Learning about latent dynamic trading demand*

Xiao Chen
Department of Mathematics
Rutgers University
110 Frelinghuysen Road
Piscataway, NJ 08854-8019, USA

Jin Hyuk Choi
Department of Mathematical Sciences
Ulsan National Institute of Science and Technology (UNIST)
UNIST-gil 50
Ulsan 689-798, Republic of Korea

Kasper Larsen
Department of Mathematics
Rutgers University
110 Frelinghuysen Road
Piscataway, NJ 08854-8019, USA

Duane J. Seppi
Tepper School of Business
Carnegie Mellon University
5000 Forbes Avenue
Pittsburgh PA 15213, USA

June 2, 2021

*The third author has been supported by the National Science Foundation under Grant No. DMS 1812679 (2018 - 2021). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation (NSF).
The corresponding author is Duane J. Seppi. Xiao Chen has email: xc206@math.rutgers.edu, Jin Hyuk Choi has email: jchoi@unist.ac.kr, Kasper Larsen has email: KL756@math.rutgers.edu, and Duane J. Seppi has email: ds64@andrew.cmu.edu.

Abstract: This paper presents an equilibrium model of dynamic trading, learning, and pricing by strategic investors with trading targets and price impact. Since trading targets are private, rebalancers and liquidity providers filter the child order flow over time to estimate the latent underlying parent trading demand imbalance and its expected impact on subsequent price pressure dynamics. We prove existence of the equilibrium and solve for equilibrium trading strategies and prices in terms of the solution to a system of coupled ODEs. We show that trading strategies are combinations of trading towards investor targets, liquidity provision for other investors' demands, and front-running based on learning about latent underlying trading demand imbalances and future price pressure.

JEL codes: G11, G12

Keywords: Order-splitting, optimal order execution, subgame perfect Nash equilibrium, dynamic learning, trading targets, front-running

1 Introduction

The price formation process in financial markets involves equating supply and demand for securities over time for arriving investors with heterogeneous trading preferences. In present day markets, large investors act on their underlying trading preferences, sometimes called *parent demands*, by splitting their trading into dynamic sequences of smaller orders, called *child orders* (see O'Hara (2015)) to minimize their price impact. Since the parent demands driving child-order trading are private information, investors use information from arriving child orders to form inferences over time about the dynamically evolving fundamental state of the market in terms of imbalances in the underlying aggregate parent demands. In particular, investors form forecasts of future trading demand imbalances and the associated pressure on future market-clearing prices and incorporate this information in their current child orders. Given the widespread prevalence of optimized order-splitting of parent orders into flows of child orders, dynamic learning about aggregate parent demands is a critical part of market dynamics.

This paper is the first to provide an analytically tractable equilibrium model of dynamic learning and trading with parent trading demands. We consider a continuous-time model with high-frequency trading at times $t \in [0, 1]$ over short time-horizons with $[0, 1]$ being a day or an hour. Trading occurs between price-sensitive optimizing traders with two different types of parent trading targets: One group has fixed individual targets, and the other group wants to track a stochastically evolving target over time. Since parent targets are initially not public, information about parent demand imbalances is partially revealed through market-clearing stock prices. Our analysis models the learning process and determines the endogenous investor holdings and equilibrium stock-price process.

Our main results are:

- We construct and solve two different equilibrium models: A simpler price-impact equilibrium and a subgame perfect Nash financial-market equilibrium. In the subgame perfect Nash equilibrium, price impact is endogenous. In addition, we find that these two equilibria are numerically very similar.
- A practical application of our model is that we can compute total trading costs for investors given the effects of dynamic learning and front-running by other investors.
- Our model replaces the exogenous price-elastic residual demand used in both Brunnermeier and Pedersen (2005) and Carlin, Lobo, and Viswanathan (2007) with endogenous demands coming from profit-maximizing strategic Brownian motion trackers. We find that this change leads to a combination of liquidity provision and front-running.

Our paper advances several strands of research on market microstructure. First, dynamic learning and trading have been extensively studied in the context of markets with strategic investors with long-lived asymmetric information, most notably in models based on Kyle (1985). However, equilibrium trading, learning, and pricing in markets with optimized dynamic order-splitting by large uninformed investors are less well understood. In our model, all price effects are due to price pressure to equate supply and demand rather than adverse selection. Second, Choi, Larsen, and Seppi (2019) construct an equilibrium with optimized dynamic trading and learning in a market with a strategic rebalancer with an end-of-day trading target and an informed investor, who trades on private long-lived asset-payoff information. By filtering the order flow over time, the rebalancer learns about the underlying asset payoff, the informed investor learns about the rebalancer's trading target, and market makers learn about both when setting prices. That earlier paper provides a characterization result for equilibrium and gives numerical examples but does not have an existence proof nor analytic solutions. In contrast, our model is solved analytically and gives the equilibrium in closed form. Third, Brunnermeier and Pedersen (2005) and Carlin, Lobo, and Viswanathan (2007) show how dynamic rebalancing by a large investor can lead to predatory trading. However, these papers abstract from the learning problem by assuming the parent trading needs are publicly observable. They also make an ad hoc assumption about the price sensitivity of the residual market-maker trading demand in the form of exogenous price-elastic noise traders. In contrast, our model assumes the underlying parent trading demands are private information. In addition, our prices are rationally set with no ad hoc residual demand function. Fourth, a large body of research models optimal order-splitting strategies for a single strategic investor given an exogenous pricing rule with no learning about latent trading demands of other investors (see, e.g., Almgren and Chriss (1999, 2000), Almgren (2003), and Schied and Schöneborn (2009)). In contrast, we solve for optimal trades and equilibrium pricing jointly. Van Kervel, Kwan, and Westerholm (2020) solve for optimal trading strategies for two dynamic rebalancers with learning over time about each other's latent trading demands. This leads to predictions about the effect of aggregate parent demand on individual investor child orders, which are then verified empirically. However, they assume an ad hoc linear pricing rule, and there are no existence proofs or analytic solutions. In contrast, our pricing rule is endogenously determined in equilibrium, and we solve our model analytically. In addition, trading in our model is a combination of front-running along with trading demand accommodation (as in van Kervel, Kwan, and Westerholm (2020)).

Our analysis also uses a modeling approach from the asset-pricing literature for non-dividend paying stocks that makes the mathematics of our model tractable. The simplification involves finding sufficient conditions for equilibrium price drifts that clear the market without determining the levels of market-clearing prices as discounted future cash flows. The monograph Karatzas and Shreve (1998) describes this approach. Atmaz and Basak (2021) show that non-dividend paying stocks are relevant for asset pricing. However, models using the non-dividend paying stock approach are new in the mainstream microstructure literature. Gârleanu and Pedersen (2016), Bouchard, Fukasawa, Herdegen, and Muhle-Karbe (2018), and Noh and Weston (2020) use the zero-dividend stock approach to model prices given exogenous transaction costs. Our model uses this approach with endogenous price impact.

2 Model

We model equilibrium trading, learning, and pricing in a market with a risky stock and a riskless bank account over a short time horizon $[0,1]$ (e.g., a trading day). For simplicity, the net supply of both the stock and bank account are set to zero. Since the time horizon is short, the risk-free interest rate on the bank account is set to zero. The stock differs from the bank account in two ways: First, the investors have individual parent demands for the stock. Second, the stock valuation is stochastic over time. In particular, we can view stock valuation as the sum of two components. One component is a fundamental valuation of a stream of future dividends absent price pressure from trading targets. The second component is the incremental valuation impact of parent trading demand imbalances on prices such that equilibrium investor stock holdings clear the market. This price pressure component is also random. It is the price pressure component that is the focus of our analysis. For simplicity, our analysis treats these two components as being orthogonal. Moreover, since our focus is on equilibrium price pressure, we ignore the dividend valuation component. Thus, hereafter, when we refer to the "stock price", this is shorthand, for brevity, for the "price pressure valuation component of stock prices." However, in a more complicated model, a separate fundamental dividend valuation component can be added to our stock price pressure valuation to get the full stock price.

Two different groups of traders trade in our equilibrium model.

(i) Price-sensitive rebalancers. Rebalancer $i \in \{1,...,M\}$ maximizes her expected profit subject to a parent trading target $\tilde{a}_i$ where $\tilde{a}_i$ is private information for i. The targets $(\tilde{a}_1,...,\tilde{a}_M)$ are assumed homogeneously distributed $\tilde{a}_i \sim \mathcal{N}(0,\sigma_{\tilde{a}}^2)$ for all rebalancers $i \in \{1,...,M\}$ with identical zero means and standard deviations $\sigma_{\tilde{a}}$. The aggregate target is

$$\tilde{a}_\Sigma := \sum_{i=1}^{M} \tilde{a}_i. \tag{2.1}$$

Rebalancer i's control is her stock holdings, which are denoted by $(\theta_{i,t})_{t\in[0,1]}$ for $i \in \{1,...,M\}$. For simplicity, the initial endowed holdings of both the bank account and the stock are normalized to zero for all rebalancers.

When $\tilde{a}_i = 0$, rebalancer i is called a "high-frequency" liquidity provider. Because $\tilde{a}_i$ is private information for i, other traders k, $k \neq i$, do not know whether rebalancer i has an active latent trading demand or is a pure liquidity provider.

(ii) Price-sensitive trackers. Trackers $j \in \{M+1, ..., M+\bar{M}\}$ all want to track a Brownian motion process w_t over time $t \in [0,1]$ where their dynamic target w_t is modeled by the exogenously process

$$w_t := w_0 + w_t^\circ, \quad t \in (0,1], \tag{2.2}$$

where the initial target is $w_0 \sim \mathcal{N}(0, \sigma_{w_0}^2)$, the drift is zero, and $(w_t^\circ)_{t\in[0,1]}$ is a standard Brownian motion (i.e., w_t° starts at zero, has a zero drift, and a unit volatility).[1] While trackers observe the same w_t at time $t \in [0,1]$, rebalancers do not and instead filter w_t over time $t \in [0,1]$. Tracker j's control is her stock holdings, which are denoted by $(\theta_{j,t})_{t\in[0,1]}$ for $j \in \{M+1, ..., M+\bar{M}\}$. Their initial stock and money market holdings are also normalized to zero.

We assume that the random variables $(\tilde{a}_1, ..., \tilde{a}_M)$, w_0, and $(w_t^\circ)_{t\in[0,1]}$ are independent.

Randomness in the stock prices in our model (i.e., in the price pressure valuation effect) comes from learning about the traders' parent targets (which are initially private information of the individual rebalancers and the trackers) and from random changes over time in the trackers' target w_t. As we shall see, trackers will be able to infer the aggregate target $\tilde{a}_\Sigma$ in (2.1) from the initial stock price, and so trackers have no need to filter the rebalancers' individual targets $(\tilde{a}_1, ..., \tilde{a}_M)$. The situation is different for rebalancer i, who can only observe her own target $\tilde{a}_i$ and past and current stock prices. Because these observations are insufficient to observe $\tilde{a}_\Sigma$ and w_t seperately, rebalancer i filters based on $\tilde{a}_i$ and on past and current stock price observations to learn about the underlying latent parent demands $\tilde{a}_\Sigma$ and w_t.

In the following, index k denotes any generic trader, index i denotes a rebalancer, and index j denotes a tracker. This allows us to express the stock-market clearing condition as

$$0 = \sum_{k=1}^{M+\bar{M}} \theta_{k,t} = \sum_{i=1}^{M} \theta_{i,t} + \sum_{j=M+1}^{M+\bar{M}} \theta_{j,t}, \quad t \in [0,1]. \tag{2.3}$$

2.1 Individual maximization problems

This section introduces the individual maximization problems and the corresponding state processes needed to describe them. In the next sections, we will be using different fil-

[1] Adding a volatility coefficient σ_w in front of w_t° in (2.2) does not increase model flexibility because — as we shall see — the stock volatility γ is a free model parameter and γ and σ_w would play identical roles. Moreover, our model can be extended to include a drift term $\mu_w t$ in (2.2).

trations depending on the application. We denote by $\mathcal{F}_{k,t}$ a generic filtration for trader $k \in \{1, ..., M + \bar{M}\}$. As an example, $\mathcal{F}_{i,t}$ and $\mathcal{F}_{j,t}$ could denote

$$\begin{aligned} &\sigma(\tilde{a}_i, S_{i,u})_{u\in[0,t]}, \quad t \in [0,1], \quad i \in \{1, ..., M\}, \\ &\sigma(w_u, S_{j,u})_{u\in[0,t]}, \quad t \in [0,1], \quad j \in \{M+1, ..., M+\bar{M}\}, \end{aligned} \tag{2.4}$$

where $(S_{i,t})_{t\in[0,1]}$ and $(S_{j,t})_{t\in[0,1]}$ denote perceived stock-price processes for a rebalancer i and a tracker j. Several different perceived stock-price processes are specified in the next sections.

A generic trader's optimal stock holdings are determined in terms of a trade-off between expected terminal wealth $X_{k,1}$ and a penalty for deviations of their holdings $\theta_{k,t}$ over time from their parent target $\tilde{a}_i$ (rebalancers) or Brownian motion w_t (trackers). An exogenous continuous (deterministic) function $\kappa : [0, 1] \to (0, \infty]$ models the severity of the target penalty.[2] The traders' objectives are

$$\begin{aligned} &\sup_{\theta_{i,t}\in\mathcal{F}_{i,t}} \mathbb{E}\Big[X_{i,1} - \int_0^1 \kappa(t)(\tilde{a}_i - \theta_{i,t})^2 dt \Big| \sigma(\tilde{a}_i, S_{i,0})\Big], \quad i \in \{1, ..., M\}, \\ &\sup_{\theta_{j,t}\in\mathcal{F}_{j,t}} \mathbb{E}\Big[X_{j,1} - \int_0^1 \kappa(t)(w_t - \theta_{j,t})^2 dt \Big| \sigma(w_0, S_{j,0})\Big], \quad j \in \{M+1, ..., M+\bar{M}\}. \end{aligned} \tag{2.5}$$

Since traders' endowed stock holdings are normalized to zero, $\tilde{a}_i$ is the ideal holdings for rebalancer i and w_t is the ideal holdings for tracker j. The suprema in (2.5) are taken over progressively measurable holding processes $\theta_{i,t}$ and $\theta_{j,t}$ with respect to traders' filtrations $\mathcal{F}_{i,t}$ and $\mathcal{F}_{j,t}$ that are square integrable (to rule out doubling strategies)

$$\mathbb{E}\left[\int_0^1 \theta_{k,t}^2 dt\right] < \infty, \quad k \in \{1, ..., M + \bar{M}\}. \tag{2.6}$$

Terminal wealth $X_{k,1}$ in (2.5) is generated by trader k's perceived wealth process

$$dX_{k,t} := \theta_{k,t} dS_{k,t}, \quad X_{k,0} := 0, \quad k \in \{1, ..., M + \bar{M}\}, \tag{2.7}$$

which is affected by k's holdings $\theta_{k,t}$ both directly and also indirectly via the impact of k's holdings on an associated perceived stock-price process $S_{k,t}$. As a result of the price impact of $\theta_{k,t}$ on $S_{k,t}$, trader k's holdings $\theta_{k,t}$ are price sensitive. In (2.7), the zero initial wealth $X_{k,0} = 0$ is because trader k's initial endowed money market and stock holdings are normalized to zero. Given the objectives in (2.5), trading reflects a combination of motives: Investors seek to have stock holdings close to their own targets a_i and w_t, but they also seek to increase their expected terminal wealth by trading on price pressure from other investors

[2] Our analysis can be extended to allow for different penalty functions for the two groups of traders.

trading on their targets. Thus, traders demand liquidity (to come close to their targets) and supply liquidity for markets to clear (by being willing to deviate from their targets so that other traders can trade towards their targets, given the appropriate price incentives), and front-run future predictable price pressure.

2.2 State processes

Before considering specific stock-price perceptions in Sections 3 and 4 below, we describe a set of conjectured state processes $(Y_t, \eta_t, q_{i,t}, w_{i,t})$ for rebalancer $i \in \{1, ..., M\}$. These processes are all endogenous in equilibrium. However, in constructing the equilibrium, it is convenient to describe their informational properties first, before showing how they arise in equilibrium. The processes (Y_t, η_t) are public in the sense that they are adapted to $\mathcal{F}_{k,t}$ for all traders $k \in \{1, ..., M + \bar{M}\}$. Furthermore, η_t will be adapted to $\sigma(Y_u)_{u\in[0,t]}$. The state processes $(q_{i,t}, w_{i,t})$ are specific to rebalancer i and are only guaranteed to be adapted to i's filtration defined in $\mathcal{F}_{i,t}$ and they are not adapted to trader k's filtration for $k \neq i$. There is a significant informational difference between trackers and rebalancers. Each tracker $j \in \{M + 1, ..., M + \bar{M}\}$ can directly observe w_t in (2.2) and — as we shall see — can therefore infer the aggregate target $\tilde{a}_\Sigma$ in (2.1) from the initial stock price. In contrast, rebalancer $i \in \{1, ..., M\}$ learns about w_t and $\tilde{a}_\Sigma$ using dynamic filtering.

The state process Y_t in our model has the form

$$Y_t := w_t - B(t)\tilde{a}_\Sigma, \quad t \in [0, 1], \tag{2.8}$$

where the aggregate target $\tilde{a}_\Sigma$ is defined in (2.1) and $B : [0, 1] \to \mathbb{R}$ is a smooth deterministic function of time that is endogenously determined in equilibrium. The process Y_t is not directly observable for the rebalancers, but Lemma 3.1 below shows that Y_t can be inferred from stock prices. Because rebalancer $i \in \{1, ..., M\}$ also knows her own target $\tilde{a}_i$, by observing Y_t over time $t \in [0, 1]$, she equivalently observes

$$\begin{aligned} Y_{i,t} :&= Y_t + B(t)\tilde{a}_i \\ &= w_t - B(t)(\tilde{a}_\Sigma - \tilde{a}_i). \end{aligned} \tag{2.9}$$

Unlike Y_t in (2.8), the process $Y_{i,t}$ is independent of rebalancer i's private trading target $\tilde{a}_i$ and satisfies

$$\sigma(\tilde{a}_i, Y_u)_{u\in[0,t]} = \sigma(\tilde{a}_i, Y_{i,u})_{u\in[0,t]}, \quad t \in [0, 1]. \tag{2.10}$$

For a continuously differentiable function $B:[0,1]\to\mathbb{R}$, we define the processes

$$
\begin{aligned}
q_{i,t} &:= \mathbb{E}\left[\tilde{a}_\Sigma - \tilde{a}_i \,\middle|\, \sigma(Y_{i,u})_{u\in[0,t]}\right],\\
dw_{i,t} &:= dw_t - B'(t)\big(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t}\big)dt, \quad w_{i,0} := Y_{i,0},
\end{aligned}
\tag{2.11}
$$

for $t\in[0,1]$. In addition, the function $\Sigma(t)$ denotes the remaining variance

$$
\Sigma(t) := \mathbb{V}[\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t}] = \mathbb{E}[(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t})^2], \quad t\in[0,1], \tag{2.12}
$$

where the second equality follows from the zero-mean assumptions placed on $(\tilde{a}_1,...,\tilde{a}_M)$ and w_0.

The following result is a special case of the Kalman-Bucy result from filtering theory.

Lemma 2.1 (Kalman-Bucy)**.** *For a continuously differentiable function $B:[0,1]\to\mathbb{R}$, the process $w_{i,t}$ is independent of $\tilde{a}_i$, is a Brownian motion, and satisfies (modulo $\mathbb{P}$ null sets)*

$$
\sigma(\tilde{a}_i, Y_{i,u})_{u\in[0,t]} = \sigma(\tilde{a}_i, w_{i,u})_{u\in[0,t]}, \quad t\in[0,1]. \tag{2.13}
$$

♢

Our equilibrium construction combines the stock-market clearing condition (2.3) with the following decomposition result:

Lemma 2.2. *Let $B:[0,1]\to\mathbb{R}$ be a continuously differentiable function. Then, the decomposition*

$$
\sum_{i=1}^{M} q_{i,t} = \eta_t + A(t)\tilde{a}_\Sigma, \quad t\in[0,1], \tag{2.14}
$$

holds with the process η_t being adapted to $\sigma(Y_u)_{u\in[0,t]}$ with Y_t in (2.8) and

$$
\begin{aligned}
A'(t) &= -\big(B'(t)\big)^2\Sigma(t)\big(A(t)+1\big), \quad A(0) = -\tfrac{(M-1)B(0)^2\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2 + B(0)^2(M-1)\sigma_{\tilde{a}}^2},\\
\Sigma(t) &= \frac{1}{\frac{1}{\mathbb{V}[\tilde{a}_\Sigma - \tilde{a}_i - q_{i,0}]} + \int_0^t \big(B'(u)\big)^2 du},\\
d\eta_t &= -\big(B'(t)\big)^2\Sigma(t)\eta_t dt - MB'(t)\Sigma(t)dY_t, \quad \eta_0 = -\tfrac{M(M-1)B(0)\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2 + B(0)^2(M-1)\sigma_{\tilde{a}}^2}Y_0.
\end{aligned}
\tag{2.15}
$$

♢

Because the targets $(\tilde{a}_1,...,\tilde{a}_M)$ are assumed independent and homogeneously distributed $\mathcal{N}(0,\sigma_{\tilde{a}}^2)$, the initial variance $\Sigma(0)=\mathbb{E}[(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,0})^2]$ is identical across all rebalancers $i\in\{1,...,M\}$. This property and the formula for $\Sigma(t)$ in (2.15) imply that $\Sigma(t)$ is also independent of index $i\in\{1,...,M\}$ for all $t\in[0,1]$.

3 Price-impact equilibrium

Investor perceptions of the impact of their trading on stock prices are a key part of the optimizations in (2.5) and the resulting market equilibrium. We consider two specifications of investor stock-price perceptions. This section presents a simpler model in which price impact is exogenous. This approach is analogous to the exogenous price impact used in, e.g., van Kerval, Kwan, and Westerholm (2020). We then solve for the endogenous stock-price process that clears the market (and also satisfies some weak consistency conditions) and the associated optimized investor holding processes. Section 4 presents a richer model of endogenous price impact in which investor price perceptions and price impacts are endogenized in a subgame perfect Nash financial-market equilibrium.

3.1 Stock-price perceptions

Rebalancers optimize (2.5) with respect to perceived stock-price processes of the form

$$
\begin{aligned}
dS^f_{i,t} &:= \Big\{ f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\theta_{i,t} \Big\} dt + \gamma dw_{i,t}, \\
S^f_{i,0} &:= Y_0, \quad i \in \{1, ..., M\},
\end{aligned}
\tag{3.1}
$$

where $f_0, f_1, f_2, f_3 : [0,1] \to \mathbb{R}$ are continuous (deterministic) functions of time $t \in [0,1]$ and (α, γ) are constants. In particular, the drift in (3.1) is perceived by rebalancers to be sensitive to the two aggregate state variables Y_t and η_t. Consistent with intuition, we will see in equilibrium that $f_0(t)$ and $f_3(t)$ are negative. The other coefficients describe the perceived impact of rebalancer i on the stock-price drift. The "f" superscript indicates that the perceived price $S^f_{i,t}$ is defined with respect to particular coefficient functions f in (3.1). Theorem 3.5 below endogenously determines (f_0, f_1, f_2, f_3) in equilibrium. The innovations in the rebalancers' perceived stock prices $dw_{i,t}$ come from new information rebalancer i learns over time about the underlying parent demand state variable Y_t, which has both a direct effect on the future stock-price drift and an additional indirect effect via its effect on η_t since η_t is adapted to $\sigma(Y_u)_{u\in[0,t]}$ from Lemma 2.2. The technical goal of our analysis — given rebalancer stock-price perceptions of the form in (3.1) with an aggregate demand state variable Y_t process of the form in (2.8) (and the associated η_t process) — is to construct a function $B(t)$ that gives an equilibrium.

Our modeling approach for price pressure follows the zero-dividend stock valuation literature (see, e.g., Karatzas and Shreve (1998)) in that we model perceived and equilibrium price drifts rather than price levels. In particular, the price pressure is not the valuation of future dividends. Instead, it is a price component needed to clear the market given investors' trading demands.[3] One consequence of this modeling approach is that, in (3.1),

[3]Our model features asymmetric information and learning about parent demands. However, there is no

the stock's volatility and initial value are model input parameters. For simplicity, we set the volatility to be a constant $\gamma > 0$ (i.e., increased parent demand w_t increases prices) and the initial price to be Y_0 in (3.1). However, many other choices would work equally well (e.g., $\gamma(t)$ or $g(Y_0)$). The price-impact parameter α is also an exogenous model input. The exogenous parameters (α, γ) can be found by calibrating model output to empirical data. A competitive market is a special case with $\alpha := 0$, whereas the empirically relevant case is $\alpha < 0$ such that buy (sell) orders decrease (increase) the future stock price drifts.

The next result shows that $w_{i,t}$ is rebalancer i's innovations process in the sense that $w_{i,t}$ is a Brownian motion relative to i's filtration defined with perceived stock prices $S^f_{i,t}$ in (3.1) and such that $S^f_{i,t}$ and $w_{i,t}$ generate the same information.

Lemma 3.1. *Let $f_0, f_1, f_2, f_3 : [0,1] \to \mathbb{R}$ be continuous functions and let $B : [0,1] \to \mathbb{R}$ be a continuously differentiable function. For a rebalancer $i \in \{1, ..., M\}$, let $\theta_{i,t}$ satisfy* (2.6) *and be progressively measurable with respect to $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S^f_{i,u})_{u\in[0,t]}$ with $S^f_{i,t}$ defined in* (3.1) *and Y_t defined in* (2.8)*. Then, modulo $\mathbb{P}$-null sets, we have*

$$\sigma(\tilde{a}_i, w_{i,u})_{u\in[0,t]} = \sigma(\tilde{a}_i, S^f_{i,u})_{u\in[0,t]}, \quad t \in [0,1], \quad i \in \{1, ..., M\}. \tag{3.2}$$

$\diamondsuit$

Thus, given a path of perceived prices generated by a process $S^f_{i,t}$ of the form in (3.1) and her personal target $\tilde{a}_i$, rebalancer i can infer the path of $w_{i,t}$. Furthermore, given the path $w_{i,t}$, rebalancer i can infer $Y_{i,t}$ using (2.13) and, thus, can infer Y_t from (2.10). Consequently, rebalancer i can infer $(q_{i,t}, \eta_t)$ where we recall from Lemma 2.2 that η_t is adapted to $\sigma(Y_t)_{t\in[0,1]}$.

Trackers have different information in that they observe w_t directly and can infer $\tilde{a}_\Sigma$ from the initial stock price. Therefore, their perceived stock prices differ from those of the rebalancers. In our model, the stock-price process perceived by tracker $j \in \{M+1, ..., M+\bar{M}\}$ takes the form:

$$\begin{aligned} dS_{j,t} &:= \Big\{\bar{f}_3(t)\eta_t + \bar{f}_4(t)\tilde{a}_\Sigma + \bar{f}_5(t)w_t + \alpha\theta_{j,t}\Big\}dt + \gamma dw_t, \\ S_{j,0} &:= Y_0, \end{aligned} \tag{3.3}$$

where $\bar{f}_3, \bar{f}_4, \bar{f}_5 : [0,1] \to \mathbb{R}$ are continuous (determinstic) functions and the α is a constant.[4] Theorem 3.5 below endogenously determines $(\bar{f}_3, \bar{f}_4, \bar{f}_5)$ in equilibrium, whereas (α, γ) are exogenous model inputs. Again, $\alpha := 0$ is the special case of a competitive market.

An important difference between rebalancer and tracker perceived prices in (3.1) and (3.3) is that rebalancer price dynamics are based on the innovations $dw_{i,t}$ (i.e., what rebal-

asymmetric information related to future stock cashflows.

[4]Our model can be extended to allow for a different price-impact coefficient α for the trackers.

ancers learn from the state variable Y_t), whereas tracker price dynamics are based on dw_t (i.e., the trackers' target). Reconciling the price perceptions of rebalancers and trackers will impose important restrictions on equilibrium price perceptions and holdings and will rely on the relation between $dw_{i,t}$ and dw_t in (2.11).

The proof of Lemma 3.2 shows that pointwise quadratic maximization gives the maximizers for (2.5) for rebalancers and trackers for arbitrary f functions.

Lemma 3.2. *Let $f_0, f_1, f_2, f_3, \bar{f}_3, \bar{f}_4, \bar{f}_5 : [0,1] \to \mathbb{R}$ and $\kappa : [0,1] \to (0,\infty]$ be continuous functions, let $B : [0,1] \to \mathbb{R}$ be continuously differentiable, let $\alpha \leq 0$, and let the perceived stock-price process in the wealth dynamics* (2.7) *be as in* (3.1) *and* (3.3)*. Then, for $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S^f_{i,u})_{u\in[0,t]}$ and $\mathcal{F}_{j,t} := \sigma(w_u, S^f_{j,u})_{u\in[0,t]}$, and, provided the holding processes*

$$\begin{aligned} \hat{\theta}_{i,t} &:= \frac{f_0(t)}{2(\kappa(t)-\alpha)}Y_t + \frac{f_1(t)+2\kappa(t)}{2(\kappa(t)-\alpha)}\tilde{a}_i + \frac{f_2(t)}{2(\kappa(t)-\alpha)}q_{i,t} + \frac{f_3(t)}{2(\kappa(t)-\alpha)}\eta_t, \\ \hat{\theta}_{j,t} &:= \frac{\bar{f}_3(t)}{2(\kappa(t)-\alpha)}\eta_t + \frac{\bar{f}_5(t)+2\kappa(t)}{2(\kappa(t)-\alpha)}w_t + \frac{\bar{f}_4(t)}{2(\kappa(t)-\alpha)}\tilde{a}_\Sigma, \end{aligned} \tag{3.4}$$

satisfy (2.6)*, the traders' maximizers for* (2.5) *are $\hat{\theta}_{i,t}$ for rebalancer $i \in \{1,...,M\}$ and $\hat{\theta}_{j,t}$ for tracker $j \in \{M+1,...,M+\bar{M}\}$.* $\diamondsuit$

To summarize, stock-price perceptions play two interconnected roles in our model. First, rebalancers and trackers solve their optimization problems in (2.5) based on their perceptions in (3.1) and (3.3) for how hypothetical orders $\theta_{i,t}$ and $\theta_{j,t}$ affect price dynamics. Second, investor stock-price perceptions affect how they learn from observed prices. In particular, Lemma 3.1 shows that rebalancers use their stock-price perceptions of prices in (3.1) to infer the aggregate demand state variable Y_t based on past and current stock prices. In other words, dynamic learning by rebalancers depends critically on their stock-price perceptions. Trackers also use their stock-price perceptions in (3.3) to infer the aggregate parent demand $\tilde{a}_\Sigma$ from the initial price at time $t = 0$. However, thereafter, there is no additional learning from prices by the trackers since they directly observe changes in their target w_t.

3.2 Equilibrium

The notion of equilibrium in our first construction is relatively simple, being based just on market clearing and consistency of investor price perceptions.

Definition 3.3. Deterministic functions of time $f_0, f_1, f_2, f_3, \bar{f}_3, \bar{f}_4, \bar{f}_5 : [0,1] \to \mathbb{R}$ constitute a *price-impact equilibrium* if:

(i) Maximizers $\hat{\theta}_{k,t}$ for (2.5) exist for traders $k \in \{1,...,M+\bar{M}\}$ given the stock-price perceptions (3.1) and (3.3) for filtrations $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S^f_{i,u})_{u\in[0,t]}$ and $\mathcal{F}_{j,t} := \sigma(w_u, S^f_{j,u})_{u\in[0,t]}$.

(ii) Inserting trader k's maximizer $\hat{\theta}_{k,t}$ into the perceived stock-price processes (3.1) and (3.3) produces identical stock-price processes across all traders $k \in \{1, ..., M + \bar{M}\}$. This common equilibrium stock-price process is denoted by $\hat{S}_t$.

(iii) The money and stock markets clear. $\diamondsuit$

Definition 3.2 places only minimal restrictions on the perceived stock-price coefficient functions in (3.1) and (3.3): Markets must clear and result in consistent perceived stock-price processes when all investors use their equilibrium strategies. Section 4 below considers a subgame perfect Nash extension of our basic model that imposes more restrictions on allowable off-equilibrium stock-price perceptions such as market clearing.

In equilibrium, Definition 3.3(ii) requires that rebalancers and trackers perceive identical stock-price dynamics when using their equilibrium holdings. However, rebalancers and trackers have different information (i.e., rebalancers form non-perfect inferences about w_t and $\tilde{a}_\Sigma$, whereas trackers observe w_t directly and infer $\tilde{a}_\Sigma$ at time 0). The resolution of this apparent paradox is investors' different information sets: While traders agree on $d\hat{S}_t$, they disagree on how to decompose $d\hat{S}_t$ into drift and volatility components. Because the trackers observe w_t, they can use dw_t in their decomposition of $d\hat{S}_t$. However, w_t is not adapted to the rebalancers' filtrations and can therefore not be used in their $d\hat{S}_t$ decompositions. Instead, rebalancers use their innovations processes $dw_{i,t}$ when decomposing $d\hat{S}_t$ into drift and volatility. By replacing $dw_{i,t}$ in $dS^f_{i,t}$ in (3.1) with the decomposition of $dw_{i,t}$ in terms of dw_t from (2.11), we can rewrite $dS^f_{i,t}$ in (3.1) as

$$\begin{aligned} dS^f_{i,t} = \Big\{ & f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\theta_{i,t} \\ & - B'(t)\big(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t}\big)\gamma \Big\} dt + \gamma dw_t, \quad i \in \{1, ..., M\}. \end{aligned} \tag{3.5}$$

Therefore, to ensure identical equilibrium stock-price perceptions for all traders $k \in \{1, ..., M + \bar{M}\}$, it suffices to match the drift of $dS^f_{j,t}$ in (3.3) for $j \in \{M+1, ..., M+\bar{M}\}$ with the drift of $dS^f_{i,t}$ in (3.5) for the equilibrium holdings $\theta_{i,t} := \hat{\theta}_{i,t}$, $i \in \{1, ..., M\}$. This produces the following requirement:

$$\begin{aligned} & f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\hat{\theta}_{i,t} - B'(t)\big(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t}\big)\gamma \\ & = \bar{f}_3(t)\eta_t + \bar{f}_4(t)\tilde{a}_\Sigma + \bar{f}_5(t)w_t + \alpha\hat{\theta}_{j,t}, \end{aligned} \tag{3.6}$$

for all rebalancers $i \in \{1, ..., M\}$ and all trackers $j \in \{M+1, ..., M+\bar{M}\}$. We note that the right-hand side of (3.6) does not depend on the rebalancer index i. Matching up coefficients in front of $(\tilde{a}_i, \tilde{a}_\Sigma, q_{i,t}, \eta_t, w_t)$ in (3.6) using $\hat{\theta}_{i,t}$ and $\hat{\theta}_{j,t}$ in (3.4) produces a system of equations that gives the coefficient functions (A.1) in Appendix A.

Our equilibrium existence result is based on the following technical lemma. It guarantees the existence of a solution to an autonomous system of coupled ODEs. The exogenous price-impact coefficient α plays no role in this result.

Lemma 3.4. *Let $\kappa : [0,1] \to [0,\infty]$ be a continuous and integrable function (i.e., $\int_0^1 \kappa(t)dt < \infty$). For an initial constant $B(0) \in \mathbb{R}$, the coupled ODEs*

$$\begin{aligned} B'(t) &= \frac{2\kappa(t)(\bar{M}B(t)+1)}{\gamma(A(t)+\bar{M}+1)}, \\ A'(t) &= -\big(B'(t)\big)^2\Sigma(t)\big(A(t)+1\big), \quad A(0) = -\tfrac{(M-1)B(0)^2\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2+B(0)^2(M-1)\sigma_{\tilde{a}}^2}, \\ \Sigma'(t) &= -\big(B'(t)\big)^2\Sigma(t)^2, \quad \Sigma(0) = \tfrac{(M-1)\sigma_{\tilde{a}}^2\sigma_{w_0}^2}{B(0)^2(M-1)\sigma_{\tilde{a}}^2+\sigma_{w_0}^2}, \end{aligned} \tag{3.7}$$

have unique solutions with $\Sigma(t) \geq 0$, $\Sigma(t)$ decreasing, $A(t) \in [-1,0]$, $A(t)$ decreasing for $t \in [0,1]$, and $B(t), B'(t) < 0$ when $\bar{M}B(0)+1 < 0$. ♢

The ODEs for $A(t)$ and $\Sigma(t)$ in (3.7) are consistent with the expressions in (2.15).

The following theoretical result gives the price-impact equilibrium in terms of the ODEs (3.7). In this theorem, the price-impact parameter α, volatility γ, and initial value $B(0)$ are free parameters. The intuition for $B(0)$ being free is discussed after our equilibrium construction in Theorem 3.5.

Theorem 3.5. *Let $\kappa : [0,1] \to (0,\infty)$ be continuous, let the functions (B, A, Σ) be as in Lemma 3.4, and let $\alpha \leq 0$. Then, we have:*

(i) A price-impact equilibrium exists and is given by the price-perception functions (A.1) *in Appendix A.*

(ii) Equilibrium holdings $\hat{\theta}_{i,t}$ for rebalancer i and $\hat{\theta}_{j,t}$ for tracker j are

$$\begin{aligned} \hat{\theta}_{i,t} &= \tfrac{\gamma B'(t)-2\kappa(t)}{\alpha-2\kappa(t)}\tilde{a}_i + \tfrac{\gamma B'(t)}{\alpha-2\kappa(t)}q_{i,t} \\ &\quad - \tfrac{\gamma B'(t)}{(M+\bar{M})(\alpha-2\kappa(t))}\eta_t + \tfrac{2\bar{M}\kappa(t)}{(M+\bar{M})(\alpha-2\kappa(t))}Y_t, \quad i \in \{1,...,M\}, \\ \hat{\theta}_{j,t} &= -\tfrac{\gamma B'(t)}{(M+\bar{M})(\alpha-2\kappa(t))}\eta_t - \tfrac{2M\kappa(t)}{(M+\bar{M})(\alpha-2\kappa(t))}w_t \\ &\quad + \tfrac{\gamma(A(t)-M+1)B'(t)-2\kappa(t)}{(M+\bar{M})(2\kappa(t)-\alpha)}\tilde{a}_\Sigma, \quad j \in \{M+1,...,M+\bar{M}\}. \end{aligned} \tag{3.8}$$

(iii) The equilibrium stock-price process has dynamics

$$\begin{aligned} d\hat{S}_t &:= \Big\{\tfrac{\gamma B'(t)}{M+\bar{M}}\eta_t - \tfrac{2\bar{M}\kappa(t)}{M+\bar{M}}w_t + \tfrac{\gamma(A(t)-M+1)B'(t)-2\kappa(t)}{M+\bar{M}}\tilde{a}_\Sigma\Big\}dt + \gamma dw_t, \\ \hat{S}_0 &:= w_0 - B(0)\tilde{a}_\Sigma. \end{aligned} \tag{3.9}$$

♢

Several observations follow from Theorem 3.5:

1. Lemma 3.1 ensures that rebalancer i can infer her innovations process $w_{i,t}$ from perceived prices $S^f_{i,t}$ and $\tilde{a}_i$, but rebalancer i cannot infer the tracker target w_t from the equilibrium prices $\hat{S}_t$ in (3.9). This is because the aggregate target $\tilde{a}_\Sigma$ also appears in the drift of $d\hat{S}_t$ and $\tilde{a}_\Sigma$ is not observed by individual rebalancers.

2. The equilibrium holdings (3.8) follow from inserting (A.1) into (3.4). Thus, the holdings in (3.8) are expressed in terms of the investors' state variables. However, the state variables are not orthogonal to each other. For example, $(q_{i,t}, \eta_t, Y_t)$ all depend on past stock prices, which, in turn, depend on $(\tilde{a}_\Sigma, w_t)$. This affects the interpretation of model comparative statics.

3. Because the exogenous price-impact coefficient α does not appear in the ODEs (3.7), α is irrelevant for the equilibrium stock-price dynamics (3.9). However, α does affect the equilibrium holdings in (3.8).

4. The stock-price volatility γ affects the stock-price drift and holdings via its impact on $B(t)$ in (3.7) and, thus, on (A.1).

5. The equilibrium stock-price dynamics (3.9) depend on w_t and $\tilde{a}_\Sigma$. Because $(w_t, \tilde{a}_\Sigma)$ is adapted to the trackers' filtrations, the trackers' equilibrium stock-price dynamics are those in (3.9). However, rebalancers have different information, so, with respect to their filtrations $\mathcal{F}_{i,t}$, their equilibrium stock-price dynamics differ from (3.9). To derive the rebalancers' drift and volatility, we use (2.11) to replace dw_t with $dw_{i,t}$ and a drift term in (3.9) to get rebalancer i's equilibrium stock-price dynamics

$$d\hat{S}_t = \Big\{ \tfrac{\gamma B'(t)}{M+\bar{M}}\eta_t - \tfrac{2\bar{M}\kappa(t)}{M+\bar{M}} w_t + \tfrac{\gamma(A(t)-M+1)B'(t)-2\kappa(t)}{M+\bar{M}}\tilde{a}_\Sigma + B'(t)\big(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t}\big)\gamma \Big\} dt + \gamma dw_{i,t}, \quad i \in \{1, ..., M\}. \tag{3.10}$$

This gives rebalancer i's equilibrium stock-price drift with respect to $\mathcal{F}_{i,t}$

$$-\gamma B'(t)\big(\tilde{a}_i + q_{i,t}\big) + \tfrac{\gamma B'(t)}{M+\bar{M}}\eta_t - \tfrac{2\bar{M}\kappa(t)}{M+\bar{M}} Y_t, \quad i \in \{1, ..., M\}. \tag{3.11}$$

Thus, rebalancers perceive the same equilibrium price process but perceive a different drift and volatility compared to the trackers.[5]

6. For arbitrary holdings $\theta_{i,t}$, rebalancer i's perceived stock-price drift in (3.1) can be

[5] Rebalancers and trackers both start with private information so their filtrations are not nested. However, in equilibrium, stock-price dynamics depend on w_t and $\tilde{a}_\Sigma$. Because the trackers infer $\tilde{a}_\Sigma$ from $S_{j,0} = Y_0$, they have no need to filter at later times. On the other hand, rebalancer i only has noisy dynamic predictions $\mathbb{E}[\tilde{a}_\Sigma | \mathcal{F}_{i,t}] = q_{i,t} + \tilde{a}_i$ of the aggregate parent imbalance $\tilde{a}_\Sigma$ given their inferences based on their individual parent targets $\tilde{a}_i$ and stock prices.

decomposed as

$$\begin{aligned} &f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\theta_{i,t} \\ &= -\gamma B'(t)\big(\tilde{a}_i + q_{i,t}\big) + \tfrac{\gamma B'(t)}{M+\bar{M}}\eta_t - \tfrac{2\bar{M}\kappa(t)}{M+\bar{M}}Y_t + \alpha(\theta_{i,t} - \hat{\theta}_{i,t}), \end{aligned} \tag{3.12}$$

where we have used the formulas for (f_0, f_1, f_2, f_3) in (A.1) in Appendix A. Likewise, for arbitrary holdings $\theta_{j,t}$, tracker j's perceived stock-price drift in (3.3) is

$$\begin{aligned} &\bar{f}_3(t)\eta_t + \bar{f}_4(t)\tilde{a}_\Sigma + \bar{f}_5(t)w_t + \alpha\theta_{j,t} \\ &= \tfrac{\gamma B'(t)}{M+\bar{M}}\eta_t - \tfrac{2\bar{M}\kappa(t)}{M+\bar{M}}w_t + \tfrac{\gamma(A(t)-M+1)B'(t)-2\kappa(t)}{M+\bar{M}}\tilde{a}_\Sigma + \alpha(\theta_{j,t} - \hat{\theta}_{j,t}), \end{aligned} \tag{3.13}$$

where we have used the formulas for $(\bar{f}_1, \bar{f}_2, \bar{f}_3)$ in (A.1) in Appendix A.

Thus, investors' off-equilibrium drifts differ from their equilibrium drifts due to the differences $\theta_{k,t} - \hat{\theta}_{k,t}$ between their off-equilibrium and equilibrium holdings.[6] Continuity is a reasonable property of investor perceptions. The representation of the perceived rebalancer drift in (3.12) relative to $\hat{\theta}_{i,t}$ from (3.8) also explains the presence of the rebalancer-specific terms $(\tilde{a}_i, q_{i,t})$ in the rebalancers' perceptions in (3.1).

The function $B(t)$ from (3.7) is key both in constructing the equilibrium and for interpreting the equilibrium price and holding processes. First, there is the issue that the initial value $B(0)$ is a free input in Theorem 3.5. The intuition is that our model determines equilibrium price drifts, but not price levels. As can be seen in (3.9), $B(0)$ controls the initial price level in our model. Second, the relation between $B(t)$ and price levels allows us to impose additional structure on $B(t)$. In particular, w_t and $\tilde{a}_\Sigma$ represent different types of demand imbalances. Thus, if $B(t) < 0$, then Y_t in (2.8) plays the role of an aggregate demand state variable. How the two component quantities w_t and $\tilde{a}_\Sigma$ are mixed in the aggregate demand state variable Y_t is different given the two components' different informational dynamics (i.e., $\tilde{a}_\Sigma$ is fixed after time 0 while w_t changes randomly over time) and the different impacts on investor demands (i.e., each rebalancer only knows their personal $\tilde{a}_i$ component of $\tilde{a}_\Sigma$ where other rebalancers' targets do not affect investor i's parent demand whereas w_t affects both an individual tracker's parent demands and is also information about other trackers' parent demands). It seems reasonable that the sign of the impact of w_t and $\tilde{a}_\Sigma$ on the price level should be the same, which imposes the additional restriction that $B(t) < 0$. From Lemma 3.4, a sufficient condition for $B(t) < 0$ for all $t \in [0,1]$ is $\bar{M}B(0) + 1 < 0$.[7]

With the economically reasonable parametric restriction that $B'(t) < 0$ and given that

[6]Eqs. (3.12) and (3.13) are similar to Eq. (3.14) in Choi, Larsen, and Seppi (2021).

[7]This sufficient condition follows because the denominator in (3.7) is positive given that $A(t) \in [-1, 0]$ so that the numerator in (3.7) determines the sign of $B'(t)$.

$\alpha \leq 0$ so that $\alpha - 2\kappa(t) < 0$, we can sign the impact of various quantities in the model on holdings and prices, which leads to the following comparative statics:

1. The equilibrium holdings $\hat{\theta}_{i,t}$ of rebalancers are positively related to their parent targets $\tilde{a}_i$. This is intuitive because rebalancers want holdings close to $\tilde{a}_i$. Rebalancer holdings $\hat{\theta}_{i,t}$ are also negatively related to the aggregate demand imbalance state variable Y_t. The fact that $\theta_{i,t}$ is decreasing in Y_t is consistent with the theoretical results and empirical evidence in van Kerval, Kwan and Westerholm (2020) that investors buy less when there is a positive parent demand imbalance for other investors in the market. However, the impact of $q_{i,t}$ on $\hat{\theta}_{i,t}$ is positive. The intuition is that when rebalancer i expects the other remaining rebalancers (given i's ability to filter using her private target information $\tilde{a}_i$) to have a net positive parent demand imbalance $\mathbb{E}[\tilde{a}_\Sigma - \tilde{a}_i|\mathcal{F}_{i,t}]$ from (2.11), she buys at time t to front-run the resulting anticipated future price pressure.

2. Tracker j's holdings $\hat{\theta}_{j,t}$ are increasing in w_t (which reflects both her own parent demand and also information about the parent demands of other trackers). Tracker holdings $\hat{\theta}_{j,t}$ are also decreasing in information η_t about imbalances in rebalancers' aggregate parent demand expectations. The effect of η_t is consistent with the van Kerval, Kwan, and Westerholm (2020) liquidity-provision result and empirical evidence. However, the impact of $\tilde{a}_\Sigma$ is ambiguous in (3.8), and numerical calculations in Section 5 show that the sign is positive. This is again consistent with front-running future predicted price pressure due to the tracker's superior information about aggregated latent parent demand imbalances.

3. The equilibrium price drift in (3.9) is decreasing in the tracker parent demand w_t. However, the impact of $\tilde{a}_\Sigma$ in the price drift is again ambiguous, which is related to information about $\tilde{a}_\Sigma$ being useful in forecasting future price pressure.

3.3 Tractability and model structure

This section outlines the key model components that make our model tractable. First, we assume all traders seek to maximize their individual objectives in (2.4). Linear-quadratic objectives have been used extensively in the literature because of their tractability. Such objectives have been used in, e.g., Kyle (1985), Brunnermeier and Pederson (2005), and Carlin, Lobo, and Viswanathan (2007).

Second, our stock does not pay dividends, which means that only the stock drift can be endogenously determined in equilibrium. Models with non-dividend paying stocks have been used extensively in the literature. The monograph Karatzas and Shreve (1998) gives an overview.[8] In particular, non-dividend paying stock models have been used for short horizon models like ours where consumption only takes place at the terminal time.[9] The rebalancers' dynamic learning produces forward-running filtering equations and by considering a non-dividend paying stock, we circumvent having additional backward-running equations. Equilibrium models with both forward and backward-running equations include Kyle (1985), Foster and Viswanathan (1994, 1996), Back, Cao, and Willard (2000), and Choi, Larsen, and Seppi (2019).

Third, instead of exogenous noise traders, we use optimizing trackers. Grossman and Stiglitz (1980) and Kyle (1985) are standard references, which use an exogenous Gaussian stock supply. Gaussian noise traders are also used in the predatory trading models in Brunnermeier and Pederson (2005) and Carlin, Lobo, and Viswanathan (2007). In our setting, we could eliminate trackers by setting $\bar{M} := 0$ and replace the stock-market clearing condition (2.3) by using w_t to model the exogenous stock supply as in

$$w_t = \sum_{i=1}^{M} \theta_{i,t}, \quad t \in [0,1]. \tag{3.14}$$

Including noise traders as in (3.14) in the model would be tractable in the price-impact equilibrium. However, surprisingly, exogenous noise-traders complicate constructing a Nash equilibrium with dynamic learning, whereas — as we show in Section 4 — optimizing trackers and market learning in (2.3) produce a subgame perfect Nash financial-market equilibrium in closed form. The models in Sannikov and Skrzypacz (2016) and Choi, Larsen, and Seppi (2021) have optimizing trackers but no dynamic learning.

Fourth, the linear-quadratic objectives (2.5) allow us to solve for the optimal holdings in Lemma 3.2 using quadratic pointwise optimization. Thus, dynamic programming and pointwise optimization are equivalent here in the price-impact equilibrium in that they

[8]Similar to a money market account, a non-dividend paying stock is a *financial asset* in the sense that holding one stock at time $t = 1$, gives one unit of consumption at $t = 1$. Likewise, being short one stock at $t = 1$, means the trader provides one unit of consumption at $t = 1$. Both the money market account and the non-dividend paying stock have exogenous initial prices and volatilities. It is custom for the money market account's initial price to be one and its volatility to be zero. For the non-dividend paying stock, we set the initial price to be Y_0, its volatility to be a positive constant γ, and determine endogenously the drift.

[9]There are long-lived non-dividend paying stocks too as; see, for example, Atmaz and Basak (2021) write: "For example, Hartzmark and Solomon (2013) find that over the long-sample of 1927-2011, the average proportion of no-dividend stocks is around 35% and accounts for 21.3% of the aggregate US stock market capitalization. Similarly, by taking into account of rising share repurchase programs since the mid-1980ies, Boudoukh et al. (2007) report that over the 1984-2003 period, the average proportion of no-dividend stocks is 64% and no-payout stocks, i.e., no dividends or no share repurchases, is 51% with the relative market capitalizations of 16.4% and 14.2%, respectively."

produce the same optimal response holdings.

4 Subgame perfect Nash equilibrium

This section builds on the analysis in Section 3 by endogenizing stock-price perceptions and price impact. In particular, we model the impact of on-equilibrium and hypothetical off-equilibrium investor holdings on market-clearing stock prices based on investor perceptions of how other investors in the market perceive prices and on other investors' resulting optimal response functions to an investor's on-equilibrium and off-equilibrium holdings.

A subgame perfect market-clearing Nash equilibrium involves describing how each trader k_0 (who might be a rebalancer i_0 or a tracker j_0 with their different information sets) perceives market-clearing stock prices given stock-price perceptions for other traders $k \neq k_0$ (where k can be rebalancers i or trackers j). In our Nash model, a generic trader k_0 perceives that other rebalancers and trackers have stock-price perceptions of the form

$$\begin{aligned} dS^Z_{i,t} &:= \Big\{ Z_t + \mu_1(t)\tilde{a}_i + \mu_2(t)q_{i,t} + \mu_3(t)\eta_t + \alpha\theta_{i,t} \Big\}dt + \gamma dW_{i,t}, \\ S^Z_{i,0} &:= Z_0, \quad i \in \{1, ..., M\}, \\ dS^Z_{j,t} &:= \Big\{ Z_t + \bar{\mu}_4(t)\tilde{a}_\Sigma + \bar{\mu}_5(t)w_t + \alpha\theta_{j,t} \Big\}dt + \gamma dW_{j,t}, \\ S^Z_{j,0} &:= Z_0, \quad j \in \{M+1, ..., M+\bar{M}\}, \end{aligned} \tag{4.1}$$

where $W_{k,t}$ is a Brownian motion for each trader $k \in \{1, ..., M+\bar{M}\}$ and Z_t is an arbitrary Itô process. The "Z" superscript in (4.1) indicates that the perceived stock prices are defined with respect to a particular Itô process Z_t (i.e., a process given as a sum of drift and volatility). We use the market-clearing condition (2.3) to construct two such Itô processes in (4.5) and (4.8) below. These Z_t processes differ from Y_t in (3.1) and (3.3) in that we use Z_t to capture the effect of arbitrary off-equilibrium stock holdings by trader k_0 on market-clearing prices given optimal responses by other investors k, $k \neq k_0$. We then go on to determine the deterministic functions $(\mu_1, \mu_2, \mu_3, \bar{\mu}_4, \bar{\mu}_5)$ in equilibrium in Theorem 4.5 below.

The major difference between the price-impact equilibrium in Section 3 and the following Nash equilibrium analysis lies in the traders' stock-price perceptions. In the price-impact equilibrium, the forms of the stock-price perceptions (3.1) and (3.3) were conjectured with no additional justification beyond them leading to equilibrium existence. In contrast, for a subgame perfect Nash equilibrium, these perceptions must be such that:

(i) Trader k_0's own stock-price perceptions must be consistent with market-clearing for any off-equilibrium holdings $\theta_{k_0,t}$ used by k_0, when other traders' holding responses are optimal given the stock-price dynamics k_0 perceives other traders $k \neq k_0$ to have. This off-equilibrium market-clearing requirement can be found in, e.g., Vayanos (1999).

(ii) Trader k_0's equilibrium holdings are found by solving her optimization problem using her own market-clearing stock-price dynamics from (i).

(iii) Any trader's optimizer from (i) must be consistent with that trader's equilibrium holdings in (ii).

4.1 Optimal off-equilibrium responses

Lemma 4.1 gives trader k's optimal response to an arbitrary Itô process Z_t and is the Nash equilibrium analogue of Lemma 3.2.

Lemma 4.1 (Optimal responses to Z_t)**.** *Let $\mu_1, \mu_2, \mu_3, \bar{\mu}_4, \bar{\mu}_5 : [0,1] \to \mathbb{R}$ and $\kappa : [0,1] \to (0,\infty]$ be continuous functions, let $\alpha \leq 0$, let $(Z_t)_{t\in[0,1]}$ be an Itô process, and let the perceived stock-price process in the wealth dynamics* (2.7) *be as in* (4.1)*. Then, Z_t is adapted to both $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, Y_u, W_{i,u}, S^Z_{i,u})_{u\in[0,t]}$ and $\mathcal{F}_{j,t} := \sigma(\tilde{a}_\Sigma, w_u, Y_u, W_{j,u}, S^Z_{j,u})_{u\in[0,t]}$ and, provided*

$$\begin{aligned} \theta^Z_{i,t} &:= -\tfrac{1}{2\alpha-2\kappa(t)} Z_t - \tfrac{2\kappa(t)+\mu_1(t)}{2\alpha-2\kappa(t)} \tilde{a}_i - \tfrac{\mu_2(t)}{2\alpha-2\kappa(t)} q_{i,t} - \tfrac{\mu_3(t)}{2\alpha-2\kappa(t)} \eta_t, \\ \theta^Z_{j,t} &:= -\tfrac{1}{2\alpha-2\kappa(t)} Z_t - \tfrac{2\kappa(t)+\bar{\mu}_5(t)}{2\alpha-2\kappa(t)} w_t - \tfrac{\bar{\mu}_4(t)}{2\alpha-2\kappa(t)} \tilde{a}_\Sigma, \end{aligned} \tag{4.2}$$

satisfy (2.6)*, the traders' maximizers for* (2.5) *are $\theta^Z_{i,t}$ for rebalancer $i \in \{1, ..., M\}$ and $\theta^Z_{j,t}$ for tracker $j \in \{M+1, ..., M+\bar{M}\}$.* ♢

Similar to Lemma 3.2, Lemma 4.1 is proven using pointwise quadratic maximization. Unlike Y_t in Lemma 3.2, there is no Markov structure imposed on Z_t in Lemma 4.1, which makes dynamical programming inapplicable. Therefore, the simplicity of the linear-quadratic objectives in (2.5) is crucial for the proof of the optimality of $\theta^Z_{i,t}$ and $\theta^Z_{j,t}$ in (4.2).

4.2 Market-clearing stock-price perceptions

Investor k_0 does not care per se about other investors' stock-price perceptions except to the extend that other investors' optimal holdings given their perceptions affect the market-clearing stock prices at which k_0 trades. Thus, when solving for trader k_0's individual equilibrium holdings, we require k_0's perceived stock-price process $S^\nu_{k_0,t}$ to clear the stock market for arbitrary holdings $\theta_{k_0,t}$. We assume that a given trader $k_0 \in \{1, ..., M + \bar{M}\}$ perceives that other traders $k \neq k_0$ perceive the stock-price processes in (4.1). Hence, trader k_0 perceives that other traders k optimally hold $\theta^Z_{k,t}$ in (4.2) shares of stock. Given this, we then find market-clearing $Z_{k_0,t}$ processes associated with arbitrary hypothetical holdings $\theta_{k_0,t}$ for trader k_0.

First, consider a trader k_0 who is a rebalancer $k_0 := i_0 \in \{1, ..., M\}$. We construct a process $(Z_{i_0,t})_{t\in[0,1]}$ such that the stock market clears in the sense

$$0 = \sum_{i=1, i\neq i_0}^{M} \theta_{i,t}^{Z_{i_0}} + \sum_{j=M+1}^{\bar{M}} \theta_{j,t}^{Z_{i_0}} + \theta_{i_0,t}, \quad t \in [0,1], \tag{4.3}$$

where $\theta_{i_0,t}$ denotes an arbitrary stock-holdings process for rebalancer i_0 and other investors' responses $\theta_{k,t}^{Z_{i_0}}$ are from (4.2) for $Z_t := Z_{i_0,t}$. Clearly, any solution $Z_{i_0,t}$ of (4.3) is specific for rebalancer i_0. To describe one particular solution, we consider a specific continuously differentiable function $B : [0,1] \to \mathbb{R}$ satisfying

$$B(t) = -\frac{A(t)\mu_2(t) + \bar{M}\bar{\mu}_4(t) + 2\kappa(t) + \mu_1(t)}{2\bar{M}\kappa(t) + \bar{M}\bar{\mu}_5(t)}, \tag{4.4}$$

where $A(t)$ is as in (2.15). Because $A(t)$ in (2.15) depends on $B(t)$, Eq. (4.4) is a fixed point requirement for $B(t)$. Below, we show that the coupled ODEs (4.19) characterize (A, B) in (4.4), and we give conditions ensuring that (4.19) has a solution. Given a solution $B(t)$ to (4.4), we can define $Y_t := w_t - B(t)\tilde{a}_\Sigma$ as in (2.8), which allows us to express a solution of (4.3) as[10]

$$\begin{aligned} Z_{i_0,t} &:= \tfrac{2(\alpha-\kappa(t))}{M+\bar{M}-1}\theta_{i_0,t} + \tfrac{2\kappa(t)+\mu_1(t)}{M+\bar{M}-1}\tilde{a}_{i_0} + \tfrac{\mu_2(t)}{M+\bar{M}-1}q_{i_0,t} \\ &\quad - \tfrac{(M-1)\mu_3(t)+\mu_2(t)}{M+\bar{M}-1}\eta_t - \tfrac{\bar{M}(2\kappa(t)+\bar{\mu}_5(t))}{M+\bar{M}-1}Y_t, \quad t \in [0,1]. \end{aligned} \tag{4.5}$$

The process $Z_{i_0,t}$ in (4.5) captures the impact of arbitrary holdings $\theta_{i_0,t}$ by rebalancer i_0 on market-clearing stock prices given i_0's perceptions of how other traders optimally respond using $\theta_{k,t}^{Z_{i_0}}$.

We then describe rebalancer i_0's own stock-price perceptions for $i_0 \in \{1, ..., M\}$. Rebalancer i_0 filters based on her own target $\tilde{a}_i$ and on observations of past and current perceived market-clearing stock prices $(S_{i_0,u}^{\nu})_{u\in[0,t]}$ defined by

$$\begin{aligned} dS_{i_0,t}^{\nu} &:= \Big\{\nu_0(t)Z_{i_0,t} + \nu_1(t)\tilde{a}_{i_0} + \nu_2(t)q_{i_0,t} + \nu_3(t)\eta_t + \alpha\theta_{i_0,t}\Big\}dt + \gamma dw_{i_0,t}, \\ S_{i_0,0}^{\nu} &:= Y_0, \quad i_0 \in \{1, ..., M\}, \end{aligned} \tag{4.6}$$

where $(\tilde{a}_{i_0}, \theta_{i_0,t})$ are known and $(Z_{i_0,t}, q_{i_0,t}, \eta_{t_0})$ are inferred by rebalancer i_0. The "ν" superscript in (4.6) indicates that the perceived prices are defined with respect to a particular set of deterministic functions $(\nu_0, \nu_1, \nu_2, \nu_3)$, which we endogenously determine in Theorem 4.5 below. More specifically, by observing $\tilde{a}_{i_0}$ and $(S_{i_0,u}^{\nu})_{u\in[0,t]}$ defined in (4.6), rebalancer i_0 infers $Y_t := w_t - B(t)\tilde{a}_\Sigma$ from (2.8) using the Volterra argument behind Lemma 3.1.

[10]The specific $B(t)$ function in (4.4) lets us combine w_t and $\tilde{a}_\Sigma$ terms from (4.3) into the Y_t term in (4.5) using $Y_t = w_t - B(t)\tilde{a}_\Sigma$ from (2.8).

To see this, we insert (4.5) into (4.6) to produce rebalancer i_0's perceived market-clearing stock-price dynamics

$$\begin{aligned} dS^{\nu}_{i_0,t} = \Big\{ & \Big(\tfrac{\nu_0(t)(2\kappa(t)+\mu_1(t))}{M+\bar{M}-1} + \nu_1(t)\Big)\tilde{a}_{i_0} + \big(\tfrac{\mu_2(t)\nu_0(t)}{M+\bar{M}-1} + \nu_2(t)\big)q_{i_0,t} \\ & + \big(\nu_3(t) - \tfrac{\nu_0(t)((M-1)\mu 3(t)+\mu_2(t))}{M+\bar{M}-1}\big)\eta_t - \tfrac{\bar{M}\nu_0(t)(2\kappa(t)+\bar{\mu}_5(t))}{M+\bar{M}-1}Y_t \\ & + \big(\alpha + \tfrac{2\nu_0(t)(\alpha-\kappa(t))}{M+\bar{M}-1}\big)\theta_{i_0,t}\Big\}dt + \gamma dw_{i_0,t}. \end{aligned} \tag{4.7}$$

Because the expressions multiplying $(\tilde{a}_{i_0}, q_{i_0,t}, \eta_t, Y_t, \theta_{i_0,t})$ in (4.7) are continuous (deterministic) functions of time $t \in [0,1]$, Lemma 3.1 applies and shows that by observing $\tilde{a}_{i_0}$ and $(S^{\nu}_{i_0,u})_{u\in[0,t]}$ in (4.7) over time $t \in [0,1]$, rebalancer i_0 can infer $w_{i_0,t}$. Subsequently, rebalancer i_0 can use (2.10) and (2.13) to also infer Y_t over time $t \in [0,1]$.

Next, consider an investor k_0 who is tracker $k_0 := j_0 \in \{M+1, ..., M+\bar{M}\}$. For arbitrary off-equilibrium holdings $\theta_{j_0,t}$, the market-clearing solution $Z_{j_0,t}$ from

$$0 = \sum_{i=1}^{M} \theta^{Z_{j_0}}_{i,t} + \sum_{j=M+1, j\neq j_0}^{\bar{M}} \theta^{Z_{j_0}}_{j,t} + \theta_{j_0,t}, \quad t \in [0,1], \tag{4.8}$$

is given by

$$\begin{aligned} Z_{j_0,t} := & \tfrac{2(\alpha-\kappa(t))}{M+\bar{M}-1}\theta_{j_0,t} - \tfrac{M\mu_3(t)+\mu_2(t)}{M+\bar{M}-1}\eta_t - \tfrac{(\bar{M}-1)(2\kappa(t)+\bar{\mu}_5(t))}{M+\bar{M}-1}w_t \\ & - \tfrac{A(t)\mu_2(t)+(\bar{M}-1)\bar{\mu}_4(t)+2\kappa(t)+\mu_1(t)}{M+\bar{M}-1}\tilde{a}_\Sigma. \end{aligned} \tag{4.9}$$

Once again, $Z_{j_0,t}$ captures tracker j_0's perceptions of the impact of her holdings $\theta_{j_0,t}$ on market-clearing stock prices given j_0's perceptions of other investors' responses $\theta^{Z_{j_0}}_{k,t}$ to $\theta_{j_0,t}$.

Tracker j_0's perceived market-clearing stock-price process is

$$\begin{aligned} dS^{\nu}_{j_0,t} &:= \Big\{Z_{j_0,t} + \bar{\nu}_3(t)\eta_t + \bar{\nu}_4(t)\tilde{a}_\Sigma + \bar{\nu}_5(t)w_t + \alpha\theta_{j_0,t}\Big\}dt + \gamma dw_t, \\ S^{\nu}_{j_0,0} &:= Y_0, \quad j \in \{M+1, ..., M+\bar{M}\}, \end{aligned} \tag{4.10}$$

where $\bar{\nu}_3, \bar{\nu}_4, \bar{\nu}_5 : [0,1] \to \mathbb{R}$ are deterministic functions of time (endogenously determined Theorem 4.5 below). Inserting (4.9) into (4.10) gives tracker j_0's perceived market-clearing stock-price dynamics

$$\begin{aligned} dS^{\nu}_{j_0,t} = \Big\{ & \Big(\bar{\nu}_3(t) - \tfrac{M\mu 3(t)+\mu_2(t)}{M+\bar{M}-1}\Big)\eta_t \\ & + \Big(\bar{\nu}_5(t) - \tfrac{(\bar{M}-1)(2\kappa(t)+\bar{\mu}_5(t))}{M+\bar{M}-1}\Big)w_t \\ & + \Big(\bar{\nu}_4(t) - \tfrac{A(t)\mu_2(t)+(\bar{M}-1)\bar{\mu}_4(t)+2\kappa(t)+\mu_1(t)}{M+\bar{M}-1}\Big)\tilde{a}_\Sigma \\ & + \tfrac{\alpha(M+\bar{M}+1)-2\kappa(t)}{M+\bar{M}-1}\theta_{j,t}\Big\}dt + \gamma dw_t. \end{aligned} \tag{4.11}$$

We note that tracker j_0's perceived market-clearing stock-price dynamics $dS^\nu_{j_0,t}$ in (4.11) are driven by the exogenous Brownian motion w_t from (2.2) whereas rebalancer i_0's prices $dS^\nu_{i_0,t}$ in (4.7) are driven by i_0's innovations process $dw_{i_0,t}$ from (2.11). This is due to the different information sets of rebalancers and trackers.

Unlike the price-impact equilibrium in Theorem 3.5, we see from (4.7) and (4.11) that, even if the direct price impacts vanish in the sense $\alpha := 0$ in (4.6) and (4.10), the remaining net price impacts $\frac{2\nu_0(t)\kappa(t)}{1-M-\bar{M}}$ and $\frac{2\kappa(t)}{1-M-\bar{M}}$ of $\theta_{i,t}$ and $\theta_{j,t}$ are nonzero. This is because price pressure in (4.7) and (4.11) clears the stock market for arbitrary holdings $\theta_{i,t}$ and $\theta_{j,t}$.

The next result gives the optimal holdings $\theta^*_{k,t}$ for all traders $k_0 := k \in \{1, ..., M + \bar{M}\}$ given their perceptions of market-clearing stock prices in (4.7) and (4.11). While both $\theta^*_{k,t}$ and the optimal response holdings $\theta^Z_{k,t}$ in (4.2) maximize (2.5), they differ because they are based on different perceived stock-price processes. On one hand, the optimal responses $\theta^Z_{k,t}$ in (4.2) are based on the stock-price perceptions in (4.1). On the other hand, the optimizer $\theta^*_{k,t}$ is based on the market-clearing stock-price perceptions in (4.7) and (4.11).

Lemma 4.2 (Trader k's maximizer for market-clearing stock-price perceptions)**.** *Let* $\nu_0, \nu_1, \nu_2, \nu_3, \bar{\nu}_3, \bar{\nu}_4, \bar{\nu}_5 : [0,1] \to \mathbb{R}$ *and* $\kappa : [0,1] \to (0,\infty]$ *be continuous functions with* $\nu_0 > 0$ *and assume* $\alpha \leq 0$*. Let the perceived market-clearing stock-price processes in the wealth dynamics* (2.7) *be given by* (4.7) *and* (4.11) *with corresponding filtrations* $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S^\nu_{i,u})_{u\in[0,t]}$ *and* $\mathcal{F}_{j,t} := \sigma(w_u, S^\nu_{j,u})_{u\in[0,t]}$ *for* $i \in \{1, ..., M\}$ *and* $j \in \{M+1, ..., M+\bar{M}\}$*. Then, provided the holding processes*

$$\begin{aligned}
\theta^*_{i,t} :=& -\tfrac{2\kappa(t)(M+\bar{M}+\nu_0(t)-1)+(M+\bar{M}-1)\nu_1(t)+\mu_1(t)\nu_0(t)}{2(\alpha-\kappa(t))(M+\bar{M}+2\nu_0(t)-1)}\tilde{a}_i \\
&- \tfrac{(M+\bar{M}-1)\nu_2(t)+\mu_2(t)\nu_0(t)}{2(\alpha-\kappa(t))(M+\bar{M}+2\nu_0(t)-1)}q_{i,t} \\
&+ \tfrac{\nu_0(t)((M-1)\mu_3(t)+\mu_2(t))-(M+\bar{M}-1)\nu_3(t)}{2(\alpha-\kappa(t))(M+\bar{M}+2\nu_0(t)-1)}\eta_t \\
&+ \tfrac{\bar{M}\nu_0(t)(2\kappa(t)+\bar{\mu}_5(t))}{2(\alpha-\kappa(t))(M+\bar{M}+2\nu_0(t)-1)}Y_t, \\
\theta^*_{j,t} :=& \tfrac{-(M+\bar{M}-1)\bar{\nu}_3(t)+M\mu_3(t)+\mu_2(t)}{2(M+\bar{M}+1)(\alpha-\kappa(t))}\eta_t \\
&+ \tfrac{-(M+\bar{M}-1)\bar{\nu}_5(t)-2M\kappa(t)+(\bar{M}-1)\bar{\mu}_5(t)}{2(M+\bar{M}+1)(\alpha-\kappa(t))}w_t \\
&+ \tfrac{A(t)\mu_2(t)-(M+\bar{M}-1)\bar{\nu}_4(t)+(\bar{M}-1)\bar{\mu}_4(t)+2\kappa(t)+\mu_1(t)}{2(M+\bar{M}+1)(\alpha-\kappa(t))}\tilde{a}_\Sigma,
\end{aligned} \tag{4.12}$$

satisfy (2.6)*, the traders' maximizers for* (2.5) *are* $\theta^*_{i,t}$ *for rebalancer* $i \in \{1, ..., M\}$ *and* $\theta^*_{j,t}$ *for tracker* $j \in \{M+1, ..., M+\bar{M}\}$*.* ♢

From Lemma 4.2, we note that a generic rebalancer i_0 has filtration $\sigma(\tilde{a}_{i_0}, S^\nu_{i_0,u})_{u\in[0,t]}$ whereas she perceives that other rebalancers $i \neq i_0$ have filtrations $\sigma(\tilde{a}_i, Y_u, W_{i,u}, S^Z_{i,u})_{u\in[0,t]}$ as in Lemma 4.1. Because these are i_0's off-equilibrium perceptions, this is allowable as long as they are consistent with i's equilibrium holdings. We require this consistency in Definition 4.3(iii) below. We also note from Lemma 4.1 that rebalancer i can infer $Z_{i_0,t}$ in

(4.5). In turn, this allows rebalancer i to infer the process

$$\tfrac{2(\alpha-\kappa(t))}{M+\bar{M}-1}\theta_{i_0,t} + \tfrac{2\kappa(t)+\mu_1(t)}{M+\bar{M}-1}\tilde{a}_{i_0} + \tfrac{\mu_2(t)}{M+\bar{M}-1}q_{i_0,t}. \tag{4.13}$$

However, knowing (4.13) is insufficient for rebalancer i to infer rebalancer i_0's private target $\tilde{a}_{i_0}$.

4.3 Equilibrium

Definition 4.3. Deterministic functions of time $\mu_1, \mu_2, \mu_3, \bar{\mu}_4, \bar{\mu}_5, \nu_0, \nu_1, \nu_2, \nu_3, \bar{\nu}_4, \bar{\nu}_5 : [0,1] \to \mathbb{R}$ constitute a *subgame perfect Nash financial-market equilibrium* if:

(i) For $k \in \{1, ..., M + \bar{M}\}$, trader k's maximizer $\theta^*_{k,t}$ for (2.5) exists given the market-clearing stock-price perceptions (4.7) and (4.11).

(ii) For $k \in \{1, ..., M + \bar{M}\}$, inserting trader k's maximizer $\theta^*_{k,t}$ into the perceived market-clearing stock-price processes (4.7) and (4.11) produces identical stock-price processes across all traders. This common equilibrium stock-price process is denoted by S^*_t.

(iii) Optimizers and equilibrium holdings must be consistent in the sense that trader k's perceived response to trader k_0's maximizer $\theta^*_{k_0,t}$ is trader k's maximizer $\theta^*_{k,t}$.

(iv) The money and stock markets clear. ♢

The identical stock-price requirement in Definition 4.3(ii) is similar to (3.5). We see from the rebalancers' perceptions (4.6) that both the drifts and the martingale terms have i dependence. We replace $dw_{i,t}$ in $dS^{\nu}_{i,t}$ in (4.6) with the decomposition of $dw_{i,t}$ in terms of dw_t from (2.11) and rewrite $dS^{\nu}_{i,t}$ in (4.6) as

$$\begin{aligned} dS^{\nu}_{i,t} = \Big\{ &\nu_0(t)Z_{i,t} + \nu_1(t)\tilde{a}_i + \nu_2(t)q_{i,t} + \nu_3(t)\eta_t + \alpha\theta_{i,t} \\ &- B'(t)\big(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t}\big)\gamma \Big\} dt + \gamma dw_t, \quad i \in \{1, ..., M\}. \end{aligned} \tag{4.14}$$

Therefore, to ensure identical equilibrium stock-price perceptions for all traders $k \in \{1, ..., M + \bar{M}\}$, it suffices to match the drift of $dS^{\nu}_{j,t}$ in (4.10) for $j \in \{M+1, ..., M+\bar{M}\}$ with the drift of $dS^{\nu}_{i,t}$ for $\theta_{i,t} := \theta^*_{i,t}$ for the optimal holdings $i \in \{1, ..., M\}$ in (4.14). This produces the requirement (the right-hand side of (4.15) does not depend on the rebalancer index i)

$$\begin{aligned} &\nu_0(t)Z^*_{i,t} + \nu_1(t)\tilde{a}_i + \nu_2(t)q_{i,t} + \nu_3(t)\eta_t + \alpha\theta^*_{i,t} - B'(t)\big(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,t}\big)\gamma \\ &= \bar{\nu}_3(t)\eta_t + \bar{\nu}_4(t)\tilde{a}_\Sigma + \bar{\nu}_5(t)w_t + \alpha\theta^*_{j,t}, \end{aligned} \tag{4.15}$$

for all rebalancers $i \in \{1, ..., M\}$ and all trackers $j \in \{M+1, ..., M+\bar{M}\}$. In (4.15), the process $Z^*_{i,t}$ is (4.5) evaluated at $\theta_{i,t} := \theta^*_{i,t}$, and $Z^*_{j,t}$ is (4.9) evaluated at $\theta_{j,t} := \theta^*_{j,t}$ so that:

$$\begin{aligned}
Z^*_{i,t} &:= \tfrac{2(\alpha-\kappa(t))}{M+\bar{M}-1}\theta^*_{i,t} + \tfrac{2\kappa(t)+\mu_1(t)}{M+\bar{M}-1}\tilde{a}_i + \tfrac{\mu_2(t)}{M+\bar{M}-1}q_{i,t} \\
&\quad - \tfrac{(M-1)\mu_3(t)+\mu_2(t)}{M+\bar{M}-1}\eta_t - \tfrac{\bar{M}(2\kappa(t)+\bar{\mu}_5(t))}{M+\bar{M}-1}Y_t, \\
Z^*_{j,t} &:= \tfrac{2(\alpha-\kappa(t))}{M+\bar{M}-1}\theta^*_{j,t} - \tfrac{M\mu_3(t)+\mu_2(t)}{M+\bar{M}-1}\eta_t \\
&\quad - \tfrac{(\bar{M}-1)(2\kappa(t)+\bar{\mu}_5(t))}{M+\bar{M}-1}w_t - \tfrac{A(t)\mu_2(t)+(\bar{M}-1)\bar{\mu}_4(t)+2\kappa(t)+\mu_1(t)}{M+\bar{M}-1}\tilde{a}_\Sigma,
\end{aligned} \tag{4.16}$$

for rebalancers $i \in \{1, ..., M\}$ and trackers $j \in \{M+1, ..., M+\bar{M}\}$.

As for the consistency requirement in Definition 4.3(iii), we first fix a rebalancer $i_0 \in \{1, ..., M\}$. We require that the response holdings in (4.2) are consistent with $\theta^*_{i_0,t}$ in the sense that

$$\begin{aligned}
\theta^*_{i,t} &= -\tfrac{1}{2\alpha-2\kappa(t)}Z^*_{i_0,t} - \tfrac{2\kappa(t)+\mu_1(t)}{2\alpha-2\kappa(t)}\tilde{a}_i - \tfrac{\mu_2(t)}{2\alpha-2\kappa(t)}q_{i,t} - \tfrac{\mu_3(t)}{2\alpha-2\kappa(t)}\eta_t, \\
\theta^*_{j,t} &= -\tfrac{1}{2\alpha-2\kappa(t)}Z^*_{i_0,t} - \tfrac{2\kappa(t)+\bar{\mu}_5(t)}{2\alpha-2\kappa(t)}w_t - \tfrac{\bar{\mu}_4(t)}{2\alpha-2\kappa(t)}\tilde{a}_\Sigma,
\end{aligned} \tag{4.17}$$

for rebalancers $i \in \{1, ..., M\} \setminus \{i_0\}$ and trackers $j \in \{M+1, ..., M+\bar{M}\}$. Second, we fix a tracker $j_0 \in \{M+1, ..., M+\bar{M}\}$ and require that the response holdings in (4.2) must be consistent with $\theta^*_{j_0,t}$ in the sense that

$$\begin{aligned}
\theta^*_{i,t} &= -\tfrac{1}{2\alpha-2\kappa(t)}Z^*_{j_0,t} - \tfrac{2\kappa(t)+\mu_1(t)}{2\alpha-2\kappa(t)}\tilde{a}_i - \tfrac{\mu_2(t)}{2\alpha-2\kappa(t)}q_{i,t} - \tfrac{\mu_3(t)}{2\alpha-2\kappa(t)}\eta_t, \\
\theta^*_{j,t} &= -\tfrac{1}{2\alpha-2\kappa(t)}Z^*_{j_0,t} - \tfrac{2\kappa(t)+\bar{\mu}_5(t)}{2\alpha-2\kappa(t)}w_t - \tfrac{\bar{\mu}_4(t)}{2\alpha-2\kappa(t)}\tilde{a}_\Sigma,
\end{aligned} \tag{4.18}$$

for rebalancers $i \in \{1, ..., M\}$ and trackers $j \in \{M+1, ..., M+\bar{M}\} \setminus \{j_0\}$.

Similar to the price-impact equilibrium, our Nash equilibrium existence result is based on a technical lemma, which guarantees the existence of a solution to an autonomous system of coupled ODEs.

Lemma 4.4. *Let $\kappa : [0,1] \to (0,\infty]$ be a continuous and integrable function (i.e., $\int_0^1 \kappa(t)dt < \infty$), let $M + \bar{M} > 2$, and let $\alpha \le 0$. For a constant $B(0) \in \mathbb{R}$, the coupled ODEs*

$$\begin{aligned}
B'(t) &= \frac{\begin{array}{l}\Big\{2\kappa(t)\Big(\bar{M}B(t)(M+\bar{M}-1)\big(\alpha(M+\bar{M}) - 2(M+\bar{M}-1)\kappa(t)\big) \\ \quad +(M+\bar{M}-2)\big(\alpha(M+\bar{M}+1) - 2(M+\bar{M})\kappa(t)\big)\Big)\Big\}\end{array}}{\begin{array}{l}\Big\{\gamma\Big(A(t)(M+\bar{M}-2)\big(\alpha(M+\bar{M}+1) - 2(M+\bar{M})\kappa(t)\big) \\ +\alpha\big((M^2+M-1)\bar{M} + M^2 + 2M\bar{M}^2 - M + \bar{M}^3 - 2\big) \\ -2\left((M^2-1)\bar{M} + (2M-1)\bar{M}^2 + (M-2)M + \bar{M}^3\right)\kappa(t)\Big)\Big\}\end{array}}, \\
A'(t) &= -\big(B'(t)\big)^2\Sigma(t)\big(A(t)+1\big), \quad A(0) = -\frac{(M-1)B(0)^2\sigma^2_{\tilde{a}}}{\sigma^2_{w_0} + B(0)^2(M-1)\sigma^2_{\tilde{a}}}, \\
\Sigma'(t) &= -\big(B'(t)\big)^2\Sigma(t)^2, \quad \Sigma(0) = \frac{(M-1)\sigma^2_{\tilde{a}}\sigma^2_{w_0}}{B(0)^2(M-1)\sigma^2_{\tilde{a}} + \sigma^2_{w_0}},
\end{aligned} \tag{4.19}$$

have unique solutions with $\Sigma(t) \ge 0$, $\Sigma(t)$ decreasing, $A(t) \in [-1, 0]$, and $A(t)$ decreasing *for* $t \in [0,1]$. ◊

The affine ODE for $B(t)$ in (4.19) is more complicated than the corresponding affine ODE in (3.7) because the Nash equilibrium has the additional fixed point requirement in (4.4) that is absent in the price-impact equilibrium. However, both ODEs for $B(t)$ are affine.

Our main theoretical result gives a Nash equilibrium in terms of the ODEs (4.19). In this theorem, the price-impact parameter α, volatility γ, and initial value $B(0)$ are free parameters.

Theorem 4.5. *Let* $\kappa : [0,1] \to (0,\infty)$ *be continuous, and let the functions* (B, A, Σ) *be as in Lemma 4.4, let* $M + \bar{M} > 2$*, and let* $\alpha \leq 0$*. Then, we have:*

(i) A subgame perfect Nash financial-market equilibrium exists and is given by the functions in (A.2) *in Appendix A.*

(ii) Equilibrium holdings are

$$\begin{aligned}\theta^*_{i,t} :=& -\frac{(M+\bar{M}-2)\,(2\kappa(t)-\gamma B'(t))}{\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t)}\tilde{a}_i \\ &+ \frac{\gamma(M+\bar{M}-2)B'(t)}{\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t)}q_{i,t} \\ &- \frac{\Big\{\gamma(M+\bar{M}-2)^2B'(t)(\alpha(M+\bar{M}+1)-2(M+\bar{M})\kappa(t))\Big\}}{\begin{array}{l}\Big\{(\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t))\big(\alpha\big((3M-1)\bar{M}^2+M(3M-2)\bar{M} \\ +(M-2)M(M+1)+\bar{M}^3\big)-2\left((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\right)\kappa(t)\big)\Big\}\end{array}}\eta_t, \\ &+ \frac{\Big\{2\bar{M}(M+\bar{M}-2)(M+\bar{M}-1)\kappa(t)\Big\}}{\begin{array}{l}\Big\{\alpha\left((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\right) \\ -2\left((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\right)\kappa(t)\Big\}\end{array}}Y_t,\end{aligned} \tag{4.20}$$

$$\begin{aligned}\theta^*_{j,t} :=& -\frac{\gamma(M+\bar{M}-2)(M+\bar{M}-1)B'(t)}{\alpha\big((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\big)-2\big((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\big)\kappa(t)}\eta_t \\ &- \frac{2M(M+\bar{M}-2)(M+\bar{M}-1)\kappa(t)}{\alpha\big((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\big)-2\big((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\big)\kappa(t)}w_t \\ &+ \frac{(M+\bar{M}-2)(M+\bar{M}-1)(\gamma(-A(t)+M-1)B'(t)+2\kappa(t))}{\alpha\big((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\big)-2\big((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\big)\kappa(t)}\tilde{a}_\Sigma,\end{aligned}$$

for rebalancers $i \in \{1,...,M\}$ *and trackers* $j \in \{M+1,...,M+\bar{M}\}$.

(iii) The equilibrium stock-price process has dynamics

$$
\begin{aligned}
dS_t^* &:= \Big\{ \frac{\gamma(M+\bar{M}-2)B'(t)(\alpha(M+\bar{M}+1)-2(M+\bar{M})\kappa(t))}{\alpha\big((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\big)-2\big((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\big)\kappa(t)}\eta_t \\
&\quad - \frac{2\bar{M}(M+\bar{M}-1)\kappa(t)(\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t))}{\alpha\big((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\big)-2\big((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\big)\kappa(t)}w_t \\
&\quad - \frac{(M+\bar{M}-2)(\alpha(M+\bar{M}+1)-2(M+\bar{M})\kappa(t))(\gamma(-A(t)+M-1)B'(t)+2\kappa(t))}{\alpha\big((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\big)-2\big((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\big)\kappa(t)}\tilde{a}_\Sigma \Big\} dt \\
&\quad + \gamma dw_t, \\
S_0^* &:= w_0 - B(0)\tilde{a}_\Sigma.
\end{aligned}
\tag{4.21}
$$

$\diamondsuit$

The following observations follow from Theorem 4.5:

1. The logic for the initial value $B(0)$ being a free input parameter is the same as in the price-impact equilibrium.

2. The price-impact parameter α and stock-price volatility γ affect the stock-price drift and holdings via its impact on $B(t)$ in (4.19). The dependence on α is different from the price-impact equilibrium where the corresponding $B(t)$ in (3.7) is independent of α. The reason is that α affects the perceived optimal responses in (4.2).

3. Similar to (3.12) and (3.13), for an arbitrary trader $k_0 \in \{1, ..., M + \bar{M}\}$ and her arbitrary holdings $\theta_{k_0,t}$, the optimal responses in (4.2) can be decomposed as

$$
\begin{aligned}
\theta_{i,t}^{Z_{k_0}} &= \theta_{i,t}^* - \frac{1}{M+\bar{M}-1}(\theta_{k_0,t} - \theta_{k_0,t}^*), \quad i \in \{1, ..., M\}, \\
\theta_{j,t}^{Z_{k_0}} &= \theta_{j,t}^* - \frac{1}{M+\bar{M}-1}(\theta_{k_0,t} - \theta_{k_0,t}^*), \quad j \in \{M+1, ..., M+\bar{M}\},
\end{aligned}
\tag{4.22}
$$

   where the equilibrium holdings $(\theta_{i,t}^*, \theta_{j,t}^*, \theta_{k_0,t}^*)$ are in (4.20).[11]

4. The subgame perfect Nash financial-market equilibrium is attractive because of its reasonable off-equilibrium market-clearing beliefs. However, although much of the mathematic structure is similar, the expressions for the equilibrium price and holding coefficients are algebraically more complex. Nonetheless, our numerical results in Section 5 below show that the differences between the price-impact equilibrium and the subgame perfect Nash financial-market equilibrium are quantitatively small. This, in turn, suggests that the economic logic from the price-impact equilibrium carries over to the Nash equilibrium.

[11] This is similar to Eq. (2.16) in Chen, Choi, Larsen, and Seppi (2021).

5 Numerics

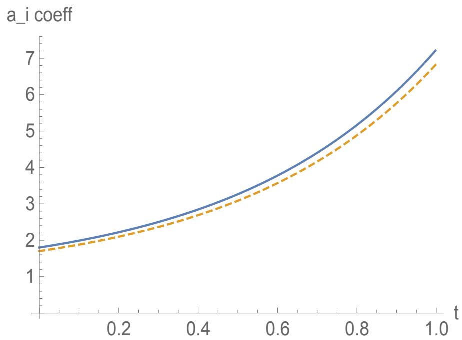
Our price-impact and subgame perfect equilibria are straightforward to compute numerically. This is because prices and holding in the model are available in closed form given the solutions to the associated coupled ODEs in (3.7) and (4.19). To illustrate our models, we compute them for several different parameterizations. In these parameterizations, there are $M := 10$ rebalancers and $\bar{M} := 10$ trackers. We assume the penalty function is a constant over the trading day at set $\kappa(t) := 1$. The target volatilities are normalized to $\sigma_{\tilde{a}} = \sigma_{w_0} = 1$. We then consider two initial values of $B(0) = -1$ (consistent with our negative $B(t)$ restriction) and also $B(0) = 0$ as a reference point. We consider two price volatility parameters $\gamma \in \{0.5, 1\}$ and a price-impact parameter $\alpha := -0.1$.

For the two equilibria, Figure 1 shows the coefficients for the equilibrium holdings for rebalancers and trackers, and Figure 2 shows the coefficients for the equilibrium prices. Figure 4 in Appendix B shows the solutions to the coupled ODEs (3.7) and (4.19).

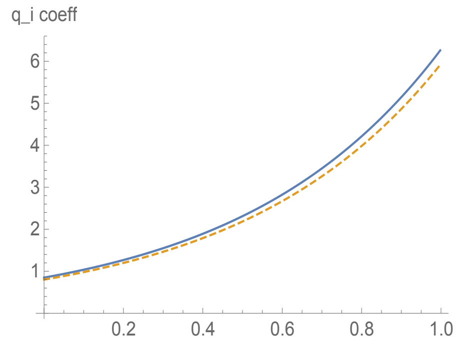
In each figure, there are pairs of lines for the two different equilibria. Interestingly, the quantitative differences between the price-impact and subgame perfect Nash equilibria are quite small despite the additional mathematical complexity of the subgame perfect Nash financial-market equilibrium. Recall here that the difference between the two equilibria is the additional restriction in the subgame perfect Nash equilibrium that investors' off-equilibrium price perceptions given hypothetical holdings must still clear the stock market given optimal order response functions based on associated stock-price perceptions for other investors. Thus, it appears that in-equilibrium market clearing has a much larger effect on equilibrium prices than the requirement of off-equilibrium market clearing.

The signs of the various price and holding coefficients in the $B(t) < 0$ case for the price-impact equilibrium are consistent with the analytic signing results in Section 3. In addition, we see that the loading on $\tilde{a}_\Sigma$ in the tracker holdings is positive, as noted in Section 3, which implies front-running on predictable future stock-price pressure. The numerical similarity of the numerical results for the two equilibria suggests that the intuitions for the signs of the various coefficients in the price-impact equilibrium carry over to the subgame perfect Nash financial-market equilibrium.

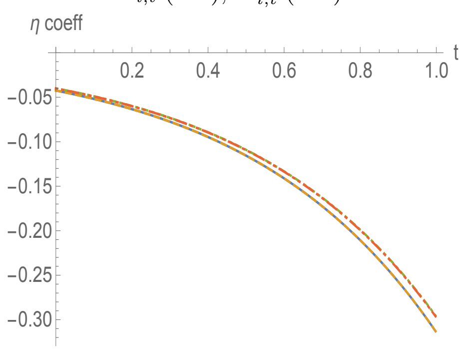
Figure 1: Plots of coefficient loadings in the price-impact equilibrium holdings $\hat{\theta}_{k,t}$ in (3.8) and in the Nash equilibrium holdings $\theta^*_{k,t}$ in (4.20). The exogenous model parameters are $\gamma := 1$, $\sigma_{w_0} := \sigma_{\tilde{a}} := 1, M := \bar{M} := 10$, $\kappa(t) := 1$, $B(0) := -1$, and $\alpha := -0.1$.



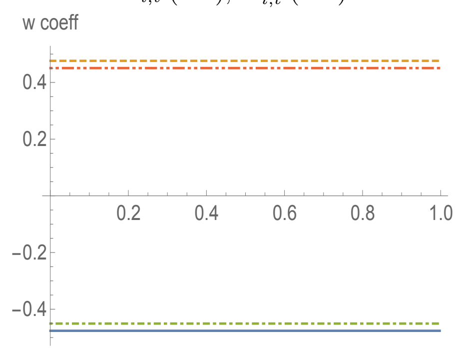
1A: $\tilde{a}_i$ coefficient
$\hat{\theta}_{i,t}$ (——), $\theta^*_{i,t}$ (−−)



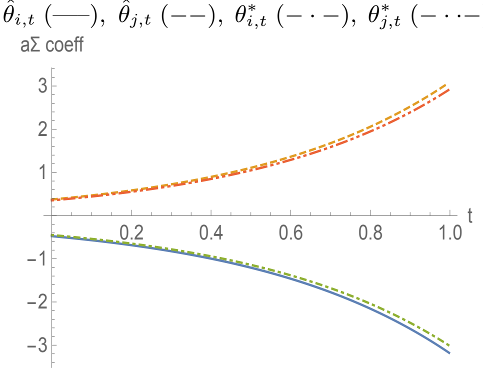
1B: $q_{i,t}$ coefficient
$\hat{\theta}_{i,t}$ (——), $\theta^*_{i,t}$ (−−)



1C: η_t coefficient
$\hat{\theta}_{i,t}$ (——), $\hat{\theta}_{j,t}$ (−−), $\theta^*_{i,t}$ (− · −), $\theta^*_{j,t}$ (− · · −)

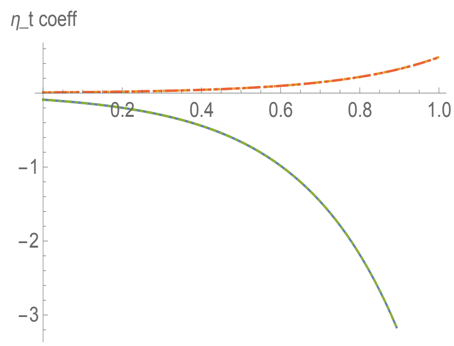


1D: w_t coefficient
$\hat{\theta}_{i,t}$ (——), $\hat{\theta}_{j,t}$ (−−), $\theta^*_{i,t}$ (− · −), $\theta^*_{j,t}$ (− · · −)

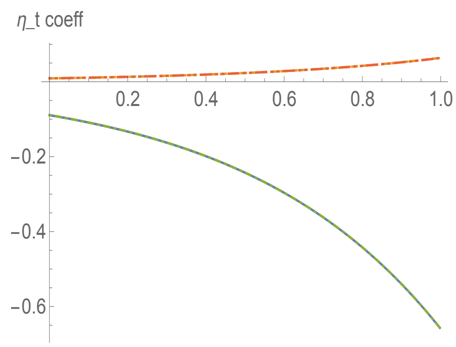


1E: $\tilde{a}_\Sigma$ coefficient
$\hat{\theta}_{i,t}$ (——), $\hat{\theta}_{j,t}$ (−−), $\theta^*_{i,t}$ (− · −), $\theta^*_{j,t}$ (− · · −)

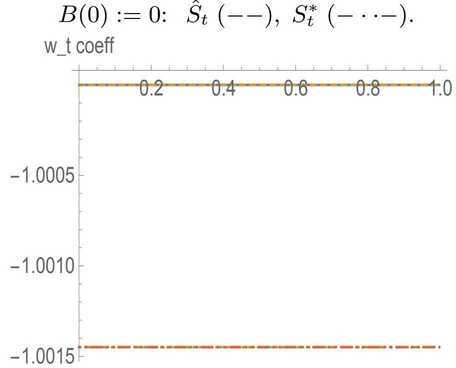
Figure 2: Plots of coefficient loadings in price-impact equilibrium stock-price dynamics $d\hat{S}_t$ in (3.9) and Nash equilibrium stock-price dynamics dS_t^* in (4.21) for $B(0) \in \{-1, 0\}$. Plots 3A, 3C, and 3E use $\gamma := 0.5$ and Plots 3B, 3D, and 3F use $\gamma := 1$. The exogenous model parameters are $\sigma_{w_0} := \sigma_{\tilde{a}} := 1, M := \bar{M} := 10,\ \alpha := -0.1$, and $\kappa(t) := 1$ for $t \in [0, 1]$.



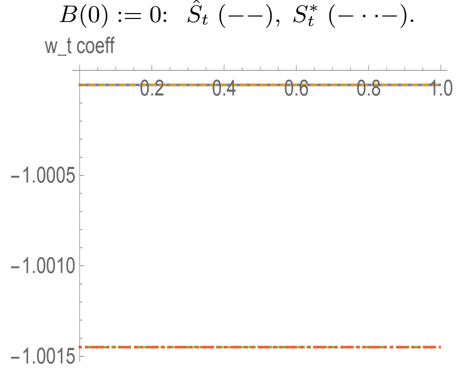
2A: η_t coefficient, $\gamma := 0.5$,
$B(0) := -1$: $\hat{S}_t$ (——), S_t^* $(- \cdot -)$,
$B(0) := 0$: $\hat{S}_t$ $(--)$, S_t^* $(- \cdot \cdot -)$.



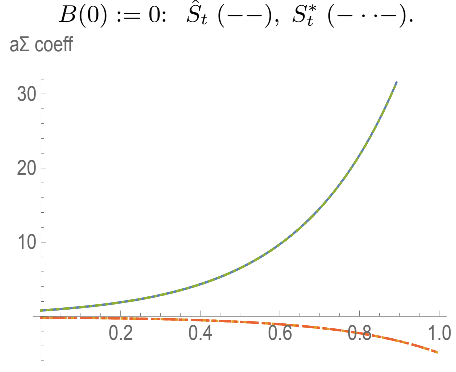
2B: η_t coefficient, $\gamma := 1$,
$B(0) := -1$: $\hat{S}_t$ (——), S_t^* $(- \cdot -)$,
$B(0) := 0$: $\hat{S}_t$ $(--)$, S_t^* $(- \cdot \cdot -)$.



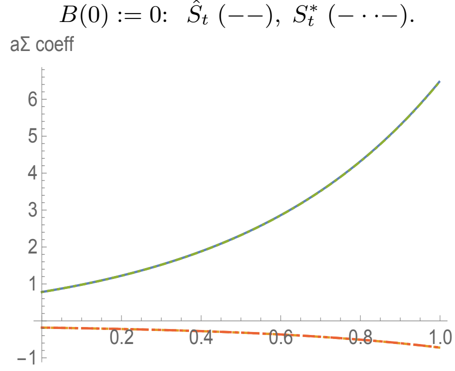
2C: w_t coefficient, $\gamma := 0.5$,
$B(0) := -1$: $\hat{S}_t$ (——), S_t^* $(- \cdot -)$,
$B(0) := 0$: $\hat{S}_t$ $(--)$, S_t^* $(- \cdot \cdot -)$.



2D: w_t coefficient, $\gamma := 1$,
$B(0) := -1$: $\hat{S}_t$ (——), S_t^* $(- \cdot -)$,
$B(0) := 0$: $\hat{S}_t$ $(--)$, S_t^* $(- \cdot \cdot -)$.



2E: $\tilde{a}_\Sigma$ coefficient, $\gamma := 0.5$,
$B(0) := -1$: $\hat{S}_t$ (——), S_t^* $(- \cdot -)$,
$B(0) := 0$: $\hat{S}_t$ $(--)$, S_t^* $(- \cdot \cdot -)$.



2F: $\tilde{a}_\Sigma$ coefficient, $\gamma := 1$,
$B(0) := -1$: $\hat{S}_t$ (——), S_t^* $(- \cdot -)$,
$B(0) := 0$: $\hat{S}_t$ $(--)$, S_t^* $(- \cdot \cdot -)$.

6 Measuring execution costs

This section gives a measure of a rebalancer's costs of rebalancing from zero endowed shares at time $t = 0$ given a target $\tilde{a}_i$. We present the measure in the price-impact equilibrium in Section 3 (the Nash analogue is logically similar and produces similar numerics). In the price-impact equilibrium, rebalancer i's value function is

$$J(\tilde{a}_i, 0, \eta_0, Y_0, q_{i,0}) := \mathbb{E}\Big[\int_0^1 \hat{\theta}_{i,t} d\hat{S}_t - \int_0^1 \kappa(t)(\tilde{a}_i - \hat{\theta}_{i,t})^2 dt \Big| \mathcal{F}_{i,0}\Big], \tag{6.1}$$

where $\hat{\theta}_{i,t}$ denotes rebalancer i's holdings in (3.8) and $\mathcal{F}_{i,t} := \sigma(\tilde{a}_i, S^f_{i,u})_{u\in[0,t]}$ where the f coefficient functions are as in (A.1) for $i \in \{1, ..., M\}$. We seek a value function $J = J(\tilde{a}_i, s, q, Y, q_i)$ such that the process

$$\begin{aligned} J(\tilde{a}_i, s, \eta_s, Y_s, q_{i,s}) + \int_0^s \Big\{ \hat{\theta}_{i,t}\Big(f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\hat{\theta}_{i,t}\Big) \\ - \kappa(t)(\tilde{a}_i - \hat{\theta}_{i,t})^2\Big\} dt, \quad s \in [0, 1], \end{aligned} \tag{6.2}$$

is a martingale with respect to $\mathcal{F}_{i,t}$. Because rebalancer i's objective in (2.5) is linear-quadratic, the value function J is quadratic in the state processes. Thus, J can be written as

$$\begin{aligned} J(\tilde{a}_i, s, \eta, Y, q_i) = J_0(s) + J_\eta(s)\eta + J_Y(s)Y + J_{q_i}(s)q_i + J_{\eta\eta}(s)\eta^2 \\ + J_{\eta Y}(s)\eta Y + J_{YY}(s)Y^2 + J_{q_iq_i}(s)q_i^2 + J_{q_i\eta}(s)q_i\eta + J_{q_iY}(s)q_iY, \end{aligned} \tag{6.3}$$

for deterministic functions of time $(J_0, J_\eta, J_Y, J_{q_i}, J_{\eta\eta}, J_{\eta Y}, J_{YY}, J_{q_iq_i}, J_{q_i\eta}, J_{q_iY})$. These functions are given by a coupled set of ODEs with zero terminal conditions (we omit the ODEs for brevity). In (6.3), the dummy variables (η, Y, q_i) are real numbers and $s \in [0, 1]$.

To quantify the costs associated with rebalancer i's trading target $\tilde{a}_i$, the quadratic mapping RC (Rebalancing Costs) defined by

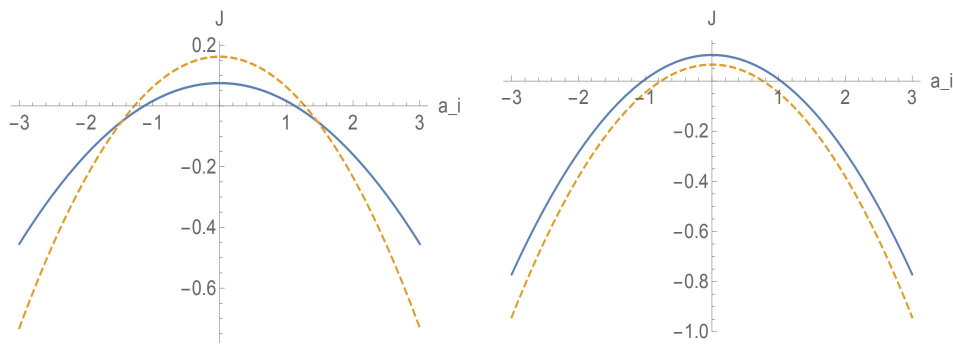
$$\mathrm{RC}(\tilde{a}_i) := J(0, 0, \eta, Y, q_i) - J(\tilde{a}_i, 0, \eta, Y, q_i), \tag{6.4}$$

measures the dependence the change in profit (i.e., change in value function) associated with a non-zero target $\tilde{a}_i$.

Figure 3 plots the rebalancer's value function J for different target values $\tilde{a}_i$ for different model parameterizations. When the target $\tilde{a}_i$ is close to zero, the rebalancers become high-frequency liquidity providers. Their value function in this case is positive due expected profit from liquidity provision and price-pressure front-running. As the target moves away from zero, the rebalancer starts to have larger holding penalties that eventually can drive the rebalancer's value function negative. Interestingly, the impact of the stock-price volatility parameter γ on the rebalancer's value function can be positive or negative. Liquidity providing rebalancers are better off with a small γ whereas rebalancers with large rebalancing targets are better off when γ is large.

The rebalancing cost RC in (6.4) for a target $\tilde{a}_i$ is computed as the difference between the value function evaluated at $\tilde{a}_i$ and the function evaluated at $\tilde{a}_i = 0$. Since J is highest at $\tilde{a}_i = 0$, RC is positive.

Figure 3: Plots of the rebalancers' value function J for various values of (γ, σ_{w_0}). The exogenous model parameters are $\sigma_{\tilde{a}} := 1, M := \bar{M} := 10,\ \alpha := -0.1,\ B(0) := -1,\ \kappa(t) := 1$ for $t \in [0,1]$, and $w_0 := B(0)(\tilde{a}_\Sigma - \tilde{a}_i)$.



3A: $\gamma := 1$ (—–), $\gamma := 0.5$ (– –), $\sigma_{w_0} := 1$ **3B:** $\gamma := 1$ (—–), $\gamma := 0.5$ (– –), $\sigma_{w_0} := 0.1$

7 Conclusion

This paper presents the first analytically tractable model of dynamic learning about parent trading demand imbalances with optimized order-splitting. In particular, we provide closed-form expressions prices and stock holdings in terms of solutions to systems of coupled ODEs in both price-impact and Nash equilibria. We then show that trading in our models reflects a combination of reaching investor's own trading targets, liquidity provision so that markets can clear, and front-running based on predictions of future price pressure.

There are many interesting directions for future research based on our analysis. First, replacing the zero-dividend stock approach with valuation based on a terminal payoff would be a significant technical step. Second, the model could be enriched by allowing for investor heterogeneity in the form of different penalty functions $\kappa(t)$ and by having multiple tracker targets (which would weaken the trackers' informational advantage). Third, it would be interesting to investigate if other off-equilibrium refinements have larger equilibrium effects. Fourth, incorporating risk-aversion into the investors' objectives would be interesting too. For example, how can Lemma 4.1 be extended if the objectives in (2.5) are changed to exponential utilities?

A Formulas

A.1 Price-impact equilibrium

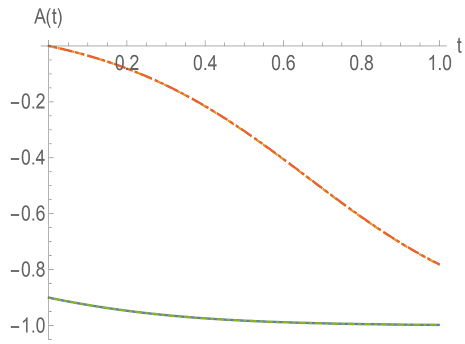
$$
\begin{aligned}
f_0(t) &:= \frac{4\bar{M}\kappa(t)(\kappa(t)-\alpha)}{(M+\bar{M})(\alpha-2\kappa(t))},\\
f_1(t) &:= \frac{2\gamma B'(t)(\kappa(t)-\alpha)+2\alpha\kappa(t)}{\alpha-2\kappa(t)},\\
f_2(t) &:= \frac{2\gamma B'(t)(\kappa(t)-\alpha)}{\alpha-2\kappa(t)},\\
f_3(t) &:= \frac{2\gamma B'(t)(\alpha-\kappa(t))}{(M+\bar{M})(\alpha-2\kappa(t))},\\
\bar{f}_3(t) &:= \frac{2\gamma B'(t)(\alpha-\kappa(t))}{(M+\bar{M})(\alpha-2\kappa(t))},\\
\bar{f}_4(t) &:= \frac{2(\alpha-\kappa(t))\left(\gamma(A(t)-M+1)B'(t)-2\kappa(t)\right)}{(M+\bar{M})(\alpha-2\kappa(t))},\\
\bar{f}_5(t) &:= \frac{2\kappa(t)(\alpha(M-\bar{M})+2\bar{M}\kappa(t))}{(M+\bar{M})(\alpha-2\kappa(t))}.
\end{aligned}
\tag{A.1}
$$

A.2 Nash equilibrium

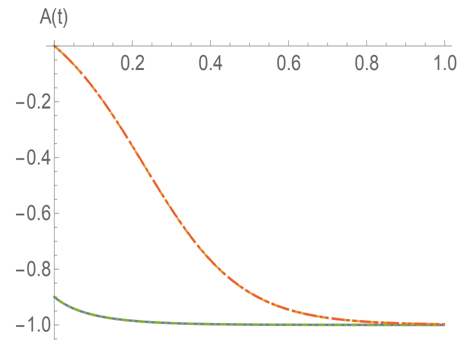
$$
\begin{aligned}
\mu_1(t) &:= \frac{2\gamma(M+\bar{M}-2)B'(t)(\kappa(t)-\alpha)+2\kappa(t)(\alpha(M+\bar{M}-4)+2\kappa(t))}{\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t)},\\
\mu_2(t) &:= -\frac{2\gamma(M+\bar{M}-2)B'(t)(\alpha-\kappa(t))}{\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t)},\\
\mu_3(t) &:= \frac{\Big\{-\big(4\gamma(M+\bar{M}-2)B'(t)(\alpha-\kappa(t))^2\big)\Big\}}{\begin{array}{l}\Big\{\big((\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t)\big)\big(\alpha((3M-1)\bar{M}^2+M(3M-2)\bar{M}\\ +(M-2)M(M+1)+\bar{M}^3)-2\left((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\right)\kappa(t)\big)\big)\Big\}\end{array}},\\
\bar{\mu}_4(t) &:= \frac{\begin{array}{l}\Big\{-\Big(2(M+\bar{M}-2)(\alpha-\kappa(t))\big(\gamma B'(t)\big(\alpha\left(-2A(t)+(M+\bar{M}-1)(M+\bar{M})^2-2\right)\\ -2\kappa(t)\left(-A(t)+(3M-2)\bar{M}^2+M(3M-4)\bar{M}+(M-1)^2M+\bar{M}^3+\bar{M}-1\right)\big)+4\kappa(t)(\alpha-\kappa(t))\big)\Big)\Big\}\end{array}}{\begin{array}{l}\Big\{\Big(\big(\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t)\big)\big(\alpha((3M-1)\bar{M}^2+M(3M-2)\bar{M}\\ +(M-2)M(M+1)+\bar{M}^3)-2\left((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\right)\kappa(t)\big)\Big)\Big\}\end{array}},\\
\bar{\mu}_5(t) &:= \frac{\begin{array}{l}\Big\{\Big(2\kappa(t)\big(\alpha\big(M^2(3\bar{M}-5)+M^3+M(\bar{M}(3\bar{M}-10)+6)\\ +(\bar{M}-4)(\bar{M}-1)\bar{M}\big)+2\left(M^2+2M(\bar{M}-1)+(\bar{M}-1)\bar{M}\right)\kappa(t)\big)\Big)\Big\}\end{array}}{\begin{array}{l}\Big\{\Big(\alpha\left((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\right)\\ -2\left((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\right)\kappa(t)\Big)\Big\}\end{array}},\\
\nu_0(t) &:= \frac{1}{M+\bar{M}-2}+1,\\
\nu_1(t) &:= \frac{2\alpha(M+\bar{M}-2)\kappa(t)-2\gamma(M+\bar{M}-1)B'(t)(\alpha-\kappa(t))}{\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t)},\\
\nu_2(t) &:= -\frac{2\gamma(M+\bar{M}-1)B'(t)(\alpha-\kappa(t))}{\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t)},\\
\nu_3(t) &:= \frac{\Big\{-(M+\bar{M}-1)4\gamma B'(t)(\alpha-\kappa(t))^2\Big\}}{\begin{array}{l}\Big\{\Big(\big(\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t)\big)\big(\alpha\big((3M-1)\bar{M}^2+M(3M-2)\bar{M}\\ +(M-2)M(M+1)+\bar{M}^3\big)-2\left((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\right)\kappa(t)\big)\Big)\Big\}\end{array}},\\
\bar{\nu}_3(t) &:= \frac{2\gamma(M+\bar{M}-2)B'(t)(\alpha-\kappa(t))}{\alpha\left((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\right)-2\left((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\right)\kappa(t)},\\
\bar{\nu}_4(t) &:= \frac{\begin{array}{l}\Big\{-\Big(2(M+\bar{M}-2)(\alpha-\kappa(t))\big(\gamma B'(t)\big(\alpha\big(-A(t)(M+\bar{M}+2)+(M+\bar{M}-1)\left(M^2+2M\bar{M}+M+\bar{M}^2\right)\\ -\bar{M}-2\big)+2\kappa(t)\left(A(t)(M+\bar{M})+M^2(1-3\bar{M})-M^3-3M(\bar{M}-1)\bar{M}+M-(\bar{M}-2)\bar{M}^2\right)\big)\\ +2\kappa(t)(\alpha(M+\bar{M}+2)-2(M+\bar{M})\kappa(t))\big)\Big)\Big\}\end{array}}{\begin{array}{l}\Big\{\Big(\big(\alpha(M+\bar{M})-2(M+\bar{M}-1)\kappa(t)\big)\big(\alpha\left((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\right)\\ -2\left((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\right)\kappa(t)\big)\Big)\Big\}\end{array}},\\
\bar{\nu}_5(t) &:= \frac{2(M+\bar{M}-1)\kappa(t)\left(\alpha\left(M^2+2M(\bar{M}-1)+(\bar{M}-4)\bar{M}\right)+2\bar{M}\kappa(t)\right)}{\alpha\left((3M-1)\bar{M}^2+M(3M-2)\bar{M}+(M-2)M(M+1)+\bar{M}^3\right)-2\left((M+\bar{M}-2)(M+\bar{M})^2+\bar{M}\right)\kappa(t)}.
\end{aligned}
\tag{A.2}
$$

B Plots of ODE solutions

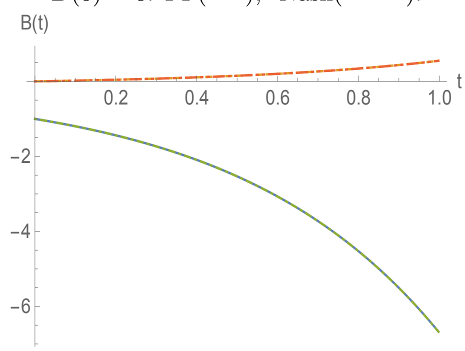
Figure 4: Plots of ODE solutions (A, B, Σ) in (3.7) and (4.19) for $B(0) \in \{-1, 0\}$. Plots 4A, 4C, and 4E use $\gamma := 1$ and Plots 4B, 4D, and 4F use $\gamma := 0.5$. The exogenous model parameters are $\sigma_{w_0} := \sigma_{\tilde{a}} := 1, M := \bar{M} := 10,\ \alpha := -0.1$, and $\kappa(t) := 1$ for $t \in [0, 1]$.



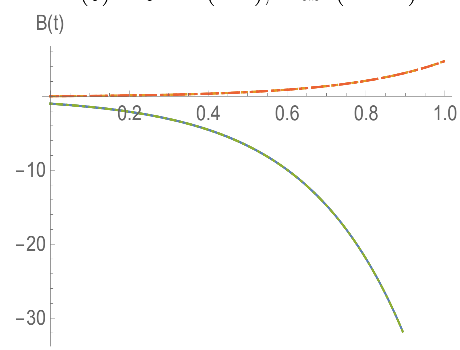
4A: $B(0) = -1$: PI (—), Nash $(- \cdot -)$, $B(0) = 0$: PI $(--)$, Nash$(- \cdot \cdot -)$.



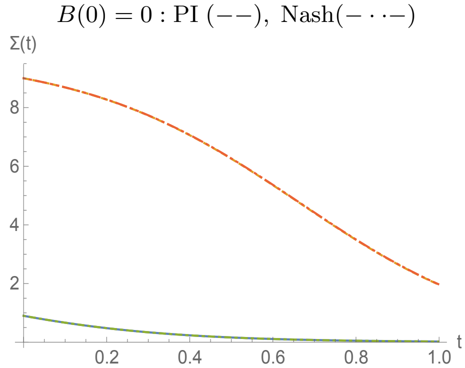
4B: $B(0) = -1$: PI (—), Nash $(- \cdot -)$, $B(0) = 0$: PI $(--)$, Nash$(- \cdot \cdot -)$.



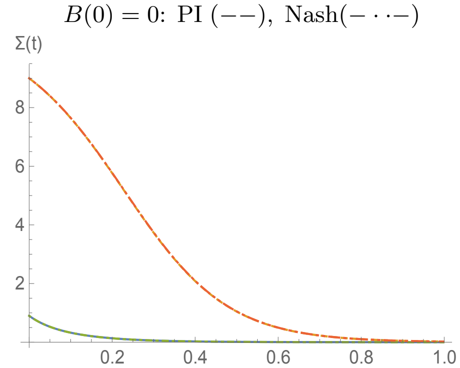
4C: $B(0) = -1$: PI (—), Nash $(- \cdot -)$, $B(0) = 0$: PI $(--)$, Nash$(- \cdot \cdot -)$



4D: $B(0) = -1$: PI (—), Nash $(- \cdot -)$, $B(0) = 0$: PI $(--)$, Nash$(- \cdot \cdot -)$



4E: $B(0) = -1$: PI (—), Nash $(- \cdot -)$, $B(0) = 0$: PI $(--)$, Nash$(- \cdot \cdot -)$



4F: $B(0) = -1$: PI (—), Nash $(- \cdot -)$, $B(0) = 0$: PI $(--)$, Nash$(- \cdot \cdot -)$

C Kalman-Bucy filtering

The proof of Lemma 2.1 follows from the well-known Kalman-Bucy result in filtering theory and can be found in, e.g., Lipster and Shiryaev (Chapter 8, 2001). We note that the solution to the Riccati equation (C.3) is given by (2.15).

Theorem C.1 (Kalman-Bucy)**.** *Let* $B : [0,1] \to \mathbb{R}$ *be a continuously differentiable function and consider the Gaussian observation process* $Y_{i,t} := w_t - B(t)(\tilde{a}_\Sigma - \tilde{a}_i)$ *from* (2.9) *with dynamics*

$$dY_{i,t} = dw_t - B'(t)\big(\tilde{a}_\Sigma - \tilde{a}_i\big)dt, \quad Y_{i,0} = w_0 - B(0)(\tilde{a}_\Sigma - \tilde{a}_i) \tag{C.1}$$

and corresponding innovations process $w_{i,t}$ *in* (2.11)*. Then,* (2.13) *holds and the filtering property in* (2.11) *holds if* $q_{i,t}$ *has dynamics given by*

$$\begin{aligned} dq_{i,t} &= -B'(t)\Sigma(t)dY_{i,t} - \big(B'(t)\big)^2\Sigma(t)q_{i,t}dt \\ &= -B'(t)\Sigma(t)dw_{i,t}, \\ q_{i,0} &= \mathbb{E}[\tilde{a}_\Sigma - \tilde{a}_i | \sigma(Y_{i,0})] \\ &= \mathbb{E}[\tilde{a}_\Sigma - \tilde{a}_i | \sigma\big(w_0 - B(0)(\tilde{a}_\Sigma - \tilde{a}_i)\big)] \\ &= -\frac{B(0)\mathbb{V}[\tilde{a}_\Sigma - \tilde{a}_i]}{\mathbb{V}[w_0] + B(0)^2\mathbb{V}[\tilde{a}_\Sigma - \tilde{a}_i]}\big(w_0 - B(0)(\tilde{a}_\Sigma - \tilde{a}_i)\big) \\ &= -\frac{(M-1)B(0)\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2 + B(0)^2(M-1)\sigma_{\tilde{a}}^2}\big(w_0 - B(0)(\tilde{a}_\Sigma - \tilde{a}_i)\big), \end{aligned} \tag{C.2}$$

and the remaining variance is given by

$$\Sigma'(t) = -\big(B'(t)\big)^2\Sigma(t)^2, \tag{C.3}$$

with initial value

$$\begin{aligned} \Sigma(0) &= \mathbb{V}[\tilde{a}_\Sigma - \tilde{a}_i - q_{i,0}] \\ &= \mathbb{E}[(\tilde{a}_\Sigma - \tilde{a}_i - q_{i,0})^2] \\ &= \mathbb{E}\left[\left(\tilde{a}_\Sigma - \tilde{a}_i + \tfrac{(M-1)B(0)\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2 + B(0)^2(M-1)\sigma_{\tilde{a}}^2}\big(w_0 - B(0)(\tilde{a}_\Sigma - \tilde{a}_i)\big)\right)^2\right] \\ &= \left(\tfrac{(M-1)B(0)\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2 + B(0)^2(M-1)\sigma_{\tilde{a}}^2}\right)^2\sigma_{w_0}^2 + \left(1 - B(0)\tfrac{(M-1)B(0)\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2 + B(0)^2(M-1)\sigma_{\tilde{a}}^2}\right)^2(M-1)\sigma_{\tilde{a}}^2 \\ &= \frac{(M-1)\sigma_{\tilde{a}}^2\sigma_{w_0}^2}{B(0)^2(M-1)\sigma_{\tilde{a}}^2 + \sigma_{w_0}^2}. \end{aligned} \tag{C.4}$$

D Remaining proofs

Proof of Lemma 2.2. To see that (2.14) holds, we use the Kalman-Bucy filter (C.2) below to write

$$q_{i,t} = q_{i,0} - \int_0^t B'(u)\Sigma(u)dw_{i,u}, \quad t \in [0,1]. \tag{D.1}$$

Then,

$$\begin{aligned}
\sum_{i=1}^M q_{i,0} &= -\frac{(M-1)B(0)\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2 + B(0)^2(M-1)\sigma_{\tilde{a}}^2}\big(Mw_0 - B(0)(M\tilde{a}_\Sigma - \tilde{a}_\Sigma)\big) \\
&= -\frac{(M-1)B(0)\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2 + B(0)^2(M-1)\sigma_{\tilde{a}}^2}\big(MY_0 + B(0)\tilde{a}_\Sigma\big), \\
\sum_{i=1}^M B'(t)\Sigma(t)dw_{i,t} &= B'(t)\Sigma(t)\Big(Mdw_t + B'(t)\Big\{\tilde{a}_\Sigma + \sum_{i=1}^M q_{i,t} - M\tilde{a}_\Sigma\Big\}dt\Big) \\
&= B'(t)\Sigma(t)\Big(MdY_t + B'(t)\Big\{\tilde{a}_\Sigma + \sum_{i=1}^M q_{i,t}\Big\}dt\Big).
\end{aligned} \tag{D.2}$$

To explicitly solve for $\sum_{i=1}^M q_{i,t}$, we note

$$\begin{aligned}
de^{\int_0^t (B'(u))^2\Sigma(u)du}\sum_{i=1}^M q_{i,t} &= e^{\int_0^t (B'(u))^2\Sigma(u)du}\Big\{(B'(t))^2\Sigma(t)\sum_{i=1}^M q_{i,t}dt - \sum_{i=1}^M B'(t)\Sigma(t)dw_{i,t}\Big\} \\
&= -e^{\int_0^t (B'(u))^2\Sigma(u)du}B'(t)\Sigma(t)\Big(MdY_t + B'(t)\tilde{a}_\Sigma dt\Big).
\end{aligned} \tag{D.3}$$

We get the solution $\sum_{i=1}^M q_{i,t}$ by integrating

$$\begin{aligned}
\sum_{i=1}^M q_{i,t} &= e^{-\int_0^t (B'(u))^2\Sigma(u)du}\sum_{i=1}^M q_{i,0} \\
&\quad - \int_0^t e^{-\int_s^t (B'(u))^2\Sigma(u)du}B'(s)\Sigma(s)\Big(MdY_s + B'(s)\tilde{a}_\Sigma ds\Big).
\end{aligned} \tag{D.4}$$

Thus, the decomposition (2.14) holds with

$$\begin{aligned}
A(t) &:= -e^{-\int_0^t (B'(u))^2\Sigma(u)du}\tfrac{(M-1)B(0)^2\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2 + B(0)^2(M-1)\sigma_{\tilde{a}}^2} - \int_0^t e^{-\int_s^t (B'(u))^2\Sigma(u)du}\big(B'(s)\big)^2\Sigma(s)ds, \\
\eta_t &:= -e^{-\int_0^t (B'(u))^2\Sigma(u)du}\tfrac{M(M-1)B(0)\sigma_{\tilde{a}}^2}{\sigma_{w_0}^2 + B(0)^2(M-1)\sigma_{\tilde{a}}^2}Y_0 - M\int_0^t e^{-\int_s^t (B'(u))^2\Sigma(u)du}B'(s)\Sigma(s)dY_s.
\end{aligned} \tag{D.5}$$

$\diamondsuit$

Proof of Lemma 3.1. The inclusion "$\supseteq$" in (3.2) follows from (2.4), (2.10), and (2.13). To see the inclusion "$\subseteq$", we use Y_t in (2.8), η_t in (D.5), and $q_{i,t}$ in (C.2) to find deterministic functions h_0, h, and H such that

$$\begin{aligned} dS^f_{i,t} - \alpha\theta_{i,t}dt &= \Big\{f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t\Big\}dt + \gamma dw_{i,t} \\ &= \Big\{h_0(t)Y_0 + h(t)\tilde{a}_i + \int_0^t H(u,t)dw_{i,u}\Big\}dt + \gamma dw_{i,t}. \end{aligned} \tag{D.6}$$

We define

$$dZ_{i,t} := dS^f_{i,t} - \Big\{\alpha\theta_{i,t} + h_0(t)Y_0 + h(t)\tilde{a}_i\Big\}dt, \quad Z_{i,0} := w_{i,0}, \tag{D.7}$$

The inclusion "$\subseteq$" in (3.2) will follow from the inclusion

$$\sigma(w_{i,u})_{u\in[0,t]} \subseteq \sigma(Z_{i,u})_{u\in[0,t]}. \tag{D.8}$$

To see (D.8), let $t_0 \in [0,t]$ be arbitrary and let $f(s)$, $s \in [0,t]$, solve the following Volterra integral equation of the second kind (such f exists by Lemma 4.3.3 in Davis (1977) because $\gamma \neq 0$):

$$\int_r^t f(s)H(r,s)ds + f(r)\gamma = 1_{[0,t_0]}(r), \quad r \in [0,t]. \tag{D.9}$$

This gives us

$$\begin{aligned} \int_0^t f(s)dZ_{i,s} &= \int_0^t f(s)\int_0^s H(r,s)dw_{i,r}ds + \int_0^t f(s)\gamma dw_{i,s} \\ &= \int_0^t \int_r^t f(s)H(r,s)dsdw_{i,r} + \int_0^t f(s)\gamma dw_{i,s} \\ &= \int_0^t \Big(\int_r^t f(s)H(r,s)ds + f(r)\sigma\Big)dw_{i,r} \\ &= \int_0^t 1_{[0,t_0]}(r)dw_{i,r} \\ &= w_{i,t_0} - w_{i,0}. \end{aligned} \tag{D.10}$$

$\diamondsuit$

Proof of Lemma 3.2. Consider a rebalancer $i \in \{1, ..., M\}$. For arbitrary holdings $\theta_{i,t}$, the expectation in the i'th objective in (2.5) is

$$\begin{aligned}&\mathbb{E}\Big[\int_0^1 \theta_{i,t}dS^f_{i,t} - \int_0^1 \kappa(t)(\tilde{a}_i-\theta_{i,t})^2dt\Big]\\&= \mathbb{E}\Big[\int_0^1 \theta_{i,t}\Big\{f_0(t)Y_t + f_1(t)\tilde{a}_i + f_2(t)q_{i,t} + f_3(t)\eta_t + \alpha\theta_{i,t}\Big\}dt - \int_0^1 \kappa(t)(\tilde{a}_i-\theta_{i,t})^2dt\Big].\end{aligned} \tag{D.11}$$

The equality in (D.11) follows from the square integrability condition (2.6), which ensures that the stochastic integral $\int_0^s \theta_{i,t}dw_{i,t}$ is a martingale with zero expectation. We can maximize the integrand in (D.11) pointwise because the second-order condition $\alpha < \kappa(t)$ holds. This gives the first formula in (3.4).

The second formula for a tracker j in (3.4) is proved similarly. ♢

Proof of Lemma 3.4. The local Lipschitz property of the ODEs (3.7) ensures that there exists a maximal interval of existence $[0,\tau)$ with $\tau \in (0,\infty]$ by the Picard-Lindelöf theorem (see, e.g., Theorem II.1.1 in Hartman 2002). We assume that $\tau < 1$ and construct a contradiction. To this end, we set

$$K := \int_0^1 \kappa(s)ds < \infty. \tag{D.12}$$

First, the Riccati ODE for $\Sigma(t)$ has the explicit solution in (2.15), which cannot explode as $t \uparrow \tau$ (even if $B(t)$ should explode as $t \uparrow \tau$).

Second, the initial value $A(0)$ in (3.7) ensures $A(0) \geq -1$ and to see that implies $A(t) \geq -1$ for all $t \in [0,\tau)$, we note

$$\frac{\partial}{\partial t}\big(A(t)+1\big) = -\big(B'(t)\big)^2\Sigma(t)\big(A(t)+1\big), \tag{D.13}$$

which implies

$$\begin{aligned}A(t)+1 &= \big(A(0)+1\big)e^{-\int_0^t (B'(s))^2\Sigma(s)ds}\\&\geq 0.\end{aligned} \tag{D.14}$$

This shows that $A(t)$ cannot explode as $t \uparrow \tau$ (even if $B(t)$ should explode as $t \uparrow \tau$).

Third, we show $B(t)$ is uniformly bounded for $t \in [0,\tau)$; hence, also $B(t)$ cannot explode as $t \uparrow \tau$. This then gives a contradiction to Theorem II.3.1 in Hartman (2002). The affine ODE for $B(t)$ in (4.19) has the explicit solution

$$B(t) = e^{\int_0^t \frac{2\bar{M}\kappa(s)}{\gamma(A(s)+1+\bar{M})}ds}\Big(B(0) + \int_0^t \frac{2\kappa(s)}{\gamma(A(s)+1+\bar{M})}e^{-\int_0^s \frac{2\bar{M}\kappa(u)}{\gamma(A(u)+1+\bar{M})}du}ds\Big). \tag{D.15}$$

Using K in (D.12) produces the bound

$$\int_0^t \frac{2\bar{M}\kappa(s)}{\gamma\big(A(s)+1+\bar{M}\big)}ds \leq \int_0^t \frac{2\bar{M}\kappa(s)}{\gamma\bar{M}}ds \leq \frac{2K}{\gamma}, \quad t\in[0,\tau). \tag{D.16}$$

In turn, the bound (D.16) and (D.15) imply

$$\begin{aligned} |B(t)| &\leq e^{\frac{2K}{\gamma}}\Big(|B(0)| + \int_0^t \frac{2\kappa(s)}{\gamma(A(s)+1+\bar{M})}ds\Big) \\ &\leq e^{\frac{2K}{\gamma}}\Big(|B(0)| + \frac{2K}{\gamma\bar{M}}\Big), \end{aligned} \tag{D.17}$$

for $t\in[0,\tau)$. Because the bound in (D.17) is uniform over $t\in[0,\tau)$, $B(t)$ does not explode as $t\uparrow\tau$.

♢

Proof of Theorem 3.5. To see that the holdings in (3.8) satisfy the square integrability condition (2.6), we insert $B'(t)$ from (3.7) to get

$$\begin{aligned} \hat{\theta}_{i,t} &= -\tfrac{2\kappa(t)(A(t)+\bar{M}(1-B(t)))}{(A(t)+\bar{M}+1)(\alpha-2\kappa(t))}\tilde{a}_i + \tfrac{2\kappa(t)(\bar{M}B(t)+1)}{(A(t)+\bar{M}+1)(\alpha-2\kappa(t))}q_{i,t} \\ &\quad - \tfrac{2\kappa(t)(\bar{M}B(t)+1)}{(M+\bar{M})(A(t)+\bar{M}+1)(\alpha-2\kappa(t))}\eta_t + \tfrac{2\bar{M}\kappa(t)}{(M+\bar{M})(\alpha-2\kappa(t))}Y_t, \\ \hat{\theta}_{j,t} &= -\tfrac{2\kappa(t)(\bar{M}B(t)+1)}{(M+\bar{M})(A(t)+\bar{M}+1)(\alpha-2\kappa(t))}\eta_t - \tfrac{2M\kappa(t)}{(M+\bar{M})(\alpha-2\kappa(t))}w_t \\ &\quad + \tfrac{2\kappa(t)(\bar{M}B(t)(-A(t)+M-1)+M+\bar{M})}{(M+\bar{M})(A(t)+\bar{M}+1)(\alpha-2\kappa(t))}\tilde{a}_\Sigma. \end{aligned} \tag{D.18}$$

Because $\kappa:[0,1]\to(0,\infty)$ is continuous, $\kappa(t)$ is uniformly bounded. This gives us that $B'(t)$ in (3.7) is also uniformly bounded. As a consequence, the variances $\mathbb{V}[q_{i,t}], \mathbb{V}[\eta_t]$, and $\mathbb{V}[Y_t]$ are also uniformly bounded functions of $t\in[0,1]$. Therefore, the holding processes in (D.18) satisfy (2.6) if the coefficient functions for $(\tilde{a}_i, q_{i,t}, \eta_t, Y_t, w_t, \tilde{a}_\Sigma)$ are square integrable over $t\in[0,1]$. For example, the coefficient function for $\tilde{a}_i$ in $\hat{\theta}_{i,t}$ is bounded because

$$\Big|\tfrac{2\kappa(t)(A(t)+\bar{M}(1-B(t)))}{(A(t)+\bar{M}+1)(\alpha-2\kappa(t))}\Big| \leq \tfrac{2|A(t)+\bar{M}(1-B(t))|}{\bar{M}}, \tag{D.19}$$

which is continuous for $t\in[0,1]$. Similarly, the remaining coefficients functions can be seen to be bounded too. The optimality in Definition 3.3(i) then follows from Lemma 3.2 and the fact that the holdings (3.8) are those in (3.4) with the f functions in (A.1) inserted.

Definition 3.3(ii)+(iii) are ensured by the specific f functions in (A.1).

♢

Proof of Lemma 4.1. Lemma A.1 in Choi, Larsen, and Seppi (2021) and the continuity of Z_t imply that Z_t is adapted to both $\mathcal{F}_{i,t}$ and $\mathcal{F}_{j,t}$. The rest of this proof is similar to the proof of Lemma 3.2 given above and is therefore omitted. ♢

Proof of Lemma 4.2. The rebalancers' second-order condition is

$$\frac{\big(\alpha-\kappa(t)\big)\big(M+\bar{M}+2\nu_0(t)-1\big)}{M+\bar{M}-1}<0, \tag{D.20}$$

whereas the trackers' second-order condition is $\alpha<\kappa(t)$. Inequality (D.20) holds because $\nu_0(t)\geq 0$ and $\alpha<\kappa(t)$. The rest of this proof is similar to the proof of Lemma 3.2 given above and is therefore omitted. ♢

Proof of Lemma 4.4. The proof only requires minor changes to the proof of Lemma 3.4. As before, we let $[0,\tau)$ be the maximal interval of existence with $\tau\in(0,\infty]$ and assume that $\tau<1$ to construct a contradiction. As in the proof of Lemma 3.4, $\Sigma(t)=\frac{1}{\frac{1}{\Sigma(0)}+\int_0^t(B'(t))^2dt}$ and $A(t)\geq -1$. Next, to show $B(t)$ is bounded on $[0,\tau)$, we rewrite the ODE for $B(t)$ in (4.19) as

$$B'(t)=\frac{2\kappa(t)\big(B(t)(c(t)+\bar{M})+1\big)}{\gamma\big(A(t)+1+\bar{M}+c(t)\big)}. \tag{D.21}$$

where the deterministic function $c(t)$ is defined as

$$c(t):=\frac{2\bar{M}\big(\kappa(t)-\alpha\big)}{(M+\bar{M}-2)\big(2(M+\bar{M})\kappa(t)-\alpha(M+\bar{M}+1)\big)},\quad t\in[0,1]. \tag{D.22}$$

Because $\alpha\leq 0$ and $\kappa(t)>0$, we have $c(t)>0$. Furthermore, $c(t)$ is bounded because

$$\begin{aligned} c(t) &\leq \frac{2\bar{M}(\kappa(t)-\alpha)}{(M+\bar{M}-2)(M+\bar{M}+1)(\kappa(t)-\alpha)}\\ &= \frac{2\bar{M}}{(M+\bar{M}-2)(M+\bar{M}+1)}\\ &=: c_0, \end{aligned} \tag{D.23}$$

where the inequality follows from $2(M+\bar{M})>(M+\bar{M}+1)$ and the positivity of $\kappa(t)$. Because $A(t)+1\geq 0$ and $c(t)>0$ we get the two estimates

$$\begin{aligned} \int_0^t \frac{2\kappa(s)(c(s)+\bar{M})}{\gamma(A(s)+1+\bar{M}+c(s))}ds &\leq \frac{2(c_0+\bar{M})}{\gamma\bar{M}}K,\\ \int_0^t \frac{2\kappa(s)}{\gamma(A(s)+1+\bar{M}+c(s))}ds &\leq \frac{2}{\gamma\bar{M}}K, \end{aligned} \tag{D.24}$$

where K is as in (D.12). Similar to (D.15), the explicit solution of (D.21) is

$$B(t)=e^{\int_0^t \frac{2\kappa(s)(c(s)+\bar{M})}{\gamma(A(s)+1+\bar{M}+c(s))}ds}B(0)+\int_0^t e^{\int_s^t \frac{2\kappa(u)(c(u)+\bar{M})}{\gamma(A(u)+1+\bar{M}+c(u))}du}\frac{2\kappa(s)}{\gamma(A(s)+1+\bar{M}+c(s))}ds. \tag{D.25}$$

Combing this expression for $B(t)$ with the bounds (D.24) produces

$$\begin{aligned} B(t) &\leq e^{\frac{2(c_0+\bar{M})K}{\gamma\bar{M}}}|B(0)| + \int_0^t e^{\frac{2(c_0+\bar{M})K}{\gamma\bar{M}}} \frac{2\kappa(s)}{\gamma(A(s)+1+\bar{M}+c(s))} ds \\ &\leq e^{\frac{2(c_0+\bar{M})K}{\gamma\bar{M}}}(|B(0)| + \frac{2K}{\gamma\bar{M}}). \end{aligned} \tag{D.26}$$

♢

Proof of Theorem 4.5. From (D.21) we see that

$$\begin{aligned} |B'(t)| &= \frac{2\kappa(t)\big(|B(t)|(c(t)+\bar{M})+1\big)}{\gamma\big(A(t)+1+\bar{M}+c(t)\big)} \\ &\leq \frac{2\kappa(t)\big(|B(t)|(c_0+\bar{M})+1\big)}{\gamma\bar{M}}, \end{aligned} \tag{D.27}$$

where c_0 is defined in (D.23). Because $\kappa(t)$ is continuous on $t \in [0,1]$, $\kappa(t)$ is bounded and from (D.17) we know that $B(t)$ is bounded too. Therefore, from (D.27), we see that $B'(t)$ is also uniformly bounded. Consequently, the variances $\mathbb{V}[q_{i,t}], \mathbb{V}[\eta_t]$, and $\mathbb{V}[Y_t]$ are also uniformly bounded functions of $t \in [0,1]$.

As before, the coefficient functions for $(\tilde{a}_i, q_{i,t}, \eta_t, Y_t, w_t, \tilde{a}_\Sigma)$ in (4.20) are all uniformly bounded for $t \in [0,1]$. Therefore, the square-integrability condition (2.6) holds.

The requirements in Definition 4.3 follow from the definition of the functions in (A.2).

♢

References

[1] Atmaz, A., and S. Basak (2021): *Stock market and no-dividend stocks*, working paper.

[2] Almgren, R. (2003): *Optimal execution with nonlinear impact functions and trading-enhanced risk*, Applied Mathematical Finance **10**, 1–18.

[3] Almgren, R., and N. Chriss (1999): *Value under liquidation*, Risk **12**, 61–63.

[4] Almgren, R., and N. Chriss (2000): *Optimal execution of portfolio transactions*, Journal of Risk **3**, 5–39.

[5] Back, K., H. Cao, and G. Willard (2000): *Imperfect competition among informed traders*, Journal of Finance **55**, 2117–2155.

[6] Bouchard, B., M. Fukasawa, M. Herdegen, and J. Muhle-Karbe (2018): *Equilibrium returns with transaction costs*, Finance and Stochastics **22**, 569–601.

[7] Boudoukh, J., R. Michaely, M. Richardson, and M. R. Roberts (2007): *On the importance of measuring payout yield: Implications for empirical asset pricing*, Journal of

Finance **62**, 877–915.

[8] Brunnermeier, M. K., and L. H. Pedersen (2005): *Predatory trading*, Journal of Finance **60**, 1825–1863.

[9] Carlin, B., M. Lobo, and S. Viswanathan (2007): Episodic liquidity crises: Cooperative and predatory trading. Journal of Finance, 62, 2235–2274.

[10] Chen, X., J. H. Choi, K. Larsen, and D. Seppi (2021): *Asset-pricing puzzles and price-impact*, working paper.

[11] Choi, J. H., K. Larsen, and D. Seppi (2019): *Information and trading targets in a dynamic market equilibrium*, Journal of Financial Economics **132**, 22–49.

[12] Choi, J. H., K. Larsen, and D. Seppi (2021): *Equilibrium effects of intraday order-splitting benchmarks*, Mathematics and Financial Economics **15**, 315–352.

[13] Davis, M. H. A. (1977): *Linear estimation and stochastic control*, Wiley.

[14] Foster, F., and S. Viswanathan (1994): *Strategic trading with asymmetrically informed traders and long-lived information*, Journal of Financial Quantitative Analysis **29**, 499–518.

[15] Foster, F., and S. Viswanathan (1996): *Strategic trading when agents forecast the forecasts of others*, Journal of Finance **51**, 1437–1478.

[16] Gârleanu, N., and L. H. Pedersen (2016): *Dynamic portfolio choice with frictions*, Journal of Economic Theory **165**, 487–516.

[17] Grossman, S. J., and J. E. Stiglitz (1980): *On the impossibility of informationally efficient markets*, American Economic Review, **70**, 393–408.

[18] Hartman, P. (2002): *Ordinary differential equations*, 2nd Ed., SIAM Classics in Applied Mathematics.

[19] Hartzmark, S. M., and D. H. Solomon (2013): *The dividend month premium*, Journal of Financial Economics **109**, 640–660.

[20] van Kervel, V., A. Kwan, and P. Westerholm (2020): *Order splitting and interacting with a counterparty*, working paper.

[21] Kyle, A. (1985): *Continuous auctions and insider trading*, Econometrica **53**, 1315–1336.

[22] Lipster, R. S., and A. N. Shiryaev (2001): *Statistics of random processes I*, Springer.

[23] Noh, E., and K. Weston (2020): *Price impact equilibrium with transaction costs and TWAP trading*, working paper.

[24] O'Hara, M. (2015): *High frequency market microstructure*, Journal of Financial Economics **116**, 257–270.

[25] Sannikov, Y., and A. Skrzypacz (2016): Dynamic trading: Price inertia and front-running, working paper.

[26] Schied, A., and T. Schöneborn (2009): *Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets*, Finance Stochastics **13**, 181–204.

[27] Vayanos, D. (1999): *Strategic trading and welfare in a dynamic market*, Review of Economic Studies **66**, 219–254.