem would be resolved.

with

$$q = W_P(X_P^{A*}, X_O^{A,*}) = \mathbb{E}_P \left[\int_{\tau_{X_P^{A,*}}}^{\tau_{X_O^{A,*}}} X_t e^{-r(t-\tau_{X_P^{A,*}})} dt + e^{-r(\tau_{X_O^{A,*}} - \tau_{X_P^{A,*}})} A \middle| X_{\tau_{X_P^{A,*}}} = X_P^{A,*} \right].$$

Using the law of iterated expectation, we see that P's perceived payoff from investing at an arbitrary time $\tau < \tau_{X_P^{A,*}}$ is

$$\mathbb{E}_P \left[-e^{-r\tau} I + \int_{\tau}^{\tau_{X_P^{A,*}}} X_t e^{-rt} dt + e^{-r\tau_{X_P^{A,*}}} A \middle| X_0 = x \right]$$

Therefore trading does not alter the payoff to P and the preferred investment policy remains identical to the no trading preferred investment threshold $X_P^{I,\text{SB}}$ defined in equation (14).

Starting at time 0 with an initial cash flow $X_0 = x > 0$, O's perceived payoff from investing at an arbitrary stopping time τ and trading at time $\tau_{X_P^{A,*}} = \inf\{t \geq \tau : X_t \leq X_P^{A,*}\}$ is

$$\mathbb{E}_O \left[e^{-r\tau} \left(\int_{\tau}^{\tau_{X_O^{A,*}}} X_t e^{-r(t-\tau)} dt + A e^{-r(\tau_{X_O^{A,*}} - \tau)} + e^{-r(\tau_{X_P^{A,*}} - \tau)} g - I \right) \middle| X_0 = x \right]$$
$$\equiv \mathbb{E}_O \left[-e^{-r\tau} I + \int_{\tau}^{\tau_{X_O^{A,*}}} X_t e^{-rt} dt + e^{-r\tau_{X_O^{A,*}}} A + e^{-r\tau_{X_P^{A,*}}} g \middle| X_0 = x \right]$$

where the constant g is the gain from trade for O given by

$$g = W_O(X_P^{A*}, X_O^{A,*}) - q = W_O(X_P^{A*}, X_O^{A,*}) - W_P(X_P^{A*}, X_O^{A,*}).$$

The perceived payoff by O from investing at an arbitrary time τ is therefore given by

$$\mathbb{E}_O \left[-e^{-r\tau} I + \int_{\tau}^{\tau_{X_O^{A,*}}} X_t e^{-rt} dt + e^{-r\tau_{X_O^{A,*}}} A \middle| X_0 = x \right] + K$$

where $K = g\mathbb{E}_O \left[e^{-r\tau_{X_P^{A,*}}} \middle| X_0 = x \right] = g \left(\frac{x}{X_P^{A,*}} \right)^{m_O} > 0$ with $m_O < 0$ defined in equation (9), is a constant that is independent from the choice of τ. Consequently, O's the preferred investment threshold remains identical to the no trading investment threshold $X_O^{I,*}$ defined by equation (12) with $n = O$.

We conclude that the group investment and abandonment policies remain identical when trading is introduced. Because O is holding up P in the abandonment decision, O has full bargaining power when buying P's shares and get all the gains from trade. Specifically, the time-0 trading allocation is $\mathcal{A}_0^{\mathrm{T}} = (a_O^{\mathrm{T}}, a_P^{\mathrm{T}})$ with

$$a_P^{\mathrm{T}} = V_{P,G}(x) \tag{B.1}$$

$$a_O^{\mathrm{T}} = V_{O,G}(x) + K. \tag{B.2}$$

Given condition (18), it follows that $a_P^{\mathrm{T}} < I$. Thus P votes against licensing and this implies that a tender offer does not solve the underinvestment problem. ∎

Proof of Proposition 11

Suppose O makes a shotgun offer to P at a price q at time $\tau_{X_P^{A,*}}$. If $q \geq A$, then P accepts the offer because the offer price is larger than A, that is, P's valuation of the project when $X_t = X_P^{A,*}$. The payoff to P is q and the payoff to O is

$$W_O(X_P^{A,*}, X_O^{A,*}) + W_O(X_P^{A,*}, X_O^{A,*}) - q.$$

If $q < A$, P rejects the offer and exercises the shotgun provision and acquire O shares at a price q . In this case the payoff to P is $2A - q$ and the payoff to O is q.

Given P's best responses, the best shotgun offer that O can make is $q = A$. The outcome of such an offer is that P sells the company to O and the $\tau_{X_P^{A,*}}$-allocation is $\mathcal{A}^{\mathrm{T}}_{\tau_{X_P^{A,*}}} = (W_O(X_P^{A,*}, X_O^{A,*}) + (W_O(X_P^{A,*}, X_O^{A,*}) - A), A)$.

Starting with $x > 0$ at time 0 and anticipating the allocation $\mathcal{A}^{\mathrm{T}}_{\tau_{X_P^{A,*}}}$ at time $\tau_{X_P^{A,*}}$, P's perceived payoff from investing at an arbitrary stopping time τ and trading with a shotgun option at time

$\tau_{X_P^{A,*}} = \inf\{t \geq \tau : X_t \leq X_P^{A,*}\}$ is

$$\mathbb{E}_P \left[e^{-r\tau} \left(\int_\tau^{\tau_{X_P^{A,*}}} X_t e^{-r(t-\tau)} dt + e^{-r(\tau_{X_P^{A,*}} - \tau)} A - I \right) \middle| X_0 = x \right].$$

Therefore, the preferred investment timing threshold for P is $X_P^{I,*}$ defined in equation (12) with $n = P$.

Starting at time 0 with an initial cash flow $X_0 = x > 0$, O's perceived payoff from investing at an arbitrary stopping time τ and trading with a shotgun option at time $\tau_{X_P^{A,*}} = \inf\{t \geq \tau : X_t \leq X_P^{A,*}\}$ is

$$\mathbb{E}_O \left[e^{-r\tau} \left(\int_\tau^{\tau_{X_O^{A,*}}} X_t e^{-r(t-\tau)} dt + Ae^{-r(\tau_{X_O^{A,*}} - \tau)} + e^{-r(\tau_{X_P^{A,*}} - \tau)} h - I \right) \middle| X_0 = x \right]$$

where the constant h is the gain from trade for O given by

$$h = W_O(X_P^{A*}, X_O^{A,*}) - A \geq 0.$$

The perceived payoff by O from investing at an arbitrary time τ is therefore given by

$$\mathbb{E}_O \left[e^{-r\tau} \left(\int_\tau^{\tau_{X_O^{A,*}}} X_t e^{-r(t-\tau)} dt + Ae^{-r(\tau_{X_O^{A,*}} - \tau)} - I \right) \middle| X_0 = x \right] + K',$$

where $K' = h\mathbb{E}_O \left[e^{-r\tau_{X_O^{A,*}}} \middle| X_0 = x \right] = h \left(\frac{x}{X_O^{A,*}} \right)^{m_O} > 0$ with $m_O < 0$ defined in equation (9), is a constant that is independent from the choice of τ. Consequently, O's the preferred investment threshold remains identical to the individually optimal investment threshold $X_O^{I,*}$ defined by equation (12) with $n = O$.

If the condition $X_O^{I,*} \leq X_P^{I,*}$ holds, the group will invest at the threshold $X_P^{I,*}$ and the time 0 trading allocation is $\mathcal{A}_0^{\mathrm{T}} = (a_O^{\mathrm{T}}, a_P^{\mathrm{T}})$ with

$$a_P^{\mathrm{T}} = V_P^*(x) \tag{B.3}$$

$$a_O^{\mathrm{T}} = V_O(x, X_P^{I,*}, X_O^{A,*}) + K'. \tag{B.4}$$

When the underinvestment condition (18) holds, $a_P^{\mathrm{T}} = V_P^*(x) \geq L$. Thus P votes for licensing. From Proposition 4, we have $V_O(x, X_P^{I,*}, X_O^{A,*}) > V_P^*(x)$ and hence $a_O^{\mathrm{T}} = V_O(x, X_P^{I,*}, X_O^{A,*}) + K' \geq V_O(x, X_P^{I,*}, X_O^{A,*}) \geq V_P^*(x) \geq L$. Consequently, O also votes for licensing at time 0. The group will then acquire the licence at time 0 and the underinvestment problem is resolved.

When $X_P^{I,*} < X_O^{I,*}$, the group will invest at the threshold $X_O^{I,*}$ and the time 0 trading allocation is $\mathcal{A}_0^{\mathrm{T}} = (a_O^{\mathrm{T}}, a_P^{\mathrm{T}})$ with

$$a_P^{\mathrm{T}} = V_P(x, X_O^{I,*}, X_P^{A,*}) \tag{B.5}$$

$$a_O^{\mathrm{T}} = V_O^*(x) + K'. \tag{B.6}$$

We have $V_P(x, X_O^{I,*}, X_P^{A,*}) < V_P^*(x)$ and the underinvestment will be solved for $L \in [V_{P,\mathrm{G}}(x), V_P(x, X_O^{I,*}, X_P^{A,*})]$ but if $L \in (V_P(x, X_O^{I,*}, X_P^{A,*})\ ,\ V_P^*(x)]$, the underinvestment problem will not be solved.

∎